Int. J. Math. And Appl., 6(1–D)(2018), 747–755
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Coupled Fixed Point Theorems in Ordered Non-Archimedean Intuitionistic Fuzzy Metric Space Using k-Monotone Property

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Abstract: Michlelet [17] proved a theorem which assures the existence of a fixed point for fuzzy ψ-contractive mappings in the framework of complete non-Archimedean fuzzy metric spaces. Motivated by this, we introduce notion of k-monotone property in Intuitionistic fuzzy metric space and proved coupled fixed point theorems for the map satisfying the mixed monotone property in partially ordered complete non-Archimedean intuitionistic fuzzy metric space.

MSC: 46S40, 47H10, 54H25

Keywords: Non- Archimedean property, k-monotone property, mixed monotone mappings, coupled fixed point, Fuzzy metric space, Intuitionistic Fuzzy metric space, Cauchy sequence, complete fuzzy metric space.

1. Introduction

Fuzzy set theory, a generalization of crisp set theory, was first introduced by Zadeh [25] in 1965 to describe situations in which data are imprecise or vague or uncertain. Kramosil and Michalek [14] introduced the concept of fuzzy metric spaces in 1975, which opened an avenue for further development of analysis in such spaces. Later on it is modified that a few concepts of mathematical analysis have been generalized by George and Veeramani [12]. Afterwards, many articles have been published on fixed point theorems under different contractive condition in fuzzy metric spaces.

Atanassov [3] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Coker [9] introduced the concepts of the so called "Intuitionistic fuzzy topological spaces". Park [21], using the idea of intuitionistic fuzzy sets, define the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norm and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [12].

Bhaskar and Lakshmikantham [5] discussed the mixed monotone mappings and gave some coupled fixed point theorems which can be used to discuss the existence and uniqueness of solution for a periodic boundary value problem.

Hu [13] studied common coupled fixed point theorems for contractive mappings in fuzzy metric space, and Park et al. [21] defined an IFMS and proved a fixed point theorem in IFMS. Chandok et al. [6], Choudhury et al [7], Cric and Laxmikantam [5], Nguyen et al. [19] studied and give the results on common coupled fixed point theorems in different metric spaces.

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Berinde [4] Generalized coupled fixed point theorems for mixed monotone mappings in partially ordered metric spaces, Recently Luong et.al. [15] proved coupled fixed points in partially ordered metric spaces. Mohinta and Samantha [18] and Park [22] prove the coupled fixed point theorem in non-Archimedean intuitionistic fuzzy metric space. Erduran F. Sola et al. proved A common fixed point theorem in weak non-Archimedean intuitionistic fuzzy metric spaces.

Now, we briey describe our reasons for being interested in results of this kind. The applications of fixed point theorems are remarkable in different disciplines of mathematics, engineering and economics in dealing with problems arising in approximation theory, game theory and many. Consequently, many researchers, following the Banach contraction principle, investigated the existence of weaker contractive conditions or extended previous results under relatively weak hypotheses on the metric space. The starting point of our paper is to follow this trend by introducing, with the denition of non-Archimedean fuzzy metric space, a more general setting than non-Archimedean Intuitionistic fuzzy metric space. The reader is referred to [2] for some discussion and applications on non-Archimedean metric spaces and its induced topology.

In this paper, we define non-Archimedean intuitionistic fuzzy metric space, and prove a coupled fixed point theorems for the map satisfying the mixed monotone property in partially ordered complete non-Archimedean intuitionistic fuzzy metric space.

2. Preliminaries

Definition 2.1 ([23]). *A binary operation* $*[0,1] \times [0,1] \to [0,1]$ *is a continuous t-norms if " * " is satisfying conditions:*

*(1). * is an commutative and associative.*

*(2). * is continuous.*

(3). $a * 1 = a$ *for all* $a \in [0,1]$.

(4). $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, *and* $a, b, c, d \in [0,1]$.

Basic example of t–norm are the Lukasiewicz t–norm $T_1, T_1(a,b) = \max(a+b-1, 0)$, *t–norm* $T_p, T_p(a,b)$, *and t–norm* $T_M, T_M(a,b) = \min\{a,b\}$.

Definition 2.2 ([14]). *A 3-tuple* $(X, M, *)$ *is said to be non-Archimedean fuzzy metric space if* X *is an arbitrary set, * is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$, $s, t > 0$,

(F_1) $M(x,y,0) = 0$.

(F_2) $M(x,y,t) = 1$ *if and only if* $x = y$.

(F_3) $M(x,y,t) = M(y,x,t)$.

(F_4) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$.

(F_5) $M(x,y,\cdot) : [0,\infty) \to [0,1]$ *is left continuous.*

Then M *is called a fuzzy metric space.*

Remark 2.3. *In the above definition, if the triangular inequality* (F_4) *is replaced by*

$$M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s) \text{ for all } x,y,z \in X \text{ and } t,s > 0 \qquad (NA-I)$$

Then the triple $(X, M, *)$ *is called a non-Archimedean fuzzy metric space.*

Remark 2.4. *It is easy to check that the triangle inequality* $(NA-I)$ *implies* (F_4), *that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space.*

Example 2.5. *Let $X = [0, +\infty]$, $a * b \le ab$ for every $a, b \in [0,1]$ and d be the usual metric. Define $M(x, y, t) = e^{-d(x,y)/t}$ for all $x, y \in X$. Then $(X, M, *)$ is a non-Archimedean fuzzy metric space. Clearly, $(X, M, *)$ is also a fuzzy metric space.*

Remark 2.6. *In Definition 2.2, if the triangular inequality (F_4) is replaced by the following:*

$$M(x, z, t) \ge \max\{M(x, y, t) * M(y, z, t/2), M(x, y, t/2) * M(y, z, t)\} \qquad (WNA - I)$$

for all $x, y, z \in X$ and $t > 0$, then the triple $(X, M, *)$ is called a weak non-Archimedean fuzzy metric space. Obviously every non-Archimedean fuzzy metric space is itself a weak non-Archimedean fuzzy metric space.

Remark 2.7. *Condition $(WNA - I)$ does not implies that $M(x, y, \cdot)$ is non-decreasing and thus a weak non-Archimedean fuzzy metric space is not necessarily a fuzzy metric space. If $M(x, y, \cdot)$ is non-decreasing, then a weak non-Archimedean fuzzy metric space is a fuzzy metric space.*

Example 2.8. *Let $X = [0, +\infty), a * b = ab$ for every $a, b \in [0, 1]$. Define $M(x, y, t) by : M(x, y, 0) = 0$,*
$M(x, x, t) = 1$ *for all* $t > 0$,
$M(x, y, t) = t$ *for* $x \ne y$ *and* $0 < t \le 1$,
$M(x, y, t) = t/2$ *for* $x \ne y$ *and* $1 < t \le 2$,
$M(x, y, t) = 1$ *for* $x \ne y$ *and* $t > 2$.
Then $(X, M, *)$ is a weak non-Archimedean fuzzy metric space, but it is not a fuzzy metric space.

Lemma 2.9. *Let $(X, M, *)$ non-Archimedean fuzzy metric space, then M is a continuous function on $X^2 \times (0, \infty)$.*

Remark 2.10. *Since * is continuous, it follows from (F_4) that the limit of the sequence in fuzzy metric space is uniquely determined. Let$(X, M, *)$ be a fuzzy metric space with the following condition:*
(F_6) $\lim_{t\to\infty} M(x, y, t) = 1$ *for all* $x, y \in X$.

Definition 2.11 ([23])**.** *A binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-co norms if "$\diamond$" is satisfying conditions:*

(1). $\diamond$ *is commutative and associative;*

(2). $\diamond$ *is continuous;*

(3). $a \diamond 0 = a$ *for all* $a \in [0, 1]$;

(4). $a \diamond b \le c \diamond d$ *whenever* $a \le c$ *and* $b \le d$, *and* $a, b, c, d \in [0, 1]$.

Note 2.12. *The concepts of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and unions, respectively. These concepts were originally introduced by Menger [16] in his study of statistical metric spaces.*

Definition 2.13 ([1])**.** *A 5-tuple $(X, M, N, *, \diamond)$ is said to be intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$,*
$(IFM - 1)$ $M(x, y, t) + N(x, y, t) \le 1$;
$(IFM - 2)$ $M(x, y, t) > 0$;
$(IFM - 3)$ $M(x, y, t) = 1$ *if and only if* $x = y$;
$(IFM - 4)$ $M(x, y, t) = M(y, x, t)$;

$(IFM-5)$ $M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s)$ for all $x,y,z \in X,\ s,t > 0$;

$(IFM-6)$ $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;

$(IFM-7)$ $N(x,y,t) > 0$;

$(IFM-8)$ $N(x,y,t) = 0$ if and only if $x = y$;

$(IFM-9)$ $N(x,y,t) = N(y,x,t)$;

$(IFM-10)$ $N(x,z,min\{t,s\}) \leq N(x,y,t) \diamond N(y,z,s)$ for all $x,y,z \in X,\ s,t > 0$;

$(IFM-11)$ $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X, the function $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non nearness between x and y with respect to 't' respectively.

Remark 2.14. *In the above definition the triangular inequality $(IFM-5)$ and $(IFM-10)$ are equivalent to $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$ and*

$$N(x,z,t) \leq N(x,y,t) \diamond N(y,z,t) \quad \text{for all } x,y,z \in X,\ s,t > 0 \tag{NA-II}$$

Then the triple $(X,M,N,*,\diamond)$ is called a non-Archimedean Intuitionistic fuzzy metric space $(NAIFMS)$.

Remark 2.15. *It is easy to check that the triangular inequality $(NA-II)$ implies, that every non-Archimedean Intuitionistic fuzzy metric space is intuitionistic fuzzy metric space.*

Definition 2.16. *Let $(X,M,N,*,\diamond)$ be a non-Archimedean Intuitionistic fuzzy metric space .*

(a). A sequence $\{x_n\}$ in X is called an Cauchy sequence, if for each $\varepsilon \in (0,1)$ and $t > 0$ there exists $n_0 \in N$ such that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ and $\lim_{n\to\infty} N(x_n, x_{n+p}, t) = 0$ for all $p = 0,1,2,\ldots$

(b). A sequence $\{x_n\}$ in a non-Archimedean Intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is said to be convergent to $x \in X$ $\lim_{n\to\infty} M(x_n,x,t) = 1, \lim_{n\to\infty} N(x_n,x,t) = 0$ for all $t > 0$.

(c). A non-Archimedean Intuitionistic fuzzy metric space $(X,M,N,*,\diamond)$ is called complete if every Cauchy sequence is convergent in X.

Definition 2.17 ([24])**.** *A partially ordered set is a set P and a binary relation $\preccurlyeq$, denoted by $(X,\preccurlyeq)$ such that for all $a,b,c \in P$,*

(a). $a \preccurlyeq a$ $(reflexivity)$,

(b). $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$ $(transitivity)$, $\preccurlyeq$

(c). $a \preccurlyeq b$ and $b \preccurlyeq a$ implies $a = b$ $(anti-symmetry)$.

Definition 2.18 ([24])**.** *Let $(X,\preccurlyeq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have k-monotone property if $x_0 \preccurlyeq x_1,\ y_0 \succcurlyeq y_1 \Rightarrow F(x_0,y_0) \preccurlyeq F(x_1,y_1)$ and $F(y_0,x_0) \preccurlyeq F(y_1,x_1)$ for all $x_0,x_1,y_0,y_1 \in X$.*

Definition 2.19 ([5])**.** *Let $(X,\preccurlyeq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have mixed monotone property if $F(x,y)$ is monotone non-decreasing in first coordinate and is monotone non-increasing in second coordinate. i.e. for any $x,y \in X$, $x_0 \preccurlyeq x_1 \Rightarrow F(x_0,y) \preccurlyeq F(x_1,y)$ and $y_0 \preccurlyeq y_1 \Rightarrow F(x,y_0) \preccurlyeq F(x,y_1)$ for all $x_0,x_1,y_0,y_1 \in X$.*

Remark 2.20. *Thus mixed monotone property is particular case of k-monotone property.*

Example 2.21. *Let $X = [2, 64]$ on the set X, we consider following relation $x \preccurlyeq y \Leftrightarrow x \leq y$, where $\preccurlyeq$ is a usual ordering, $(X, \preccurlyeq)$ a partial order set. We define $F : X \times X \to X$ as $F(x, y) = x + [1/y]$, where $[k]$ represents greatest integer just less than or equal to k. One can verify that $F(x, y)$ follows k-monotone property.*

Definition 2.22 ([5])**.** *An element $(x, y) \in X \times X \to X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

3. Main Results

Theorem 3.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and $(X, M, N, *, \diamond)$ is a complete Non-Archimedean Intuitionistic fuzzy metric space. Let $F : X \times X \to X$ be a continuous mapping having k-monotone property on X. Assume that for every $\varepsilon \in (0, 1)$ with*

$$\begin{aligned} M(F(x,y),F(u,v),t) &\geq 1-\frac{\varepsilon}{2}\max\{M(F(x,y),x,t),M(x,F(u,v),t),M(F(x,y),u,t),M(u,F(u,v),t)\} \\ N(F(x,y),F(u,v),t) &\leq 1-\frac{\varepsilon}{2}\min\{N(F(x,y),x,t),N(x,F(u,v),t),N(F(x,y),u,t),N(u,F(u,v),t)\} \end{aligned} \tag{1}$$

for all $x, y, u, v \in X$ with $x \succcurlyeq u$ and $y \preccurlyeq v$. If there exists $x_0, y_0, x_1, y_1 \in X$, such that $x_0 \preccurlyeq x_1, y_0 \succcurlyeq y_1$, where $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$ then there exists $x, y, \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Proof. Let $x_0, x_1, y_0, y_1 \in X$ be such that $x_0 \preccurlyeq x_1, y_0 \succcurlyeq y_1$, where $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$ we construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows

$$x_{n+1} = F(x_n, y_n) \;\; \& \;\; y_n + 1 = F(y_n, x_n) \; for \; all \; n \geq 0$$

We shall show that $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n \geq 0$. Since $x_0 \preccurlyeq x_1$, $y_0 \succcurlyeq y_1$, therefore by k-monotone property $x_1 = F(x_0, y_0) \preccurlyeq F(x_1, y_1) = x_2$ and $y_1 = F(y_0, x_0) \succcurlyeq F(y_1, x_1) = y_2$ i.e. $x_1 \preccurlyeq x_2$, $y_1 \succcurlyeq y_2$, again applying the same property we have, $x_2 = F(x_1, y_1) \preccurlyeq F(x_2, y_2) = x_3$ and $y_2 = F(y_1, x_1) \succcurlyeq F(y_2, x_2) = y_3$. Continue in this manner we shall have, $x_0 \preccurlyeq x_1 \preccurlyeq x_2 \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots$ and $y_0 \succcurlyeq y_1 \succcurlyeq y_2 \succcurlyeq \cdots \succcurlyeq y_n \succcurlyeq y_{n+1} \succcurlyeq \ldots$. Since $x_{n-1} \preccurlyeq x_n$ and $y_{n-1} \succcurlyeq y_n$, from (1) we have,

$$\begin{aligned} M(F(x_n,y_n),F(x_{n-1},y_{n-1}),t) &\geq 1-\frac{\varepsilon}{2}\max\left\{\begin{array}{l} M(F(x_n,y_n),x_n,t),M(x_n,F(x_{n-1},y_{n-1}),t, \\ M(F(x_n,y_n),x_{n-1},t),M(x_{n-1},F(x_{n-1},y_{n-1}),t \end{array}\right\} \\ &= 1-\frac{\varepsilon}{2}\max\left\{\begin{array}{l} M(x_{n+1},x_n,t),M(x_n,x_n,t), \\ M(x_{n+1},x_{n-1},t),M(x_{n-1},x_n,t) \end{array}\right\} \\ &= 1-\frac{\varepsilon}{2}\max\left\{ M(x_{n+1},x_n,t),1,M(X_{n+1},x_{n-1},t),M(x_{n-1},x_n,t) \right\} \\ &= 1-\frac{\varepsilon}{2} > 1-\varepsilon \end{aligned}$$

i.e. $M(x_{n+1}, x_n, t) > 1 - \varepsilon$ and

$$\begin{aligned} N(F(x_n,y_n),F(x_{n-1},y_{n-1}),t) &\leq 1-\frac{\varepsilon}{2}\min\left\{\begin{array}{l} N(F(x_n,y_n),x_n,t),N(x_n,F(x_{n-1},y_{n-1}),t, \\ N(F(x_n,y_n),x_{n-1},t),N(x_{n-1},F(x_{n-1},y_{n-1}),t \end{array}\right\} \\ &= 1-\frac{\varepsilon}{2}\min\left\{\begin{array}{l} N(x_{n+1},x_n,t),N(x_n,x_n,t), \\ N(x_{n+1},x_{n-1},t),N(x_{n-1},x_n,t) \end{array}\right\} \end{aligned}$$

$$= 1 - \frac{\varepsilon}{2} \min \left\{ N(x_{n+1}, x_n, t), 0, N(X_{n+1}, x_{n-1}, t), N(x_{n-1}, x_n, t) \right\}$$
$$= 1 - \frac{\varepsilon}{2} < 1 - \varepsilon$$

i.e. $N(x_{n+1}, x_n, t) < 1 - \varepsilon$. Similarly we can show that $M(x_{n+1}, x_{n+2}, t) > 1 - \varepsilon$. So for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$ and $t > 0$ we have

$$M(x_n, x_m, t) \geq M(x_n, x_{n+1}, t) * M(x_{n+1}, x_{n+2}, t) * \ldots M(x_{m-1}, x_m, t)$$
$$M(x_n, x_m, t) \geq (1 - \varepsilon) * (1 - \varepsilon) * (1 - \varepsilon) * \cdots * (1 - \varepsilon)$$
$$\Rightarrow M(y_{n+1}, y_{n+2}, t) > 1 - \varepsilon$$

and

$$N(x_n, x_m, t) \leq N(x_n, x_{n+1}, t) \diamond M(x_{n+1}, x_{n+2}, t) \diamond \cdots \diamond M(x_{m-1}, x_m, t)$$
$$N(x_n, x_m, t) \leq (1 - \varepsilon) \diamond (1 - \varepsilon) \diamond (1 - \varepsilon) \diamond \cdots \diamond (1 - \varepsilon)$$
$$\Rightarrow N(y_{n+1}, y_{n+2}, t) < 1 - \varepsilon$$

This shows that the sequence $\{x_n\}$ is Cauchy sequence in X and since X is complete fuzzy metric space it converges to a point $x \in X$ i.e. $\lim_{n\to\infty} x_n = x$. Again since $y_{n-1} \succcurlyeq y_n, x_{n-1} \preccurlyeq x_n$, from (1) we have,

$$M(F(y_{n-1}, x_{n-1}), F(y_n, x_n), t) \geq 1 - \frac{\varepsilon}{2} \max \left\{ \begin{array}{l} M(F(y_{n-1}, x_{n-1}), y_{n-1}, t) \\ M(y_{n-1}, F(y_n, x_n), t), M(F(y_{n-1}, x_{n-1}), y_n, t) M(y_n, F(y_n, x_n), t) \end{array} \right\}$$
$$= 1 - \frac{\varepsilon}{2} \max \left\{ \begin{array}{l} M(y_n, y_{n-1}, t) M(y_{n-1}, y_{n+1}, t), \\ M(y_n, y_n, t), M(y_n, y_{n+1}, t) \end{array} \right\}$$
$$= 1 - \frac{\varepsilon}{2} \max \left\{ M(y_n, y_{n-1}, t) M(y_{n-1}, y_{n+1}, t), 1, M(y_n, y_{n+1}, t) \right\}$$
$$= 1 - \frac{\varepsilon}{2} > 1 - \varepsilon$$
$$M(y_{n+1}, y_n, t) > 1 - \varepsilon$$

Similarly we can show that $M(y_{n+1}, y_{n+2}, t) > 1 - \varepsilon$. So for all $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $m > n > n_0$ and $t > 0$ we have

$$M(y_n, y_m, t) \geq M(y_n, y_{n+1}, t) * M(y_{n+1}, y_{n+2}, t) * \cdots * M(y_{m-1}, y_m, t)$$
$$M(y_n, y_m, t) > (1 - \varepsilon) * (1 - \varepsilon) * \cdots * (1 - \varepsilon)$$

and $y_1 = F(y_0, x_0)$,

$$N(y_n, y_m, t) \geq N(y_n, y_{n+1}, t) \diamond N(y_{n+1}, y_{n+2}, t) \diamond \cdots \diamond N(y_{m-1}, y_m, t)$$
$$N(y_n, y_m, t) < (1 - \varepsilon) \diamond (1 - \varepsilon) \diamond \cdots \diamond (1 - \varepsilon)$$

This shows that the sequence y_n is Cauchy sequence in X and since X is complete fuzzy metric space it converges to a point $y \in X$ i.e. $\lim_{n\to\infty} y_n = y$. Since F is given continuous therefore using convergence of x_n and y_n we have, $F(x, y) = x$ and $F(x, y) = y$. □

Now we shall define a partial order relation over non-Archimedean fuzzy metric space and prove a coupled fixed point theorem using that relation.

Lemma 3.2. *Let $(X, M, N, *, \diamond)$ be a a non-Archimedean Intuitionistic fuzzy metric space with $a * b \geq max\{a + b - 1, 0\}$ and $a \diamond b \leq min\{a + b - 1, 0\}$ with $\phi : X \times X \times [0, \infty) \to R$, define the relation "$\preccurlyeq$" on X as follows $x \preccurlyeq u, y \succcurlyeq v \Leftrightarrow M(x, u, t)M(y, v, t) \geq 1 + \phi(x, y, t) - \phi(u, v, t)$ for all $t > 0$ then "$\preccurlyeq$" is partial order on X, called the partial order induced by ϕ.*

Proof. The relation "$\preccurlyeq$" is a reflexive relation: let $x, y \in X$ be any element. Since $M(x, x, t)M(y, y, t) = 1 = 1+\phi(x, y, t)-\phi(u, v, t)$ for all $x, y \in X$. Therefore

$$\preccurlyeq \text{ is a reflexive relation} \tag{i}$$

For any $x, y, u, v \in X$ suppose that $x \preccurlyeq u, y \succcurlyeq v, x \succcurlyeq u, y \preccurlyeq v$ then we have.

$$x \preccurlyeq u, y \succcurlyeq v \Leftrightarrow M(x, u, t)M(y, v, t) \geq 1 + \phi(x, y, t) - \phi(u, v, t) \tag{I}$$

$$x \succcurlyeq u, y \preccurlyeq v \Leftrightarrow M(u, x, t)M(v, y, t) \geq 1 + \phi(u, v, t) - \phi(x, y, t) \tag{II}$$

Adding (I) & (II), we get,

$$\begin{aligned} & 2M(x, u, t)M(y, v, t) \geq 2 \\ \text{Or } & M(x, u, t)M(y, v, t) \geq 1 \\ & M(x, u, t)M(y, v, t) = 1 \Rightarrow M(x, u, t) = 1, \ M(y, v, t) = 1 \end{aligned}$$

i.e. $x = u$ and $y = v$. Therefore

$$\preccurlyeq \text{ is anti symmetric relation} \tag{ii}$$

If $x \preccurlyeq u, y \succcurlyeq v, u \preccurlyeq u, v \succcurlyeq v$, we have,

$$\begin{aligned} M(x, u', t)M(y, v', t) &\geq M(x, u, t)M(y, v, t) * M(u, u', t)M(v, v', t) \\ &= \max[M(x, u, t)M(y, v, t) + M(u, u', t)M(v, v', t) - 1, 0] \\ &= \max[1 + \phi(x, y, t) - \phi(u, v, t) + 1 + \phi(u, v, t) - \phi(u', v', t) - 1, 0] \\ &= \max[1 + \phi(x, y, t) - \phi(u', v', t), 0] \\ &= 1 + \phi(x, y, t) - \phi(u', v', t) i.e. x \preccurlyeq u', y \succcurlyeq v' \end{aligned}$$

and

$$\begin{aligned} N(x, u', t)N(y, v', t) &\leq N(x, u, t)N(y, v, t) \diamond N(u, u', t)N(v, v', t) \\ &= \max[N(x, u, t)N(y, v, t) + N(u, u', t)N(v, v', t) - 1, 0] \\ &= \max[1 + \phi(x, y, t) - \phi(u, v, t) + 1 + \phi(u, v, t) - \phi(u', v', t) - 1, 0] \\ &= \max[1 + \phi(x, y, t) - \phi(u', v', t), 0] \\ &= 1 + \phi(x, y, t) - \phi(u', v', t) i.e. x \preccurlyeq u', y \succcurlyeq v' \end{aligned}$$

Thus

$$\preccurlyeq \text{ is transitive relation.} \tag{iii}$$

□

Theorem 3.3. *Let $(X, M, N, *, \diamond)$ be a non-Archimedean Intuitionistic fuzzy metric space with $a * b \geq max\{a + b - 1, 0\}$ and $a \diamond b \leq min\{a + b - 1, 0\}$ with $\phi : X \times X \times [0, \infty) \to R$, bounded from above "$\preccurlyeq$" the partial order induced by ϕ if $F : X \times X \to X$ follows k-monotone property over X and there are $x_0, y_0, x_1, y_1 \in X$,such that $x_0 \preccurlyeq x_1, y_0 \succcurlyeq y_1$, where $x_1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$ then there exists $x, y, \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.*

Proof. Let $x_0, y_0, x_1, y_1 \in X$, such that $x_0 \preccurlyeq x_1, y_0 \succcurlyeq y_1$, where ${}_x1 = F(x_0, y_0)$ and $y_1 = F(y_0, x_0)$. We construct sequences $\{x_n\}$ and $\{y_n\}$ in X as follows $x_{n+1} = F(x_n, y_n)$ and $y_{n+1} = F(y_n, x_n)$ for all $n \geq 0$. We shall show that $x_n \preccurlyeq x_{n+1}$ and $y_n \succcurlyeq y_{n+1}$ for all $n \geq 0$. Since $x_0 \preccurlyeq x_1, y_0 \succcurlyeq y_1$, therefore by k-monotone property $x_1 = F(x_0, y_0) \preccurlyeq F(x_1, y_1) = x_2$ and $y_1 = F(y_0, x_0) \succcurlyeq F(y_1, x_1) = y_2$ i.e. $x_1 \preccurlyeq x_2, y_1 \succcurlyeq y_2$, again applying the same property we have $x_2 = F(x_1, y_1) \preccurlyeq F(x_2, y_2) = x_3$ and $y_2 = F(y_1, x_1) \succcurlyeq F(y_2, x_2) = y_3$. Continue in this manner we shall have, $x_0 \preccurlyeq x_1 \preccurlyeq x_2 \preccurlyeq \cdots \preccurlyeq x_n \preccurlyeq x_{n+1} \preccurlyeq \ldots$ and $y_0 \succcurlyeq y_1 \succcurlyeq y_2 \succcurlyeq \cdots \succcurlyeq y_n \succcurlyeq y_{n+1} \succcurlyeq \ldots$. By the definition of "$\preccurlyeq$" we have, for all $t > 0 \phi(x_0, y_0, t) \preccurlyeq \phi(x_1, y_1, t) \preccurlyeq \phi(x_3, y_3, t) \preccurlyeq \ldots$. In other words, for all $t > 0$, the sequence $\{\phi(x_n, y_n, t)\}$ is non decreasing in R. Since ϕ is bounded above, and $\{\phi(x_n, y_n, t)\}$ is convergent and hence it is a Cauchy sequence. So, for all $\varepsilon > 0$, there exists $n_0 \in N$ so that for all $m > n > n_0$ and $t > 0$ we have,

$$|\phi(x_m, y_m, t) - \phi(x_n, y_n, t)| < \varepsilon$$

Since $x_n \preccurlyeq x_m$ & $y_n \succcurlyeq y_m$, we have

$$x_n \preccurlyeq x_m \ \&\ y_n \succcurlyeq y_m \Leftrightarrow M(x_n, x_m, t)M(y_n, y_m, t) \geq 1 + \phi(x_n, y_n, t) - \phi(x_m, y_m, t) \ \text{ for all } t > 0$$
$$1 - [\phi(x_m, y_m, t) - \phi(x_n, y_n, t)] > 1 - \varepsilon$$
$$x_n \preccurlyeq x_m \ \&\ y_n \succcurlyeq y_m \Leftrightarrow N(x_n, x_m, t)N(y_n, y_m, t) \leq 1 + \phi(x_n, y_n, t) - \phi(x_m, y_m, t) \ \text{ for all } t > 0$$
$$1 - [\phi(x_m, y_m, t) - \phi(x_n, y_n, t)] < 1 - \varepsilon$$

We claim that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequence in X, if not then there exists some $\varepsilon_1, \varepsilon_2$ such that $\varepsilon_1 < \varepsilon_2$ and $M(x_n, x_m, t) \leq (1-\varepsilon_1)$ and $M(y_n, y_m, t) \leq (1-\varepsilon_2)$. Then $M(x_n, x_m, t)M(y_n, y_m, t) \leq (1-\varepsilon 1))(1-\varepsilon_2) < (1-\varepsilon_1))^2 < (1-\varepsilon_1))$ and $N(x_n, x_m, t) \leq (1 - \varepsilon_1)$ and $N(y_n, y_m, t) \leq (1 - \varepsilon_2)$. Then $N(x_n, x_m, t)N(y_n, y_m, t) \leq (1 - \varepsilon_1))(1 - \varepsilon_2) < (1 - \varepsilon_1))^2 < (1 - \varepsilon_1))$. Which is a contradiction. This shows that the sequence $\{x_n\}$ and $\{y_n\}$ a Cauchy sequence in X, since X is complete, these converges to points x, y respectively in X consequently, by the continuity of F, we have $F(x, y) = x$ and $F(y, x) = y$. □

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