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Relation between Khalimsky Topology and Slapal's Topology

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Abstract: In this paper we study properties of both Khalimsky topology and Slapal's topology and the relation between them.

Keywords: Khalimsky topology, Slapal's topology, quotient topology, Alexandroff topological space, 4-adjacent, 8-adjacent.

1. Introduction

An important problem of digital topology is to provide the digital plane Z^2 with a convenient structure for the study of geometric and topological properties of digital images. A basic criterion for such a convenience is the validity of an analogy of the Jordan curve theorem. It was in 1990 that a topology on Z^2 convenient for the study of digital images was introduced by Khalimsky. A drawback of the Khalimsky topology is that the Jordan curves with respect to it can never turn at an acute angle. To overcome this deficiency, another topology on Z^2 was introduced by Slapal.

Notationt 1.1 ([6]). *Let $z = (x, y) \in Z^2$. Put*

$$H_2(z) = \{(x-1, y), (x+1, y)\}$$

$$V_2(z) = \{(x, y-1), (x, y+1)\}$$

$$D_4(z) = H_2(z) \cup \{(x-1, y-1), (x+1, y-1)\}$$

$$U_4(z) = H_2(z) \cup \{(x-1, y+1), (x+1, y+1)\}$$

$$L_4(z) = V_2(z) \cup \{(x-1, y-1), (x-1, y+1)\}$$

$$R_4(z) = V_2(z) \cup \{(x+1, y-1), (x+1, y+1)\}$$

Then we put

$$A_4(z) = H_2(z) \cup V_2(z)$$

$$A_8(z) = H_2(z) \cup L_4(z) \cup R_4(z)$$

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$$= V_2(z) \cup D_4(z) \cup U_4(z)$$

and $A'_4(z) = A_8(z) - A_4(z)$. $A_4(z)$ *and* $A_8(z)$ *are said to be 4-adjacent and 8-adjacent to* z *respectively.*

$$H_2(z), V_2(z), D_4(z), V_4(z), L_4(z), R_4(z)$$

and $A'_4(z)$ *are called horizontally 2-adjacent, vertically 2-adjacent, down 4-adjacent, up 4-adjacent, left 4-adjacent, right 4-adjacent and diagonally 4-adjacent to* z *respectively.*

Definition 1.2. *For any* $z = (x, y) \in Z^2$

$$V(z) = \begin{cases} \{z\} \cup A_8(z) & \text{if } x, y \text{ are even,} \\ \{z\} \cup H_2(z) & \text{if } x \text{ is even, and } y \text{ is odd} \\ \{z\} \cup V_2(z) & \text{if } x \text{ is odd and } y \text{ is even} \\ \{z\} & \text{otherwise} \end{cases}$$

The topological space (Z^2, V) *is called the Khalimsky topological space.*

Definition 1.3 ([5]). *Let* w *be the Alexandroff* $T_{1/2}$ *topology on* Z^2 *defined as follows. For any point* $z = (x, y) \in Z^2$

$$w(z) = \begin{cases} \{z\} \cup A_8(z) & \text{if } x = 4k, y = 4l, k, l \in Z \\ \{z\} \cup A'_4(z) & \text{if } x = 2 + 4k, y = 2 + 4l, k, l \in Z \\ \{z\} \cup D_4(z) & \text{if } x = 2 + 4k, y = 1 + 4l, k, l \in Z \\ \{z\} \cup U_4(z) & \text{if } x = 2 + 4k, y = 3 + 4l, k, l \in Z \\ \{z\} \cup L_4(z) & \text{if } x = 1 + 4k, y = 2 + 4l, k, l \in Z \\ \{z\} \cup R_4(z) & \text{if } x = 3 + 4k, y = 2 + 4k, k, l \in Z \\ \{z\} \cup H_2(z) & \text{if } x = 2 + 4k, y = 4l, k, l \in Z \\ \{z\} \cup V_2(z) & \text{if } x = 4k, y = 2 + 4l, k, l \in Z \\ \{z\} & \text{otherwise} \end{cases}$$

2. Quotient Topologies of w

Remark 2.1. *Given a topological space* (X, p)*, a set* Y *and a surjection* $e : X \to Y$*, a topology* q *on* Y *is said to be the quotient topology of* p *generated by* e *if* q *is the finest topology on* Y *for which* $e : (X, p) \to (Y, q)$ *is continuous. For Alexandroff topological spaces* (X, p) *and* (Y, q)*, a map* $c : (X, p) \to (Y, q)$ *is continuous if and only if* $e(p\{x\}) \subseteq q\{e(x)\}$ *for every* $x \in X$*. We need the following lemma.*

Lemma 2.2. *Let* (X, p)*,* (Y, q) *be Alexandroff topological spaces and let* $e : X \to Y$ *be a surjection. Then the following condition is sufficient for* q *to be the quotient topology of* p *generated by* e*. For any pair of points* $x, y \in Y$*,* $x \in q(y)$ *if and only if there are* $a \in e^{-1}(x)$ *and* $b \in e^{-1}(y)$ *such that* $a \in p(b)$*.*

We require the following surjection for the forthcoming theorem.

Notationt 2.3. *Let $f : Z^2 \to Z^2$ be a surjection given as follows. For every $(x, y) \in Z^2$*

$$f(x,y) = \begin{cases} (2k, 2l) & \text{if } (x,y) = (4k, 4l)\ , \\ (2k, 2l+1) & \text{if } (x,y) \in A_4(4k, 4l+2)\ , \\ (2k+1, 2l) & \text{if } (x,y) \in A_4(4k+2, 4l)\ , \\ (2k+1, 2l+1) & \text{if } (x,y) \in A'_4(4k+2, 4l+2)\ , \end{cases}$$

where $k, l \in Z$.

Theorem 2.4. *The Khalimsky topology t coincides with the quotient topology of w generated by f.*

Proof. We can show that for any points $z_1, z_2 \in Z^2, z_1 \in t(z_2)$ if and only if there are points $a \in f^{-1}(z_1)$ and $b \in f^{-1}(z_2)$ such that $a \in w(b)$. This is true if $z_1 = z_2$. Therefore suppose that $z_1 \neq z_2$. Let $z_1 \in t(z_2)$. Then z_2 is not a closed point in (Z^2, t), hence $z_2 = (x, y)$ where x or y is even. Thus we have the following three possibilities.

Case 1: $z_2 = (2k, 2l)$, for some $k, l \in Z$ and $z_1 \in A_8(z_2) - \{z_2\}$. Then $f^{-1}(z_2) = (4k, 4l)$ and we get one of the following eight cases.

(1). $z_1 = (2k+1, 2l)$ hence $f^{-1}(z) = A_4(4k+2, 4l)$, $(4k+1, 4l) \in f^{-1}(z_1)$ and we have $(4k+1, 4l) \in w\{4k, 4l\}$

(2). $z_1 = (2k-1, 2l)$ hence $f^{-1}(z_1) = A_4(4k-2, 4l)$, $(4k-1, 4l) \in f^{-1}(z_1)$ and we have $(4k-1, 4l) \in w(4k, 4l)$

(3). $z_1 = (2k, 2l+1)$ hence $f^{-1}(z_1) = A_4(4k, 4l+2)$, $(4k, 4l+1) \in f^{-1}(z_1)$ and we have $(4k, 4l+1) \in w\{(4k, 4l)\}$

(4). $z_1 = (2k, 2l-1)$ hence $f^{-1}(z_1) = A_4(4k, 4l-2)$, $(4k, 4l-1) \in f^{-1}(z_1)$ and we have $(4k, 4l-1) \in w\{(4k, 4l)\}$

(5). $z_1 = (2k+1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l+2)$, $(4k+1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k+1, 4l+1) \in w\{(4k, 4l)\}$

(6). $z_1 = (2k+1, 2l-1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l-2)$, $(4k+1, 4l-1) \in f^{-1}(z_1)$ and we have $(4k+1, 4l-1) \in w\{(4k, 4l)\}$

(7). $z_1 = (2k-1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k-2, 4l+2)$, $(4k-1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k-1, 4l+1) \in w\{(4k, 4l)\}$

(8). $z_1 = (2k-1, 2l-1)$ hence $f^{-1}(z_1) = A'_4(4k-2, 4l-2)$, $(4k-1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k-1, 4l-1) \in w\{(4k, 4l)\}$.

Case 2: $z_2 = (2k, 2l+1)$, for some $k, l \in Z$ and $z_1 \in H_2(z_2) - \{z_2\}$. Then

$$f^{-1}(z_2) = A_4(4k, 4l+2), \{(4k+1, 4l+2), (4k-1, 4l+2)\} \subset f^{-1}(z_2)$$

and we get one of the following two cases

(1). $z_1 = (2k+1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l+2)$, $(4k+1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k+1, 4l+1) \in w\{4k+1, 4l+2\}$

(2). $z_1 = (2k-1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k-2, 4l+2)$, $(4k-1, 4l+3) \in f^{-1}(z_1)$ and we have $(4k-1, 4l+3) \in w\{(4k-1, 4l+2)\}$

Case 3: $z_2 = (2k+1, 2l)$, for some $k, l \in Z$ and $z_1 \in V_2(z_2) - \{z_2\}$. Then

$$f^{-1}(z_2) = A_4(4k+2, 4l), \{(4k+2, 4l+2), (4k+2, 4l-1)\} \subseteq f^{-1}(z_2)$$

and we get one of the following two cases

(1). $z_1 = (2k+1, 2l+1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l+2)$, $(4k+1, 4l+1) \in f^{-1}(z_1)$ and we have $(4k+1, 4l+1) \in w\{4k+2, 4l+2\}$

(2). $z_1 = (2k+1, 2l-1)$ hence $f^{-1}(z_1) = A'_4(4k+2, 4l-2)$, $(4k+1, 4l-3) \in f^{-1}(z_1)$ and we have $(4k+1, 4l-1) \in w\{(4k+2, 4l-1)\}$ we have shown that whenever $z_1 \in t\{z_2\}$ there are points $a \in f^{-1}(z_1)$ and $b \in^{-1} (z_2)$ such that $a \in w(b)$.

Conversely suppose that there are points $a \in f^{-1}(z_1)$ and $b \in^{-1} (z_2)$ such that $a \in w(b)$. Then $f^{-1}(z_1)$ is not open in (Z^2, w). Therefore we have the following three possibilities.

Case 1: $f^{-1}(z_1)A_4(4k, 4l+2)$ for some $k, l \in Z$ hence $z_1 = (2k, 2l+1)$ and we get one of the following two cases

(1). $z_2 = (2k, 2l+2)$ because then $f^{-1}(z_2) = \{(4k, 4l+4)\}$, $a \in (4k, 4l+3) \in f^{-1}(z_1)$ and $b = (4k, 4l+4) \in f^{-1}(z_2)$ then we have $z_1 \in t\{z_2\}$

(2). $z_2 = (2k, 2l)$ because then $f^{-1}(z_2) = \{(4k, 4l)\}, a = (4k, 4l+1) \in f^{-1}(z_1)$ and $b = (4k, 4l) \in f^{-1}(z_2)$. So $z_1 \in t\{z_2\}$.

Case 2: $f^{-1}(z_1) = A_4(4k+2, 4l)$ for some $k, l \in Z$ hence $z_1 = (2k+1, 2l)$ and we get one of the following two cases

(1). $z_2 = (2k+2, 2l)$ because then $f^{-1}(z_2) = \{(4k+4l, 4l)\}, a = (4k+3, 4l) \in f^{-1}(z_1)$ and $b = (4k+4, 4l) \in f^{-1}(z_2)$ so we have $z_1 \in t\{z_2\}$.

(2). $z_2 = (2k, 2l)$ because then $f^{-1}(z_2) = \{(4k, 4l)\}, a = (4k+1, 4l) \in f^{-1}(z_1)$ and $b = (4k, 4l) \in f^{-1}(z_2)$, then we have $z_1 \in t\{z_2\}$.

Case 3: $f^{-1}(z_1) = A'_4(4k+2, 4l+2)$ for some $k, l \in Z$ hence $z_1 = (2k+1, 2l+1)$ and we get one of the following four cases

(1). $z_2 = (2k+2, 2l+2)$ because then $f^{-1}(z_2) = \{(4k+4, 4l+4)\}, a = (4k+3, 4l+3) \in f^{-1}(z_1)$ and $b = (4k+4, 4l+4) \in f^{-1}(z_2)$ so we have $z_1 \in t\{z_2\}$

(2). $z_2 = (2k, 2l+2)$ because then $f^{-1}(z_2) = \{(4k, 4l+4)\}, a = (4k+1, 4l+3) \in f^{-1}(z_1)$ and $b = (4k, 4l+4) \in f^{-1}(z_2), z_1 \in t\{z_2\}$

(3). $z_2 = (2k+2, 2l)$ because then $f^{-1}(z_2) = \{(4k+4, 4l)\}, a = (4k+3, 4l+1) \in f^{-1}(z_1)$ and $b = (4k+4, 4l) \in f^{-1}(z_2)$, so we have $z_1 \in t\{z_2\}$

(4). $z_2 = (2k, 2l)$ because then $f^{-1}(z_2) = \{(4k, 4l)\}, a = (4k+1, 4l+1) \in f^{-1}(z_1)$ and $b = (4k, 4l) \in f^{-1}(z_2)$, so we have $z_1 \in t\{z_2\}$

We have shown that $a \in f^{-1}(z_1), b \in f^{-1}(z_2)$ and $a \in w(b)$ imply $z_1 \in t(z_2)$. By lemma 2.2, t is the quotient topology of w generalized by f. □

Notationt 2.5. *Let $g : Z^2 \to Z^2$ be the surjection as follows. For any $(x, y) \in Z^2$*

$$g(x,y) = \begin{cases} k+l, l-k & \text{if } (x,y) \in A_8(4k,4l), k,l \in Z \\ (k+l+1, l-k) & \text{if } (x,y) = (4k+2, 4l+2), \\ & \text{for some } k,l \in Z \text{ with } k+1 \text{ odd} \end{cases}$$

or $(x, y) \in A_{12}(4k+2, 4l+2)$ for some $k, l \in Z$ with $k+l$ even, where $A_{12}(k, l) = \{(x, y) \in Z^2, x = k$ and $|y-l| \leq 3$ or $y = l$ and $|x-k| \leq 3\}$. Thus A_{12} consists of the point (k, l) and the 12 nearest points to (k, l) each of which has one co-ordinate common with (k, l).

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