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Generalized k-Quasi-Hyponormal Composition Operators On Weighted Hardy Space

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Abstract: In this paper we discuss the conditions for a composition operator and a weighted composition operator to be generalized k-quasi-hyponormal operator and also the characterization of generalized k-quasi-hyponormal composition operators on weighted Hardy space.

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Keywords: Generalized k-quasi-hyponormal operator, composition operator, weighted Hardy space.

1. Introduction

Let H be an infinite dimensional complex Hilbert and $B(H)$ denote the algebra of all bounded linear operators acting on H. Recall that an operator $T \in B(H)$ is positive,$T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in H$. An operator $T \in B(H)$ is said to be hyponormal if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and its known that hyponormal operators have many interesting properties similar to those of normal operators. An operator T is said to be p-hyponormal if $(T^*T)^p \geq (TT^*)^p$ for $p \in (0, 1]$ and an operator T is said to be log-hyponormal if T is invertible and $log \mid T \mid \geq log \mid T^* \mid$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator. An operator T on a Hilbert space H is called quasi-hyponormal, if $(T^*T)^2 \leq T^{*2}T^2$. An operator T on a Hilbert space H is called k-quasi-hyponormal, if $T^{*k}(TT^*)T^k \leq T^{*(k+1)}T^{(k+1)}$, where k is a positive integer. An operator T on a Hilbert space H is called a generalized k-quasi-hyponormal operator if for positive integer k such that $k \geq 2$ and $M > 0$, T satisfies $M^{k+1}T^{*k}(T^*T)T^k - T^{*k}T^*TT^k \geq 0$ [10]. In this paper, we are interested in generalized k-quasi-hyponormal composition operators.

1.1. Preliminaries

Let f be an analytic map on the open disk D given by the Taylor's series $f(z) = a_0 + a_1z + a_2z^2 + \dots$. Let $\beta = \{\beta_n\}_{n=0}^{\infty}$be a sequence of positive numbers with $\beta_0 = 1$ and $\dfrac{\beta_{n+1}}{\beta_n} \to 1$ as $n \to \infty$. The set $H^2(\beta)$ of formal complex power series $f(z) = \sum\limits_{n=0}^{\infty} a_nz^n$ such that $\|f\|_\beta^2 = \sum\limits_{n=0}^{\infty} \mid a_n \mid^2 \beta_n^2 < \infty$ is a Hilbert space of functions analytic in the unit disc with the inner product $\langle f, g\rangle_\beta = \sum\limits_{n=0}^{\infty} a_n\bar{b_n}\beta_n^2$ for f as above and $g(z) = \sum\limits_{n=0}^{\infty} b_nz^n$. Let D be the open unit disc in the complex plane

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and $T : D \to D$ be an analytic self-map of the unit disc and consider the corresponding composition operator C_T acting on $H^2(\beta)$, i.e., $C_T(f) = f \circ T$, $f \in H^2(\beta)$. Let w be a point on the open disk. Define

$$k_w^\beta(z) = \sum_{n=0}^{\infty} \frac{Z^n w^{-n}}{\beta_n^2}$$

Then the function k_w^β is a point evalution for $H^2(\beta)$. Then k_w^β is in $H^2(\beta)$ and $\left\|k_w^\beta\right\|^2 = \sum_{n=0}^{\infty} \frac{|\, w \,|^{2n}}{\beta_n^2}$. Thus, $\|k_w\|$ is an increasing function of $|\, w \,|$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\left\langle f, k_w^\beta \right\rangle_\beta = \sum_{n=0}^{\infty} \frac{a_n w^n \beta_n^2}{\beta_n^2} = f(w).$$

Therefore, $\langle f, k_w^\beta \rangle_\beta = f(w)$ for all f and k_w^β is known as the point evaluation kernal at w. It can be easily shown that $C_T^* k_w^\beta = k_{T(w)}^\beta$ and $k_0^\beta = 1$ (the function identically equal to 1).

2. On Generalized k-Quasi-Hyponormal Composition Operators

Let $L^2 = L^2(\Omega, A, \mu)$ denote the space of all complex-valued measurable function for which $\int_\Omega |\, f \,|^2 \leq \infty$. A composition operator on L^2, induced by a non-singular measurable transformation T, is denoted by C_T and is given by $C_T f = f \circ T$ for each $f \in L^2$. Then for $f \in L^2$ and for any positive integer k, $C_T^k f = f \circ T^k$ and $C_T^{*k} f = h_k.E(f) \circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$. In this chapter, some basic properties of generalized k-quasi-hyponormal Composition operators are discussed.

Theorem 2.1. *Let $C_T \in B(L^2)$. Then the following are equivalent:*

(i). C_T is generalized k-quasi-hyponormal.

(ii). $\left\|\sqrt{h_k.h \circ T^{-(k-1)}}.f\right\| \leq M^{\frac{k+1}{2}} \left\|\sqrt{h_k.E(h) \circ T^{-k}}.f\right\|$ for each $f \in L^2$.

(iii). $h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.E(h) \circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.

(iv). $h_{k-1}.(h \circ T^{-(k-1)})^2 \leq M^{k+1} h_{k-1}.h \circ T^{-(k-1)} E(h) \circ T^{-k}$.

(v). $h_{k-1} \circ T^{-1}.h \circ T^{-(k-1)} \leq M^{k+1} h_{k-1} \circ T^{-1}.E(h) \circ T^{-k}$.

Proof. To Prove: $(i) \equiv (ii)$. Assume that C_T is generalized k-quasi-hyponormal.

$$\begin{aligned} C_T^{*k}(C_T C_T^*)C_T^k &\leq M^{k+1} C_T^{*(k+1)} C_T^{k+1} \\ \left\langle (C_T^{*k}(C_T C_T^*)C_T^k)f, f \right\rangle &\leq M^{k+1} \left\langle C_T^{*(k+1)} C_T^{k+1} f, f \right\rangle \text{ for each } f \in L^2. \end{aligned} \tag{1}$$

Consider,

$$\begin{aligned} C_T^{*k}(C_T C_T^*)C_T^k f &= C_T^{*k}(h \circ T.E(f \circ T^k)) \\ C_T^{*k}(C_T C_T^*)C_T^k f &= h_k.h \circ T^{-(k-1)}.f \end{aligned} \tag{2}$$

Then,

$$\begin{aligned} C_T^{*(k+1)} C_T^{k+1} f &= C_T^{*k}(h.f \circ T^k) \\ C_T^{*(k+1)} C_T^{k+1} f &= h_k.E(h) \circ T^{-k} f \end{aligned} \tag{3}$$

Substitute (2) and (3) in (1),

$$h_k.h\circ T^{-(k-1)}f \leq M^{k+1}h_k.E(h)\circ T^{-k}f$$
$$\left\|\sqrt{h_k.h\circ T^{-(k-1)}}.f\right\| \leq M^{\frac{k+1}{2}}\left\|\sqrt{h_k.E(h)\circ T^{-k}}.f\right\|$$

Hence $(i)\equiv(ii)$.

To Prove: $(ii)\equiv(iii)$. Assume that $\left\|\sqrt{h_k.h\circ T^{-(k-1)}}.f\right\| \leq M^{\frac{k+1}{2}}\left\|\sqrt{h_k.E(h)\circ T^{-k}}.f\right\|$ for each $f\in L^2$.

$$h_k.h\circ T^{-(k-1)}f \leq M^{k+1}h_k.E(h)\circ T^{-k}f$$
$$h_k.h\circ T^{-(k-1)} \leq M^{k+1}h_k.E(h)\circ T^{-k}$$

Hence $(ii)\equiv(iii)$.

To Prove: $(iii)\equiv(iv)$. Assume that $h_k.h\circ T^{-(k-1)} \leq M^{k+1}h_k.E(h)\circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.

$$h_k.h\circ T^{-(k-1)} \leq M^{k+1}h_k.E(h)\circ T^{-k} \tag{4}$$
$$h_k = \mu T^{-k}(B) \tag{5}$$

We have,

$$\mu T^{-k}(B) = \mu T^{-1}(T^{-(k-1)}(B))$$
$$= \int_{T^{-(k-1)}(B)} h d\mu$$
$$\mu T^{-k}(B) = \int_B h_{k-1}.h\circ T^{-(k-1)}d\mu \tag{6}$$

Substitute (6) in (5),

$$h_k = h_{k-1}h\circ T^{-(k-1)}$$

From (4),

$$h_{k-1}.(h\circ T^{-(k-1)})^2 \leq M^{k+1}h_{k-1}h\circ T^{-(k-1)}.E(h)\circ T^{-k}$$

Hence $(iii)\equiv(iv)$.

To Prove: $(iii)\equiv(v)$. Assume that $h_k.h\circ T^{-(k-1)} \leq M^{k+1}h_k.E(h)\circ T^{-k}$, where $h_k = d\mu T^{-k}/d\mu$.

$$h_k.h\circ T^{-(k-1)} \leq M^{k+1}h_k.E(h)\circ T^{-k} \tag{7}$$
$$h_k = \mu T^{-k}(B) \tag{8}$$

We have,

$$\mu T^{-k}(B) = \int_{T^{-1}(B)} h_{k-1}d\mu$$
$$\mu T^{-k}(B) = \int_B h.h_{k-1}\circ T^{-1}d\mu \tag{9}$$

Substitute (9) in (8),

$$h_k = h.h_{k-1}\circ T^{-1}d\mu$$

From (7),

$$h_{k-1}\circ T^{-1}.h\circ T^{-(k-1)} \leq M^{k+1}h_{k-1}\circ T^{-1}.E(h)\circ T^{-k}.$$

Hence $(iii)\equiv(v)$. Hence the proof. □

Theorem 2.2. *If $T^{-1}(A) = A$ then C_T is generalized k-quasi-hyponormal if and only if $h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.h \circ T^{-k}$.*

Proof.

Case 1: Assume that C_T is generalized k-quasi-hyponormal.

$$C_T^{*k}(C_T C_T^*)C_T^k \leq M^{k+1} C_T^{*(k+1)} C_T^{k+1} \tag{10}$$

$$\left\langle (C_T^{*k}(C_T C_T^*)C_T^k)f, f \right\rangle \leq M^{k+1} \left\langle C_T^{*(k+1)} C_T^{k+1} f, f \right\rangle \quad \text{for each } f \in L^2. \tag{11}$$

Substitute (2) and (3) in (11),

$$h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.E(h) \circ T^{-k}$$

$$h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.h \circ T^{-k}$$

Case 2: Assume that

$$h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.h \circ T^{-k}$$

$$h_k.h \circ T^{-(k-1)} \leq M^{k+1} h_k.E(h) \circ T^{-k}.$$

Therefore, we have

$$C_T^{*k}(C_T C_T^*)C_T^k \leq M^{k+1} C_T^{*(k+1)} C_T^{k+1}$$

Hence C_T is generalized k-quasi-hyponormal. Hence the proof. □

Theorem 2.3. *If C_T^* is generalized k-quasi-hyponormal then $f \in L^2$,*

$$\left\langle h \circ T^k.h_k \circ T^k.E(f), f \right\rangle \leq M^{k+1} \left\langle h_{k+1} \circ T^{k+1}.E(f), f \right\rangle.$$

Proof.

Case 1: Assume that C_T^* is generalized k-quasi-hyponormal.

$$C_T^k(C_T^* C_T)C_T^{*k} \leq M^{k+1} C_T^{(k+1)} C_T^{*(k+1)} \tag{12}$$

$$\left\langle (C_T^k(C_T^* C_T)C_T^{*k})f, f \right\rangle \leq M^{k+1} \left\langle C_T^{(k+1)} C_T^{*(k+1)} f, f \right\rangle \quad \text{for each } f \in L^2. \tag{13}$$

Consider,

$$(C_T^k(C_T^* C_T)C_T^{*k})f = C_T^k(h.h_k.E(f) \circ T^{-k}) \tag{14}$$

$$(C_T^k(C_T^* C_T)C_T^{*k})f = h \circ T^k.h_k \circ T^k.E(f). \tag{15}$$

Then,

$$C_T^{(k+1)} C_T^{*(k+1)} f = C_T^{(k+1)}(h_{k+1}.E(f) \circ T^{-(k+1)}) \tag{16}$$

$$C_T^{(k+1)} C_T^{*(k+1)} f = h_{k+1} \circ T^{(k+1)} E(f). \tag{17}$$

Substitute (15) and (17) in (13),

$$\left\langle h \circ T^k.h_k \circ T^k.E(f), f \right\rangle \leq M^{k+1} \left\langle h_{k+1} \circ T^{(k+1)} E(f), f \right\rangle \text{ for each } f \in L^2.$$

Case 2: Assume that

$$\left\langle h \circ T^k.h_k \circ T^k.E(f), f \right\rangle \leq M^{k+1} \left\langle h_{k+1} \circ T^{k+1}.E(f), f \right\rangle \tag{18}$$

$$h \circ T^k.h_k \circ T^k.E(f) \leq M^{k+1} h_{k+1} \circ T^{k+1}.E(f). \tag{19}$$

Consider,

$$\begin{aligned} h \circ T^k.h_k \circ T^k.E(f) &= C_T^k(h.h_k.E(f) \circ T^{-k}) \\ &= C_T^k(C_T^* C_T)(h_k.E(f) \circ T^{-k}) \\ h \circ T^k.h_k \circ T^k.E(f) &= C_T^k(C_T^* C_T) C_T^{*k} f \end{aligned} \tag{20}$$

Then,

$$\begin{aligned} h_{k+1} \circ T^{(k+1)}.E(f) &= C_T^{(k+1)}(h_{k+1}.E(f) \circ T^{-(k+1)}) \\ h_{k+1} \circ T^{(k+1)}.E(f) &= C_T^{(k+1)} C_T^{*(k+1)} f \end{aligned} \tag{21}$$

Substitute (20) and (21) in (19),

$$C_T^k(C_T^* C_T^)C_T^{*k} \leq M^{k+1} C_T^{(k+1)} C_T^{*(k+1)}$$

Hence C_T^* is generalized k-quasi-hyponormal. □

Theorem 2.4. *If $T^{-1}(A) = A$ then C_T^* is generalized k-quasi-hyponormal if and only if*

$$\left\| \sqrt{h \circ T^k.h_k \circ T^k} f \right\| \leq M^{\frac{k+1}{2}} \left\| \sqrt{h_{k+1} \circ T^{(k+1)}} f \right\| \quad \textit{for each } f \in L^2.$$

Proof.

Case 1: Assume that C_T^* is generalized k-quasi-hyponormal.

$$C_T^k(C_T^* C_T) C_T^{*k} \leq M^{k+1} C_T^{(k+1)} C_T^{*(k+1)}$$

$$\left\langle (C_T^k(C_T^* C_T) C_T^{*k}) f, f \right\rangle \leq M^{k+1} \left\langle C_T^{(k+1)} C_T^{*(k+1)} f, f \right\rangle \quad \text{for each } f \in L^2.$$

Therefore, we have

$$h \circ T^k.h_k \circ T^k.E(f) \leq M^{k+1} h_{k+1} \circ T^{(k+1)} E(f) \text{ for each } f \in L^2.$$

$$h \circ T^k.h_k \circ T^k f \leq M^{k+1} h_{k+1} \circ T^{(k+1)} f$$

$$\left\| \sqrt{h \circ T^k.h_k \circ T^k} f \right\| \leq M^{\frac{k+1}{2}} \left\| \sqrt{h_{k+1} \circ T^{(k+1)}} f \right\| \text{ for each } f \in L^2.$$

Case 2: Assume that

$$\left\| \sqrt{h \circ T^k.h_k \circ T^k} f \right\| \leq M^{\frac{k+1}{2}} \left\| \sqrt{h_{k+1} \circ T^{(k+1)}} f \right\| \text{ for each } f \in L^2.$$

$$h \circ T^k.h_k \circ T^k.E(f) \leq M^{k+1} h_{k+1} \circ T^{(k+1)} E(f) \text{ for each } f \in L^2.$$

Therefore, we have

$$C_T^k(C_T^* C_T) C_T^{*k} \leq M^{k+1} C_T^{(k+1)} C_T^{*(k+1)}$$

Hence C_T^* is generalized k-quasi-hyponormal. □

3. On Generalized k-Quasi-Hyponormal Weighted Composition Operators

The new class of operator, k-quasi-hyponormal Weighed Composition operator has been introduced by G.Datt [1]. Quasi-hyponormal operator is an extension of hyponormal operator, quasiposinormal operator, quasiposinormal Composition operator and quasiposinormal Weighted Composition operator. Now we deal with the weighted composition operator $W = W_{(u,T)} \in B(L^2)$, $(f \mapsto u.f \circ T)$ induced by the complex-valued measurable mapping u on Ω and the measurable transformation $T : \Omega \longmapsto \Omega$. It is known that W^* is given by

$$W^* f = h.E(u.f) \circ T^{-1}, \quad \text{for each } f \in L^2.$$

For a positive integer k, we put $u_k = u.(u \circ T).(u \circ T^2)\dots(u.T^{(k.1)})$ and $\widehat{u}_k = (u \circ T^{-1}).(u \circ T^{-2})\dots(u \circ T^{-k})$. Then, $u_k \circ T^{-k} = \widehat{u}_k$. For $k = 0$, we denote $u_k = \widehat{u}_k = 1$ and $W^k = I$. However, h_k is used to denote the Radon Nikodym derivative of μT^{-k} with respect to μ and $h_1 = h$. For $f \in L^2$, $W^k f = u_k.f \circ T^k$ so that $W^{*k} f = h_k.E(u_k.f) \circ T^{-k}$. The following simple computations,

$$\begin{aligned}
W^* W^k f &= h.E(u^2).T^{-1}.W^{(k-1)} \\
W^{*(k+1)} f &= h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)} = h_{k+1}.E(u.f) \circ T^{-(k+1)}.\widehat{u}_k^2 \\
W^{*k}(WW^*)W^k f &= h_k.h \circ T^{-(k+1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}_{k-1}^2.f \\
W^{*(k+1)} W^{(k+1)} f &= h_{k+1}.E(u_{k+1}^2) \circ T^{-(k+1)}.f
\end{aligned}$$

help us to conclude the following.

Theorem 3.1. *Let $W \in B(L^2)$. Then W^* is generalized k-quasi-hyponormal if and only if*

$$\left\| u.h_k \circ T.E(u_k.f) \circ T^{-(k-1)} \right\| \leq M^{\frac{k+1}{2}} \left\| h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)} \right\| \quad \textit{for each } f \in L^2.$$

Proof.

Case 1: Assume that W^* is generalized k-quasi-hyponormal.

$$\begin{aligned}
W^k(W^*W)W^{*k} &\leq M^{k+1} W^{(k+1)} W^{*(k+1)} \\
\left\| WW^{*k} f \right\|^2 &\leq M^{k+1} \left\| W^{*(k+1)} f \right\|^2
\end{aligned} \tag{22}$$

Consider,

$$\begin{aligned}
WW^{*k} f &= u.h_k \circ T^k.E(u_k.f) \\
WW^{*k} f &= u.h_k \circ T.E(u_k.f) \circ T^{-(k-1)}
\end{aligned} \tag{23}$$

Then,

$$W^{*(k+1)} f = h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)} \tag{24}$$

Substitute (23) and (24) in (22),

$$\left\| uh_k \circ T.E(u_k.f) \circ T^{-(k-1)} \right\| \leq M^{\frac{k+1}{2}} \left\| h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)} \right\|$$

Case 2: Assume that

$$\left\|u.h_k \circ T.E(u_k.f) \circ T^{-(k-1)}\right\| \leq M^{\frac{k+1}{2}} \left\|h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)}\right\|$$
$$\left\|uh_k \circ T.E(u_k.f) \circ T^{-(k-1)}\right\|^2 \leq M^{k+1} \left\|h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)}\right\|^2 \tag{25}$$

Consider

$$h_{k+1}.E(u_{(k+1)}.f) \circ T^{-(k+1)} = W^{*(k+1)}f \tag{26}$$

Then,

$$u.h_k \circ T.E(u_k.f) \circ T^{-(k-1)} = u.h_k \circ T^k.E(u_k.f)$$
$$u.h_k \circ T.E(u_k.f) \circ T^{-(k-1)} = WW^{*k}f \tag{27}$$

Substitute (26) and (27) in (25),

$$W^k(W^*W)W^{*k} \leq M^{k+1}W^{(k+1)}W^{*(k+1)}$$

Hence W^* is generalized k-quasi-hyponormal. □

Theorem 3.2. *Let $W \in B(L^2)$. Then the following are equivalent:*

(i). W is generalized k-quasi-hyponormal.

(ii). $\left\|h.E(u^2) \circ T^{-1}.W^{(k-1)}f\right\| \leq M^{\frac{k+1}{2}} \left\|u_{k+1}.f \circ T^{(k+1)}\right\|$ *for each* $f \in L^2$.

(iii). $\left\|\sqrt{h_{k-1}}.h \circ T^{-(k-1)}.\widehat{u}_{k-1}.E(u^2) \circ T^{-k}f\right\| \leq M^{\frac{k+1}{2}} \left\|\sqrt{h_{k+1}}.\widehat{u}_{k+1}.f\right\|$ *for each* $f \in L^2$.

(iv). $h_k.h \circ T^{-(k-1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}^2_{k-1} \leq M^{k+1}h_{k+1}.E(u^2) \circ T^{-(k+1)}.\widehat{u}^2_k$
$$= M^{k+1}h_k.\widetilde{h}_k.\widehat{u}^2_k.E(u^2) \circ T^{-(k+1)},$$
where $\widetilde{h}_k = d\mu T^{-(k+1)}/d\mu T^{-k}$.

Proof. To Prove: $(i) \equiv (iv)$. Assume that W is generalized k-quasi-hyponormal.

$$W^{*k}(WW^*)W^k \leq M^{k+1}W^{*(k+1)}W^{(k+1)} \tag{28}$$

Consider

$$W^{*k}(WW^*)W^k = h_k.h \circ T^{-(k-1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}^2_{k-1} \tag{29}$$

Then,

$$W^{*(k+1)}W^{(k+1)}f = h_{k+1}.E(u^2_{k+1}) \circ T^{-(k+1)}f \tag{30}$$

Substitute (29) and (30) in (28),

$$h_k.h \circ T^{-(k-1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}^2_{k-1} \leq M^{k+1}h_{k+1}.E(u^2_{k+1}) \circ T^{-(k+1)}f$$
$$h_k.h \circ T^{-(k-1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}^2_{k-1} \leq M^{k+1}h_k.\widetilde{h}_k.\widehat{u}^2_k.E(u^2) \circ T^{-(k+1)}$$

Hence $(i) \equiv (iv)$. To Prove: $(iv) \equiv (iii)$. Assume that

$$h_k.h \circ T^{-(k-1)}.(E(u^2) \circ T^{-k})^2.\widehat{u}^2_{k-1} \leq M^{k+1}h_{k+1}.E(u^2) \circ T^{-(k+1)}.\widehat{u}^2_k$$

$$= M^{k+1}h_k.\widetilde{h}_k.\widehat{u}_k^2.E(u^2)\circ T^{-(k+1)}$$
$$h_k.h\circ T^{-(k-1)}.(E(u^2)\circ T^{-k})^2.\widehat{u}_{k-1}^2 f \leq M^{k+1}h_{k+1}.E(u_{k+1}^2)\circ T^{-(k+1)}f$$
$$h_{k-1}.h\circ T^{-(k-1)}.(E(u^2)\circ T^{-k})^2.\widehat{u}_{k-1}^2 f \leq M^{k+1}h_{k+1}.\widehat{u}_{k+1}^2 f$$
$$\left\|\sqrt{h_{k-1}}.h\circ T^{-(k-1)}.\widehat{u}_{k-1}.E(u^2)\circ T^{-k}f\right\| \leq M^{\frac{k+1}{2}}\left\|\sqrt{h_{k+1}}.\widehat{u}_{k+1}.f\right\|$$

Hence $(iv)\equiv(iii)$.

To Prove: $(iv)\equiv(ii)$. Assume that

$$h_k.h\circ T^{-(k-1)}.(E(u^2)\circ T^{-k})^2.\widehat{u}_{k-1}^2 \leq M^{k+1}h_{k+1}.E(u^2)\circ T^{-(k+1)}.\widehat{u}_k^2$$
$$= c^2h_k.\widetilde{h}_k.\widehat{u}_k^2.E(u^2)\circ T^{-(k+1)}$$
$$h_k.h\circ T^{-(k-1)}.(E(u^2)\circ T^{-k})^2.\widehat{u}_{k-1}^2 f \leq M^{k+1}h_{k+1}.E(u_{k+1}^2)\circ T^{-(k+1)}f \tag{31}$$

Consider,

$$h_k.h\circ T^{-(k-1)}.(E(u^2)\circ T^{-k})^2.\widehat{u}_{k-1}^2 f = h_{k+1}.h^2.(E(u^2)\circ T^{-1})^2.W^{2(k-1)}f \tag{32}$$

Then,

$$h_{k+1}.E(u_{k+1}^2)\circ T^{-(k+1)}f = h_{k+1}.u_{k+1}^2\circ T^{2(k+1)}f \tag{33}$$

Substitute (32) and (33) in (31),

$$\left\|h.E(u^2)\circ T^{-1}.W^{(k-1)}f\right\| \leq M^{\frac{k+1}{2}}\left\|u_{k+1}\circ T^{(k+1)}f\right\|$$

Hence the proof. □

Theorem 3.3. *Let W is generalized k-quasi-hyponormal if and only if* $\left\|h.E(u^2)\circ T^{-1}.W^{(k-1)}f\right\| \leq M^{\frac{k+1}{2}}\left\|u_{k+1}.f\circ T^{(k+1)}\right\|$ *for each $f\in L^2$.*

Proof.

Case 1: Assume that W is generalized k-quasi-hyponormal.

$$W^{*k}(WW^*)W^k \leq M^{k+1}W^{*(k+1)}W^{(k+1)}$$
$$\left\|W^*W^kf\right\|^2 \leq M^{k+1}\left\|W^{(k+1)}f\right\|^2 \tag{34}$$

Consider,

$$W^*W^kf = h.E(u^2).T^{-1}.W^{(k+1)} \tag{35}$$

Then,

$$W^{(k+1)}f = (u_{k+1}.f)\circ T^{(k+1)} \tag{36}$$

Substitute (35) and (36) in (34),

$$\left\|h.E(u^2).T^{-1}.W^{(k+1)}\right\| \leq M^{\frac{k+1}{2}}\left\|(u_{k+1}.f)\circ T^{(k+1)}\right\|$$

Case 2: Assume that

$$\left\|h.E(u^2)\circ T^{-1}.W^{(k-1)}f\right\| \leq M^{\frac{k+1}{2}}\left\|u_{k+1}.f\circ T^{(k+1)}\right\| \quad \text{for each } f\in L^2. \tag{37}$$

$$\left\| h.E(u^2).T^{-1}.W^{(k+1)} \right\|^2 \leq M^{k+1} \left\| (u_{k+1}.f) \circ T^{(k+1)} \right\|^2 \tag{38}$$

Consider,

$$h.E(u^2).T^{-1}.W^{(k+1)} = W^* W^k f \tag{39}$$

Then,

$$(u_{k+1}.f) \circ T^{(k+1)} = W^{(k+1)} f \tag{40}$$

Substitute (39) and (40) in (38),

$$\left\| W^* W^k f \right\|^2 \leq M^{k+1} \left\| W^{(k+1)} f \right\|^2$$
$$W^{*k}(WW^*)W^k \leq M^{k+1} W^{*(k+1)} W^{(k+1)}$$

Hence W is generalized k-quasi-hyponormal. □

4. Generalized k-Quasi-Hyponormal Operators on Weighted Hardy Space

The operator C_T are not necessarily defined on all of $H^2(\beta)$. They are ever where defined in some special cases in the classical Hardy space H^2. Let w be a point on the disk. Define $k^\beta_{w(z)} = \sum_{n=0}^{\infty} \frac{z^n \bar{w}^n}{\beta_n^2}$. Then the function k^β_w is a point evalution for $H^2(\beta)$. Then k^β_w is in $H^2(\beta)$ and $\left\| k^\beta_w \right\|^2 = \sum_{n=0}^{\infty} \frac{|\, w \,|^{2n}}{\beta_n^2}$. Thus, $\|k_w\|$ is an increasing function of $|\, w \,|$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $\left\langle f, k^\beta_w \right\rangle_\beta = f(w)$ for all f and k^β_w is known as the point evalution kernal at w. It can be easily shown that $C_T^* k^\beta_w = k^\beta_{T(w)}$ and $k^\beta_0 = 1$.

Theorem 4.1. *If C_T is generalized k-quasi-hyponormal operator on $H^2(\beta)$ then* $\left\| k^\beta_{T(0)} \right\|_\beta \leq M^{k+1}$.

Proof. Assume that C_T is generalized k-quasi-hyponormal operator on $H^2(\beta)$.

$$M^{k+1} C_T^{*(k+1)} C_T^{k+1} - C_T^{*k} C_T C_T^* C_T^k \geq 0$$
$$M^{k+1} \left\| C_T^{k+1} f \right\|^2 - \left\| C_T^* C_T^k f \right\|^2 \geq 0$$

Let $f = k^\beta_0$

$$M^{k+1} \left\| C_T^k C_T k^\beta_0 \right\|_\beta^2 - \left\| C_T^* C_T^k k^\beta_0 \right\|_\beta^2 \geq 0$$
$$M^{k+1} \left\| C_T^k k^\beta_0 \right\|_\beta^2 - \left\| C_T^* C_T^k k^\beta_0 \right\|_\beta^2 \geq 0$$
$$M^{k+1} \left\| k^\beta_0 \right\|_\beta^2 - \left\| C_T^* k^\beta_0 \right\|_\beta^2 \geq 0$$
$$M^{k+1} \left\| k^\beta_0 \right\|_\beta^2 - \left\| k^\beta_{T(0)} \right\|_\beta^2 \geq 0$$
$$M^{k+1}(1) - \left\| k^\beta_{T(0)} \right\|_\beta^2 \geq 0$$
$$M^{k+1} - \left\| k^\beta_{T(0)} \right\|_\beta^2 \geq 0$$
$$\left\| k^\beta_{T(0)} \right\|_\beta^2 \leq M^{k+1}$$

Hence the proof. □

Theorem 4.2. *If C_T^* is generalized k-quasi-hyponormal operator on $H^2(\beta)$ then $M^{k+1} \geq 1$.*

Proof. Assume that C_T^* is generalized k-quasi-hyponormal operator on $H^2(\beta)$.

$$M^{k+1} C_T^{k+1} C_T^{*(k+1)} - C_T^k C_T^* C_T C_T^{*k} \geq 0$$
$$M^{k+1} \left\| C_T^{*(k+1)} f \right\|^2 - \left\| C_T C_T^{*k} f \right\|^2 \geq 0$$

Let $f = k_0^\beta$

$$M^{k+1} \left\| C_T^{*(k+1)} k_0^\beta \right\|_\beta^2 - \left\| C_T C_T^{*k} k_0^\beta \right\|_\beta^2 \geq 0$$
$$M^{k+1} \left\| k_0^\beta \right\|_\beta^2 - \left\| C_T k_0^\beta \right\|_\beta^2 \geq 0$$
$$M^{k+1} \left\| k_0^\beta \right\|_\beta^2 - \left\| k_0^\beta \right\|_\beta^2 \geq 0$$
$$M^{k+1}(1) - 1 \geq 0$$
$$M^{k+1} - 1 \geq 0$$
$$M^{k+1} \geq 1$$

Hence the proof. □

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