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Fuzzy Maximal Ideals of ADL's

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Abstract: In this paper, we determine all maximal L-fuzzy ideals of a given ADL A with a maximal element by establishing a one-to-one correspondence between maximal L-fuzzy ideals of A and the pairs (M, α), where M is a maximal ideal of A and α is a dual atom in a complete lattice L satisfying the infinite meet distributive law. Also, the notion of an L-fuzzy maximal ideal of A is introduced and characterized.

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1. Introduction

L.A. Zadeh [9] introduced the notion of a fuzzy subset of a set X as a function from X into [0, 1]. Liu [2] introduced and studied the notion of fuzzy ideal of rings. Following Liu, Mukherjee and Sen [3] defined and examined fuzzy prime ideals of a ring and continued the study of fuzzy maximal, radical and primary ideals of a ring, as in [4]. Y.B. June, K.H. Kim and M.A. Ozturk [1] introduced the notion of fuzzy maximal ideals of gamma near-rings. U.M. Swamy and K.L.N. Swamy [8] introduced the concept of fuzzy prime ideal of a ring with truth values in a complete lattice satisfying the infinite meet distributive law.

Axiomatization of Boole's two valued propositional calculus led to the crucial concept of a Boolean algebra in Lattice Theory and Boolean rings in Ring Theory. Boolean algebras are used to construct and simplify electrical circuits, switching circuits, which are used in the design of computer chips. However, there are some situations in which two valued propositional calculus is not adequate. In this context, several generalizations of Boolean algebras (Boolean rings) have come up into focus. U.M. Swamy and G.C. Rao [7] have introduced the notion an Almost Distributive Lattice (abbreviated as ADL) as a common abstraction of several lattice theoretic and ring theoretic generalizations of Boolean algebras and Boolean rings. U.M. Swamy, Ch. S.S. Raj and A. Natnael Teshale [5, 6] have introduced the notion of L-fuzzy ideals of an ADL A and the notions of prime L-fuzzy ideals and L-fuzzy prime ideals of an ADL A with a maximal element with the truth values in a complete lattice L satisfying the infinite meet distributive law; that is

$$a \wedge \left(\bigvee_{s \in S} s \right) = \bigvee_{s \in S} (a \wedge s)$$

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for all $a \in L$ and $S \subseteq L$.

In this paper, Section 3 is devoted to the study of maximal L-fuzzy ideals of a given ADL A with a maximal element and determine all maximal L-fuzzy ideals of A, by proving that the maximal L-fuzzy ideals are precisely L-fuzzy ideals α_M given by

$$\alpha_M(x) = \begin{cases} 1, & \text{if } x \in M \\ \alpha, & \text{if } x \notin M \end{cases}$$

where M is a maximal ideal of A and α is a dual atom in L.

In Section 4, we introduce the notion of an L-fuzzy maximal ideal of A, which is weaker than that of a maximal L-fuzzy ideal of A. An L-fuzzy ideal λ of A is said to be an L-fuzzy maximal ideal of A if, for each $\alpha \in L$, the α-cut λ_α is either A or a maximal ideal of A.

2. Preliminaries

In this section, we recall some definitions and basic results on Almost Distributive Lattices and L-fuzzy ideals of an ADL.

Definition 2.1 ([7]). *An algebra $A = (A, \wedge, \vee, 0)$ of type $(2, 2, 0)$ is called an Almost Distributive Lattice (abbreviated as ADL) if it satisfies the following conditions for all a, b and $c \in A$.*

(1). $0 \wedge a = 0$

(2). $a \vee 0 = a$

(3). $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(4). $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5). $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$

(6). $(a \vee b) \wedge b = b$.

Any bounded below distributive lattice is an ADL, where 0 is the smallest element. Any nonempty set X can be made into an ADL by fixing an arbitrarily chosen element 0 in X and by defining the binary operations $\wedge$ and $\vee$ on X by

$$a \wedge b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases} \quad \text{and} \quad a \vee b = \begin{cases} b, & \text{if } a = 0 \\ a, & \text{if } a \neq 0. \end{cases}$$

This ADL $(X, \wedge, \vee, 0)$ is called a discrete ADL.

Definition 2.2 ([7]). *Let $A = (A, \wedge, \vee, 0)$ be an ADL. For any a and $b \in A$, define $a \leq b$ if $a = a \wedge b$ ($\Leftrightarrow a \vee b = b$). Then $\leq$ is a partial order on A with respect to which 0 is the smallest element in A.*

Theorem 2.3 ([7]). *The following hold for any a, b and c in an ADL A.*

(1). $a \wedge 0 = 0 = 0 \wedge a$ *and* $a \vee 0 = a = 0 \vee a$

(2). $a \wedge a = a = a \vee a$

(3). $a \wedge b \leq b \leq b \vee a$

(4). $a \wedge b = a \Leftrightarrow a \vee b = b$

(5). $a \wedge b = b \Leftrightarrow a \vee b = a$

(6). $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ *(i.e., $\wedge$ is associative)*

(7). $a \vee (b \vee a) = a \vee b$

(8). $a \leq b \Rightarrow a \wedge b = a = b \wedge a \ \big(\Leftrightarrow a \vee b = b = b \vee a\big)$

(9). $(a \wedge b) \wedge c = (b \wedge a) \wedge c$

(10). $(a \vee b) \wedge c = (b \vee a) \wedge c$

(11). $a \wedge b = b \wedge a \Leftrightarrow a \vee b = b \vee a$

(12). $a \wedge b = \inf\{a, b\} \Leftrightarrow a \wedge b = b \wedge a \Leftrightarrow a \vee b = \sup\{a, b\}$.

An element $m \in A$ is said to be maximal if, for any $x \in A$, $m \leq x$ implies $m = x$. It can be easily observed that m is maximal if and only if $m \wedge x = x$ for all $x \in A$.

Definition 2.4 ([7]). *Let I be a non empty subset of an ADL A. Then I is called an ideal of A if $a, b \in I \Rightarrow a \vee b \in I$ and $a \wedge x \in I$ for all $x \in A$.*

As a consequence, for any ideal I of A, $x \wedge a \in I$ for all $a \in I$ and $x \in A$. For any $S \subseteq A$, the smallest ideal of A containing S is called the ideal generated by S in A and is denoted by $(S]$. It is known that

$$(S] = \Big\{ \big(\bigvee_{i=1}^{n} x_i\big) \wedge a \mid n \geq 0, x_i \in S \text{ and } a \in A \Big\}.$$

when $S = \{x\}$, we write $(x]$ for $(\{x\}]$. Note that $(x] = \{x \wedge a \mid a \in A\}$.
Throughout this paper, L stands for a complete lattice satisfying the infinite meet distributivity.

Definition 2.5 ([6]). *An L-fuzzy subset λ of a set X is a mapping from X into L. If L is the unit interval $[0, 1]$ of real numbers, then these are the usual fuzzy subsets of X.*

For any $\alpha \in L$, the set $\lambda_\alpha = \{x \in X : \alpha \leq \lambda(x)\}$ is called the α-cut of λ.

Definition 2.6 ([6]). *An L -fuzzy subset λ of A is said to be an L-fuzzy ideal of A, if $\lambda(0) = 1$ and $\lambda(x \vee y) = \lambda(x) \wedge \lambda(y)$, for all $x, y \in A$.*

Lemma 2.7 ([6]). *Let λ be an L-fuzzy ideal of A, S a non-empty subset of A and $x, y \in A$. Then we have the following.*

(1). $x \wedge y = y$ *and* $y \wedge x = x \Rightarrow \lambda(x) = \lambda(y)$

(2). $\lambda(x \wedge y) = \lambda(y \wedge x)$

(3). $x \in (S] \Rightarrow \lambda(x) \geq \bigwedge_{i=1}^{n} \lambda(a_i)$ *for some* $a_1, a_2, \ldots, a_n \in S$

(4). $x \in (y] \Rightarrow \lambda(x) \geq \lambda(y)$

(5). If m is a maximal element in A then $\lambda(m) \leq \lambda(x)$, for all x

(6). $\lambda(m) = \lambda(n)$ *for all maximal elements m and n in A.*

Theorem 2.8 ([6]). *The set of all L-fuzzy ideals of A is a complete distributive lattice, in which the supremum $\bigvee_{i\in\Delta}\lambda_i$ and infimum $\bigwedge_{i\in\Delta}\lambda_i$ of any family $\{\lambda_i : i \in \Delta\}$ of L-fuzzy ideals of A are given by*

$$\Big(\bigvee_{i\in\Delta}\lambda_i\Big)(x) = \bigvee\Big\{\bigwedge_{a\in F}\Big(\bigvee_{i\in\Delta}\lambda_i(a)\Big) : x \in (F], F \subset\subset A\Big\}$$

$$\text{and} \quad \Big(\bigwedge_{i\in\Delta}\lambda_i\Big)(x) = \bigwedge_{i\in\Delta}\lambda_i(x)$$

3. Maximal L-Fuzzy Ideals

Let us recall from [7] that a proper ideal M of an ADL A is said to be maximal if it is not properly contained in any proper ideal of A; (equivalently, for any ideal N of A, $M \subseteq N$ implies that either $M = N$ or $N = A$). Hereafter A stands for an ADL with a maximal element.

Definition 3.1. *An L-fuzzy ideal λ of A is said to be proper if $\lambda(x) \neq 1$ for some $x \in A$. A proper L-fuzzy ideal λ of A is said to be maximal if λ is a maximal element in the set of all proper L-fuzzy ideals of A.*

In this section, we determine all the maximal L-fuzzy ideals of A by establishing a one-to-one correspondence between maximal L-fuzzy ideals of A and pairs (M, α), where M is a maximal ideal of A and α is a dual atom of L. Let us recall that an element $\alpha \neq 1$ in L is called a dual atom if there is no $\beta \in L$ such that $\alpha < \beta < 1$. Clearly, α is a dual atom if and only if α is a maximal element of $L - \{1\}$. Here 1 stands for the largest element in L. For any subset I of A and $\alpha \in L$, let us define $\alpha_I : A \to L$ by

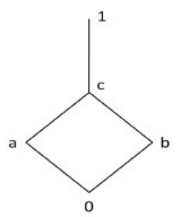
$$\alpha_I(x) = \begin{cases} 1, & \text{if } x \in I \\ \alpha, & \text{if } x \notin I \end{cases}$$

Note that α_I is an L-fuzzy ideal of A if and only if I is an ideal of A. In general, every maximal ideal (maximal element) of any bounded distributive lattice is a prime ideal (prime element). Let us recall that an element $p \neq 1$ in L is called prime if, for any $a, b \in L$, $a \wedge b \leq p$ implies either $a \leq p$ or $b \leq p$. From [5], we recall that, λ is a prime L-fuzzy ideal of A if and only if there exists a prime ideal P of A and a prime element α in L such that $\lambda = \alpha_P$. The following is an immediate consequence.

Theorem 3.2. *Every maximal L-fuzzy ideal of A is a prime L-fuzzy ideal of A.*

The converse of this is not true. For, consider the following example.

Example 3.3. *Let $A = \{0, a, b, c, 1\}$ be the lattice represented by the Hasse diagram is given below.*



Define $\mu : A \to [0,1]$ by $\mu(0) = \mu(a) = 1$ and $\mu(b) = \mu(c) = \mu(1) = 0.5$. Clearly, $\mu = \alpha_P$, where $\alpha = 0.5$ and $P = \{0, a\}$ is a prime ideal of A. Therefore, μ is a prime L-fuzzy ideal of A, but not maximal, since $0.5_P < 0.75_P$.

In the following we characterize maximal L-fuzzy ideals of an ADL A.

Theorem 3.4. *Let λ be an L-fuzzy subset of A. Then λ is a maximal L-fuzzy ideal of A if and only if there exist a maximal ideal M of A and a dual atom α in L such that $\lambda = \alpha_M$.*

Proof. Suppose that λ is a maximal L-fuzzy ideal of A. Then by Theorem 3.2, λ is a prime L-fuzzy ideal of A and hence $\lambda = \alpha_M$, for some prime ideal M of A and prime element α in L. Let N be a proper ideal of A containing M. Then $\lambda = \alpha_M \leq \alpha_N$ and α_N is a proper L-fuzzy ideal of A. By the maximality of λ, $\alpha_M = \alpha_N$ and hence $M = N$. Thus M is a maximal ideal of A. Also, if $\alpha \leq \beta < 1$ in L, then $\lambda = \alpha_M \leq \beta_M$ and β_M is a proper L-fuzzy ideal of A. Again by the maximality of λ, $\alpha_M = \beta_M$ and hence $\alpha = \beta$. Thus α is a dual atom in L.
Conversely suppose that M is a maximal ideal of A and α is a dual atom in L such that $\lambda = \alpha_M$. Since M is proper, there exists $x \in A - M$ such that $\lambda(x) = \alpha < 1$. Therefore, λ is a proper L-fuzzy ideal of A. Let μ be a proper L-fuzzy ideal of A such that $\lambda \leq \mu$. Consider $N = \{x \in A : \mu(x) = 1\}$. Clearly, $N = \mu_1$ is an ideal of A and

$$y \in M \Rightarrow 1 = \alpha_M(y) = \lambda(y) \leq \mu(y) \Rightarrow \mu(y) = 1 \Rightarrow y \in N.$$

Therefore, $M \subseteq N$. Also, since μ is proper, there exists $x \in A$ such that $\mu(x) \neq 1$ and hence $x \notin N$. Therefore, N is a proper ideal of A. By the maximality of M, it follows that $M = N$. For any $x \in A$ with $\mu(x) = \beta < 1$, we get that $x \notin N$ and hence we have

$$\alpha = \alpha_M(x) = \lambda(x) \leq \mu(x) = \beta,$$

and hence $\alpha = \beta$, since α is a dual atom. Now, for any $x \in A$,

$$x \in M \Rightarrow \lambda(x) = 1 = \mu(x)$$
$$x \notin M \Rightarrow \lambda(x) = \alpha = \beta = \mu(x).$$

Thus, $\lambda = \mu$. Therefore, λ is a maximal L-fuzzy ideal of A. □

The following is simple consequence of the above theorem.

Theorem 3.5. *$(M, \alpha) \longmapsto \alpha_M$ is a bijection correspondence between the set of pairs (M, α), where M is a maximal ideal of A and α is a dual atom in L and the set of maximal L-fuzzy ideals of A.*

4. L-fuzzy Maximal Ideals

In this section, we introduce the notion of an $L-$fuzzy maximal ideal of A, which is weaker than that of a maximal L-fuzzy ideal of A and in such a way that any ideal I of A is maximal if and only if its characteristics map χ_I is an L-fuzzy maximal ideal of A.

Definition 4.1. *A proper L-fuzzy ideal λ of A is called an L-fuzzy maximal ideal of A if, for each $\alpha \in L$, either $\lambda_\alpha = A$ or λ_α is a maximal ideal of A.*

It can be easily verified that for any L-fuzzy maximal ideal λ of A, $\lambda_1 = \{x \in A : \lambda(x) = 1\}$ is a maximal ideal of A. Mainly, in this section we give a characterization of an L-fuzzy maximal ideal of A. First we have the following lemma.

Lemma 4.2. *Any L-fuzzy maximal ideal of A attains exactly two values.*

Proof. Let λ be an L-fuzzy maximal ideal of A. Suppose that λ assumes more than two values. Then there exists $x, y \in A$ and $\alpha \neq \beta$ in $L - \{1\}$ such that $\lambda(x) = \alpha$, $\lambda(y) = \beta$ and $\lambda(0) = 1$. Then $x \in \lambda_\alpha$ and $x \notin \lambda_1$. Therefore, $\lambda_1 \subsetneqq \lambda_\alpha \subseteq A$. By the maximality of λ_1, we get that $\lambda_\alpha = A$ and in particular, $y \in \lambda_\alpha$, so that $\alpha \leq \beta$. Similarly, $\beta \leq \alpha$. Thus $\alpha = \beta$. In particular, we have $\lambda(x) = \lambda(m)$, for any maximal element m in A and $\lambda(x) \in L - \{1\}$. Thus $\lambda(0)$ and $\lambda(m)$ are the only values of λ and these are distinct, since λ is proper. □

Theorem 4.3. *A proper L-fuzzy ideal λ of A is an L-fuzzy maximal ideal of A if and only if $\lambda = \alpha_M$, for some maximal ideal M of A and $\alpha(\neq 1) \in L$.*

Proof. Suppose that λ is an L-fuzzy maximal ideal of A. Then $M = \{x \in A : \lambda(x) = 1\}$ is a maximal ideal of A. By lemma 4.2, for any $x \in A$ and any maximal element m in A, we have that $\lambda(x) = \lambda(0)$ or $\lambda(m)$. Let $\alpha = \lambda(m)$. Then $\lambda(x) = \alpha$, for each $x \in A - M$. This implies that $\lambda = \alpha_M$.
Conversely, suppose that $\lambda = \alpha_M$, for some maximal ideal M of A and $\alpha(\neq 1)$ in L. Since $\alpha < 1$ and M is proper, it follows that λ is a proper L-fuzzy ideal of A. Also, for any $\beta \in L$, $\beta \leq \alpha \Rightarrow A = \lambda_\alpha \subseteq \lambda_\beta \Rightarrow \lambda_\beta = A$ and $\beta \nleq \alpha \Rightarrow \lambda_\beta \subseteq M$ (since, $x \notin M \Rightarrow \lambda(x) = \alpha$ and hence $x \notin \lambda_\beta) \Rightarrow \lambda_\beta = M$ (since, $M = \lambda_1 \subseteq \lambda_\beta$). Thus $\lambda_\beta = A$ or M, for each $\beta \in L$. Therefore, λ is an L-fuzzy maximal ideal of A. □

Corollary 4.4. *Let M be a proper ideal of A. Then M is a maximal ideal of A if and only if the characteristic map χ_M is an L-fuzzy maximal ideal of A.*

Proof. It follows from the facts that, $\chi_M = 0_M$ and $(\chi_M)_\alpha = M$, for all $0 < \alpha < 1$ and $(\chi_M)_0 = A$. □

Corollary 4.5. *Every maximal L-fuzzy ideal of A is an L-fuzzy maximal ideal of A.*

The converse of this is not true. For, consider the following example.

Example 4.6. *Let $L = [0, 1]$, the closed unit interval of real numbers. Then L is a complete lattice satisfying the infinite meet distributive law. Let A be an ADL with a maximal element and M a maximal ideal of A. Then for any $0 \leq \alpha < 1$, α_M is an L-fuzzy maximal ideal of A (by theorem 4.3) but not maximal L-fuzzy ideal of A, since α is not a dual atom in L. In fact, L has no dual atoms.*

References

[1] Y.B.Jun, K.H.Kim and M.A. Ozturk, *Fuzzy Maximal Ideals of Gamma Near-Rings*, Turk J. Math., 25(2001), 457-463.

[2] W.J.Liu, *Fuzzy invariant subgroups and fuzzy ideals*, Fuzzy Set and Systems, 8(1982), 133-139.

[3] T.K.Mukherjee and M.K.Sen, *Fuzzy ideal of a ring (1)*, in Proceedings, Seminar on Fuzzy systems and Non-standard Logic, Calcutta, May (1984).

[4] T.K.Mukherjee and M.K.Sen, *Fuzzy Maximal, Radical and Primary ideals of a Ring*, Information Sciences, 53(1991), 237-250.

[5] Ch.Santhi Sundar Raj, A.Natnael Teshale and U.M.Swamy, *Fuzzy Prime Ideals of ADL's*, (Communicated Paper).

[6] U.M.Swamy, Ch.Santhi Sundar Raj and A.Natnael Teshale, *Fuzzy Ideals of Almost Distributive Lattices*, Annals of Fuzzy Mathematics and Informatics, 14(4)(2017), 371-379.

[7] U.M.Swamy and G.C.Rao, *Almost Distributive Lattices*, J. Australian Math. Soc., (Series A), 31(1981), 77-91.

[8] U.M.Swamy and K.L.N.Swamy, *Fuzzy Prime Ideals of Rings*, J. Math. Anal. Appl., 134(1988), 94-103.

[9] L.A.Zadeh, *Fuzzy set*, Inform and Control, 8(1965), 338-353.