Int. J. Math. And Appl., 6(1–E)(2018), 1013–1023
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Analysis of Preemptive Priority Retrial Queueing System with Starting Failure, Modified Bernoulli Vacation with Vacation Interruption, Repair, Immediate Feedback and Impatient Customers

G. Ayyappan[1] and J. Udayageetha[2,*]

1 Department of Mathematics, Pondicherry Engineering College, Puducherry, India.

2 Department of Mathematics, Perunthalaivar Kamarajar Arts College, Puducherry, India.

Abstract: This paper considers $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. The server serves two types of customers and follows the pre-emptive priority rule subject to starting failure, repair, immediate feedback, orbital search and modified Bernoulli vacation with vacation interruption. High priority customes are considered as a feedback customer. After vacation completion, service completion and repair completion if there is no high priority customers present in the system the server may go for orbital search or remains idle. The high priority customer may renege the queue and the low priority customers may balk the orbit. Retrial time, service times, repair time and vacation time are assumed to be arbitrarily distributed. Various performance measures are derived and numerical results are presented. Using the supplementary variable technique, the steady-state distributions of the server state and the number of customers in the orbit are obtained. Further, some particular cases of interest are discussed. Finally, numerical illustrations are provided.

MSC: 60K25, 68M30, 90B22.

Keywords: Priority queueing systems, Starting failure, Modified Bernoulli vacation, Vacation interruption, Immediate feedback, Reneging, Balking.

1. Introduction

A Priority mechanism in a queueing system differentiates customers based on their classes. Such differentiation appears in a number of situations of everyday life and in major engineering systems, notably, job scheduling in manufacturing, operating systems in computers etc., Correct assignment of priorities brings customer satisfaction while keeping the total workload unchanged. Priority queueing system can be broadly classified into two categories namely non pre emptive and pre emptive priority queueing discipline.

In this paper, we have consider the pre emptive queueing discipline. Jain [4] studied about the bulk arrival retrial queue with unreliable server and priority subscribers. Gao [2] described about pre emptive queueing discipline with general retrial times. One of the most important characteristic in the service facility of a queueing system is its starting failures. An arriving customer who finds the server idle must turn on the server. If the server is started successfully the customer gets the service immediately. Otherwise the repair for the server begins and the customer must join the queue or orbit. The

* *E-mail: udaya.shivani@pec.edu*

server is assumed to be reliable during the service period. Such systems with starting failures have been studied as queueing models by Yang and Li [10], Krishnakumar [7] and Ke and Chang [6].

One specific feature which has been widely discussed in retrial queueing systems is feedback of customers. After completing the service, the customer who does not satisfy with the service has to go to the server immediately one more time is known as immediate feedback, otherwise known as Re-service. The real life applications are bank counters, working ATM machines, etc. Some authors like Kalidass [5] and Rajadurai [8], have discussed the concept of immediate feedback.

Retrial queues considered by researchers so far have the characteristic that each service is preceeded and followed by an idle period. A retrial queue in which immediately after a service completion the server searches for customer from the orbit or remains idle. Sumitha [9] studied about starting failure with orbital search. Research work on retrial queueing system with orbital search is often found in literature. This motivates to study a single server retrial queue with vacation, starting failure and orbital search.

In this paper, we consider $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ general retrial queueing system with priority services. The server serves two types of customers and follows the pre-emptive priority rule subject to starting failure, repair, immediate feedback, orbital search and modified Bernoulli vacation with vacation interruption. If the server is started successfully the customer gets the service immediately. Otherwise the repair for the server begins and the high priority customer must join the queue and the low priority customer must join the orbit. The server is assumed to be reliable during the service period. After completing all high priority service in the system the server may go for a vacation. During vacation period the arriving high priority customer interrupt the vacation and make the server to serve them. Immediate feedback is given to the high priority customer. After vacation completion, service completion and repair completion the server searches for the customers in the orbit to serve or remains idle. The high priority customer may renege the queue and the low priority customer may balk the orbit. Retrial time, service times, repair time and vacation time are assumed to be generally(arbitrarily) distributed.

Further on, the structure of the paper is as follows. Section 1 is an introduction to priority retrial queueing discipline and comprises literature review. A detailed description of the model, notations used, mathematical formulation and governing equations of the model is given in Section 2. Section 3 presents the steady state solutions and the stationary joint distribution of the server state and orbit size. Section 4 demonstrates the performance measures of the model. In Section 5, the numerical results are computed and graphical studies are shown following which the conclusion is given.

2. Model Description

We consider a single server priority queuing system with two types of customers namely, high priority and low-priority customers. The basic operation of the model can be described as: **Arrival process:** Two class of customers arrive at the system in two independent compound Poisson processes with arrival rate λ_1 and λ_2 respectively. Let $\lambda_1 c_{1,i} dt$ and $\lambda_2 c_{2,i} dt$; $(i = 1, 2, 3, \dots)$ be the first order probability that a batch of 'i' customers arrives at the system during a short interval of time $(t, t+dt)$, where for $0 \le c_{1,i} \le 1$, $\sum_{i=1}^{\infty} c_{1,i} = 1$, $0 \le c_{2,i} \le 1$, $\sum_{i=1}^{\infty} c_{2,i} = 1$. The arriving high priority customer who find the server busy is queued and then is served. The arriving low-priority customer on finding the server busy, are routed to a retrial queue (orbit).

Retrial process: After joining the orbit the low priority customer follows constant retrial policy that attempts to get the service. The retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$. Let $\eta(x)dx$ be the conditional probability of completion of retrial during the interval $(x, x+dx]$ where x is the elapsed retrial time.

Service process: If a high priority customer arrives in batch and finds a low priority customer in service, they pre-empt the low priority customer who is undergoing service; thus the service of the pre-empted low priority customer begins only after the completion of service of all high priority customers present in the system. And if the high priority customer is not satisfied with the service given, they can get the reservice immediately. The service times for the high priority, feedback and low priority customers are generally(arbitrary) distributed with distribution functions $B_i(s)$ and the density functions $b_i(s)$, $i = 1, 2, 3$ respectively. Let $\mu_i(x)dx$ be the conditional probability of completion of the service during the interval $(x, x+dx]$, where x is the elapsed service time.

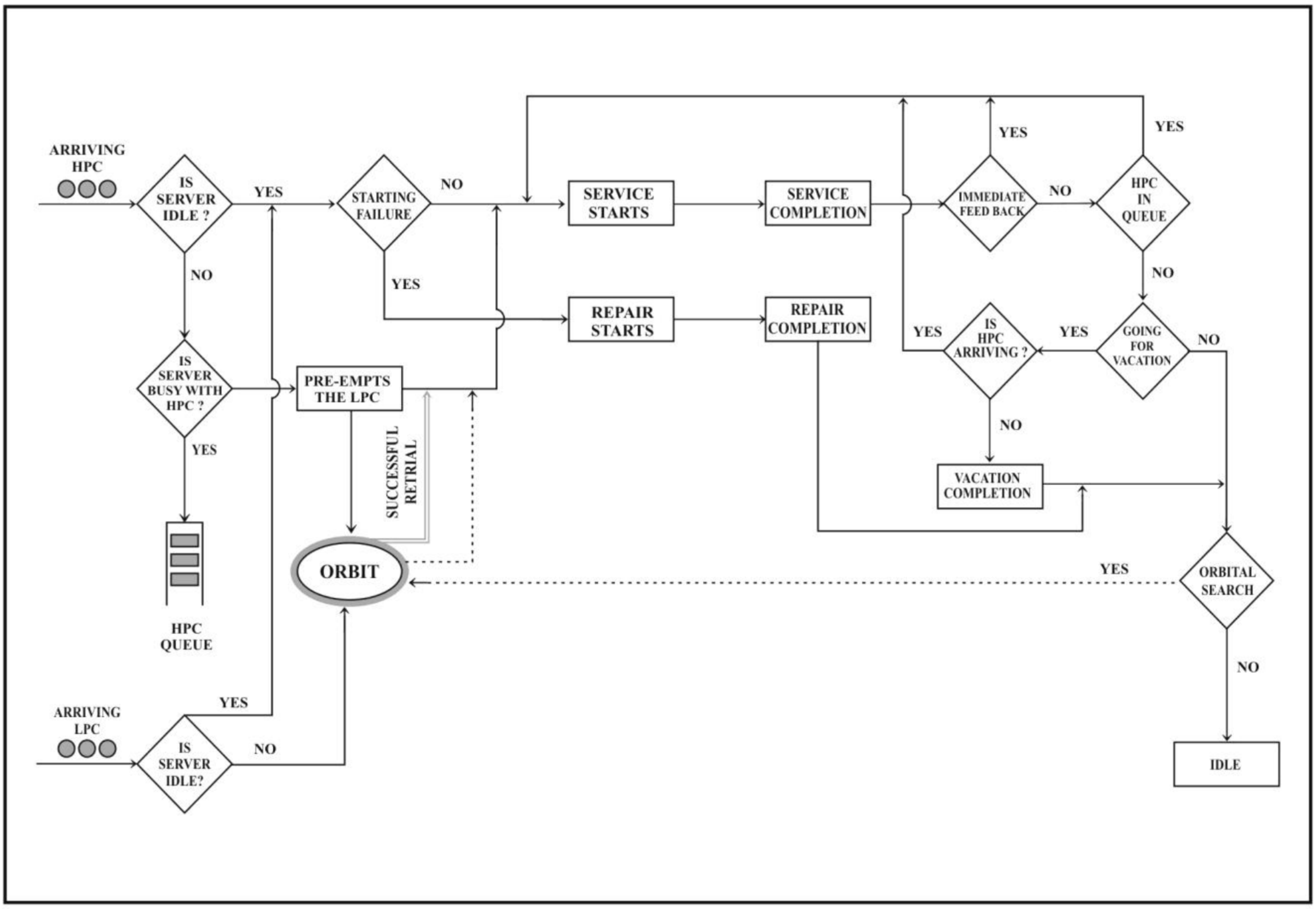

Figure 1. **Flow chart of the Model description**

Immediate feedback: After completing the primary call service, the customer who wishes to obtain another round of service has to go to the server immediately one more time with probability p or may leave the system with probability $1-p$, which is known as immediate feedback (Re-service).

Modified Bernoulli Vacation: After completing all high priority customers and every service completion of low priority customer the server may take a vacation with probability θ or continue the service to the next customer with probability $1-\theta$. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$. Let $\beta(x)dx$ be the conditional probability of completion of vacation during the interval $(x, x+dx]$ where x is the elapsed vacation time

Vacation Interruption: During Vacation period the arriving high priority customer interrupt the vacation and change the server into service mode.

Starting Failure: During the idle period the arriving customers may get the service successfully with probability α or the server is broken down with probability $\overline{\alpha}$. Repair for the server begins immediately. The server is assumed to be reliable during the service period.

Repair Process: The broken down server is sent for repair immediately. Repair time is generally distributed with distribution function $R(s)$ and the density function $r(s)$. Let $\gamma(x)dx$ be the conditional probability of completion of repair during the interval $(x, x+dx]$ where x is the elapsed repair time.

Orbital Search: After completing service, vacation and repair if there is no high priority customer present in the system the server may search the orbit to serve the low priority customer with probability r or remains idle.

Reneging: If the server is busy or unavailable in the system, the high priority customer may renege the queue exponentially with rate ξ.

Balking: If the server is busy or unavailable in the system, the arriving low priority customers may balk the orbit with probability $1-b$.

Idle State: If the server does not go for orbital search, he remains idle in the system.

2.1. Definitions and Notations

Let $N_1(t)$, $N_2(t)$ be the queue size and orbit size at time t, $B_i^0(t)$, $i = 1,2,3$, $V^0(t)$, $R^0(t)$ and $I^0(t)$ be the elapsed service time of the high priority customer, feedback customer, low priority customer, vacation time, repair time and retrial time respectively at time t. Let $Y(t)$ denote the state of the server,

$$Y(t) = \begin{cases} 0, & \text{if the server is idle;} \\ 1, & \text{if the server is providing service to the high priority customer;} \\ 2, & \text{if the server is providing service to the feedback customer;} \\ 3, & \text{if the server is providing service to the low priority customer;} \\ 4, & \text{if the server is in vacation;} \\ 5, & \text{if the server is in repair state;} \end{cases}$$

we have $I(x)$, $B_i(x)$, $V(x)$ and $R(x)$ is continuous at $x = 0$, and, $\eta(x)dx = \frac{dI(x)}{1-I(x)}$, $\mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)}$, $i = 1,2,3$, $\beta(x)dx = \frac{dV(x)}{1-V(x)}$, $\gamma(x)dx = \frac{dR(x)}{1-R(x)}$ are the first order differential (hazard rate) functions of $I(.)$, $B_i(.)$, $V(.)$ and $R(.)$ respectively.

2.2. Queue Size Distribution

Since the service time, vacation time, repair time and retrial time are not exponential, the process $\{Y(t), N_1(t), N_2(t)\}$ is non Markovian. In such case we introduce supplementary variables corresponding to elapsed times to make it Markovian [Cox(1955)]. Joint distributions of the server state, queue size and orbit size are defined as,

$$\overline{I}_{0,n}(x,s,t)dx = Pr\{Y(t) = 0, x < I^0(t) \le x+dx, N_1(t) = 0, N_2(t) = n\}, \quad n \ge 1$$

$$\overline{P}^{(1)}_{1,m,n}(x,s,t)dx = Pr\{Y(t) = 1, x < B_1^0(t) \le x+dx, N_1(t) = m, N_2(t) = n\}, \quad m \ge 0, n \ge 0,$$

$$\overline{P}^{(1)}_{2,m,n}(x,s,t)dx = Pr\{Y(t) = 2, x < B_2^0(t) \le x+dx, N_1(t) = m, N_2(t) = n\}, \quad m \ge 0, n \ge 0,$$

$$\overline{P}^{(2)}_{0,n}(x,s,t)dx = Pr\{Y(t) = 3, x < B_3^0(t) \le x+dx, N_1(t) = 0, N_2(t) = n\}, \quad n \ge 0,$$

$$\overline{V}_{0,n}(x,s,t)dx = Pr\{Y(t) = 4, x < V^0(t) \le x+dx, N_1(t) = 0, N_2(t) = n, \}, \quad n \ge 0$$

$$\overline{R}_{m,n}(x,s,t)dx = Pr\{Y(t) = 5, x < R^0(t) \le x+dx, N_1(t) = m, N_2(t) = n\}, \quad m \ge 0, n \ge 0$$

2.3. Equations Governing the System

The Kolmogorov forward equations which governs the model:

The server is providing high priority service :

$$\begin{aligned}\frac{\partial}{\partial t}P^{(1)}_{1,m,n}(x,t)+\frac{\partial}{\partial x}P^{(1)}_{1,m,n}(x,t) &= -(\lambda_1+\lambda_2+\xi+\mu_1(x))P^{(1)}_{1,m,n}(x,t)\\ &+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}P^{(1)}_{1,m-i,n}(x,t)+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n}c_{2,i}P^{(1)}_{1,m,n-i}(x,t)\\ &+\lambda_2(1-b)P^{(1)}_{1,m,n}(x,t)+\xi P^{(1)}_{1,m+1,n}(x,t);\ m\geq 0,\ n\geq 0,\end{aligned} \tag{1}$$

The server is providing feedback service :

$$\begin{aligned}\frac{\partial}{\partial t}P^{(1)}_{2,m,n}(x,t)+\frac{\partial}{\partial x}P^{(1)}_{2,m,n}(x,t) &- (\lambda_1+\lambda_2+\xi+\mu_2(x))P^{(1)}_{2,m,n}(x,t)\\ &+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}P^{(1)}_{2,m-i,n}(x,t)+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n}c_{2,i}P^{(1)}_{2,m,n-i}(x,t)\\ &+\lambda_2(1-b)P^{(1)}_{2,m,n}(x,t)+\xi P^{(1)}_{2,m+1,n}(x,t);\ m\geq 0,\ n\geq 0,\end{aligned} \tag{2}$$

The server is providing low priority service :

$$\begin{aligned}\frac{\partial}{\partial t}P^{(2)}_{0,n}(x,t)+\frac{\partial}{\partial x}P^{(2)}_{0,n}(x,t) &= -(\lambda_1+\lambda_2+\mu_3(x))P^{(2)}_{0,n}(x,t)\\ &+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n}c_{2,i}P^{(2)}_{0,n-i}(x,t)+\lambda_2(1-b)P^{(2)}_{0,n}(x,t);\ n\geq 0,\end{aligned} \tag{3}$$

The server is on vacation :

$$\begin{aligned}\frac{\partial}{\partial t}V_{0,n}(x,t)+\frac{\partial}{\partial x}V_{0,n}(x,t) &= -(\lambda_1+\lambda_2+\beta(x))V_{0,n}(x,t)\\ &+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n}c_{2,i}V_{0,n-i}(x,t)+\lambda_2(1-b)V_{0,n}(x,t);\ n\geq 0,\end{aligned} \tag{4}$$

The server is in repair process:

$$\begin{aligned}\frac{\partial}{\partial t}R_{m,n}(x,t)+\frac{\partial}{\partial x}R_{m,n}(x,t) &= -(\lambda_1+\lambda_2+\xi+\gamma(x))R_{m,n}(x,t)\\ &+(1-\delta_{m0})\lambda_1\sum_{i=1}^{m}c_{1,i}R_{m-i,n}(x,t)+(1-\delta_{0n})\lambda_2 b\sum_{i=1}^{n}c_{2,i}R_{m,n-i}(x,t)\\ &+\lambda_2(1-b)R_{m,n}(x,t)+\xi R_{m+1,n}(x,t);\ m\geq 0,\ n\geq 0,\end{aligned} \tag{5}$$

The server is in retrial state:

$$\frac{\partial}{\partial t}I_{0,n}(x,t)+\frac{\partial}{\partial x}I_{0,n}(x,t) = -(\lambda_1+\lambda_2+\eta(x))I_{0,n}(x,t);\ n\geq 1, \tag{6}$$

The server is in idle state:

$$\begin{aligned}\frac{d}{dt}I_{0,0}(t) &= -(\lambda_1+\lambda_2)I_{0,0}(t)+(1-\theta)\{\int_0^\infty qP^{(1)}_{1,0,0}(x,t)\mu_1(x)dx+\int_0^\infty P^{(1)}_{2,0,0}(x,t)\mu_2(x)dx\\ &+\int_0^\infty P^{(2)}_{0,0}(x,t)\mu_3(x)dx\}+\int_0^\infty V_{0,0}(x,t)\beta(x)dx+\int_0^\infty R_{0,0}(x,t)\gamma(x)dx.\end{aligned} \tag{7}$$

The above set of equations are to be solved under the following boundary conditions at $x=0$.

$$\begin{aligned}I_{0,n}(0,t) &= (1-r)\{\int_0^\infty V_{0,n}(x,t)\beta(x)dx+\int_0^\infty R_{0,n}(x,t)\gamma(x)dx+(1-\theta)\{\int_0^\infty qP^{(1)}_{1,0,n}(x,t)\mu_1(x)dx\\ &+\int_0^\infty P^{(1)}_{2,0,n}(x,t)\mu_2(x)dx+\int_0^\infty P^{(2)}_{0,n}(x,t)\mu_3(x)dx\}\};\ n\geq 1.\end{aligned} \tag{8}$$

$$P^{(1)}_{1,m,n}(0,t) = q\int_0^\infty P^{(1)}_{1,m+1,n}(x,t)\mu_1(x)dx + \lambda_1 c_{1,m+1}I_{0,n}(t) + (1-\delta_{0n})\lambda_1 c_{1,m+1}\int_0^\infty P^{(2)}_{0,n-1}(x,t)dx$$
$$+\int_0^\infty R_{m+1,n}(x,t)\gamma(x)dx + (1-\delta_{0n})\lambda_1 c_{1,m+1}\int_0^\infty V_{0,n}(x,t)dx;\ m\geq 0,\ n\geq 1, \quad (9)$$
$$P^{(1)}_{2,m,n}(0,t) = p\int_0^\infty P^{(1)}_{m,n}(x,t)\mu_1(x)dx;\ m\geq 0, n\geq 0, \quad (10)$$
$$P^{(2)}_{0,n}(0,t) = \alpha\int_0^\infty I_{0,n+1}(x,t)\eta(x)dx + \alpha\lambda_2 b c_{2,n+1}I_{0,0}(t) + \alpha\lambda_2 b\sum_{i=1}^{n} c_{2,i}\int_0^\infty I_{0,n+1-i}(x,t)dx$$
$$+r\{\int_0^\infty R_{0,n+1}(x,t)\gamma(x)dx + (1-\theta)\{\int_0^\infty qP^{(1)}_{1,0,n+1}(x,t)\mu_1(x)dx$$
$$+\int_0^\infty P^{(1)}_{2,0,n+1}(x,t)\mu_2(x)dx + \int_0^\infty P^{(2)}_{0,n+1}(x,t)\mu_3(x)dx\} + \int_0^\infty V_{0,n+1}(x,t)\beta(x)dx\};\ n\geq 0, \quad (11)$$
$$V_{0,n}(0,t) = \theta\{q\int_0^\infty P^{(1)}_{1,0,n}(x,t)\mu_1(x)dx + \int_0^\infty P^{(1)}_{2,0,n}(x,t)\mu_2(x)dx + \int_0^\infty P^{(2)}_{0,n}(x,t)\mu_3(x)dx\};\ n\geq 0, \quad (12)$$
$$R_{m,n}(0,t) = \overline{\alpha}\lambda_1 c_{1,m}I_{0,n}(t),\ m\geq 1\ n\geq 0, \quad (13)$$
$$R_{0,n}(0,t) = \overline{\alpha}\lambda_2 b c_{2,n}I_{0,0}(t);\ n\geq 1. \quad (14)$$

We assume that initially there are no customers in the system and the server is idle. Then the initial conditions are,

$$P^{(1)}_{1,m,n}(0) = P^{(1)}_{2,m,n}(0) = P^{(2)}_{0,n}(0) = V_{0,n}(0) = R_{m,n}(0) = I_{0,n}(0) = 0 \text{ and } I_{0,0}(0) = 1; m\geq 0, n\geq 0. \quad (15)$$

The Probability Generating Function(PGF) of this model:

$$A(x,z_2,t) = \sum_{n=1}^{\infty} z_2^n A_{0,n}(x,t),\ B(x,z_1,z_2,t) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} z_1^m z_2^n B_{m,n}(x,t)$$

where $A = P_0^{(2)}, I, V$ and $B = P_i^{(1)}, R,\ i = 1,2$. By taking Laplace transforms from equation (1) to equation (14) and solving those equations,

$$\overline{I}_0(x,s,z_2) = \overline{I}_0(0,s,z_2)[1-\overline{I}(\varphi(a,s))]e^{-\varphi(a,s)x}, \quad (16)$$
$$\overline{P}_i^{(1)}(x,s,z_1,z_2) = \overline{P}_i^{(1)}(0,s,z_1,z_2)[1-\overline{B}_i(\varphi_1(s,z_1,z_2))]e^{-\varphi_1(s,z_1,z_2)x},\ i=1,2 \quad (17)$$
$$\overline{P}_0^{(2)}(x,s,z_2) = \overline{P}_0^{(2)}(0,s,z_2)[1-\overline{B}_3(\varphi_2(s,z_2))]e^{-\varphi_2(s,z_2)x}, \quad (18)$$
$$\overline{V}(x,s,z_2) = \overline{V}(0,s,z_2)[1-\overline{V}(\varphi_2(s,z_2))]e^{-\varphi_2(s,z_2)x}, \quad (19)$$
$$\overline{R}(x,s,z_1,z_2) = \overline{R}(0,s,z_1,z_2)[1-\overline{R}(\varphi_1(s,z_1,z_2))]e^{-\varphi_1(s,z_1,z_2)x}. \quad (20)$$

where, $\varphi(a,s) = s+\lambda_1+\lambda_2$, $\varphi_1(s,z_1,z_2) = s+\lambda_1[1-C(z_1)]+\lambda_2 b[1-C(z_2)]+\xi[1-\frac{1}{z_1}]$, and $\varphi_2(s,z_2) = s+\lambda_1+\lambda_2 b[1-C(z_2)]$. By solving the above equations, we get,

$$\overline{P}^{(1)}(0,s,z_1,z_2) = \frac{\left\{\begin{array}{l}\overline{I}_0(x,s,z_2)\{\lambda_1 C_1(z_1)[\alpha+\overline{\alpha}\overline{R}(\varphi_1(s,z_1,z_2))]\overline{\zeta}_1(s,g(z_2)) - \lambda_1 C_1(g(z_2))\\ [\alpha+\overline{\alpha}\overline{R}(\varphi_1(s,g(z_2)))]\overline{\zeta}_1(s,z_1,z_2)\} + \overline{P}_0^{(2)}(0,s,z_2)\{\overline{\zeta}_2(s,z_1,z_2)\overline{\zeta}_1(s,g(z_2))\\ -\overline{\zeta}_1(s,z_1,z_2)\overline{\zeta}_2(s,g(z_2))\} + \overline{\alpha}\lambda_2 b C_2(z_2)I_{0,0}\{\overline{\zeta}_1(s,g(z_2))[\overline{R}(\varphi_1(s,z_1,z_2))\\ -\overline{R}(\varphi_1(s,z_2))] - \overline{\zeta}_1(s,z_1,z_2)[\overline{R}(\varphi_1(s,g(z_2))) - \overline{R}(\varphi_1(s,z_2))]\}\end{array}\right\}}{\left\{\ \overline{\zeta}_1(s,g(z_2))\{z_1 - (q+p\overline{B_2}(\varphi_1(s,z_1,z_2)))\overline{B_1}(\varphi_1(s,z_1,z_2))\}\ \right\}}, \quad (21)$$

$$\overline{P}_0^{(2)}(0,s,z_2) = \frac{\left\{\begin{array}{l}(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\{\overline{\zeta}_1(s,g(z_2))\overline{\zeta}_6(s,z_2) - r\overline{\zeta}_4(s,g(z_2))\}\\ +\lambda_2 b C_2(z_2)\overline{I}_{0,0}(s)(1-r)\{\alpha\overline{\zeta}_4(s,g(z_2)) + \overline{\alpha}\overline{\zeta}_5(s,g(z_2))\overline{G}_6(s,z_2)\overline{\zeta}_5(s,z_2)\}\end{array}\right\}}{\left\{\ (1-r)\{z_2\overline{\zeta}_4(s,g(z_2)) - \overline{\zeta}_3(s,z_2)\overline{\zeta}_6(s,z_2)\}\ \right\}} \quad (22)$$

$$\overline{I}_0(0,s,z_2)=\frac{\left\{\begin{array}{l}(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\{z_2\overline{\zeta}_1(s,g(z_2))-r\overline{\zeta}_3(s,z_2)\overline{\zeta}_4(s,g(z_2))\}\\ +\lambda_2 bC_2(z_2)\overline{I}_{0,0}(s)(1-r)\{\alpha\overline{\zeta}_3(s,z_2)\overline{\zeta}_4(s,g(z_2))+\overline{\alpha}z_2\overline{\zeta}_5(s,z_2)\}\end{array}\right\}}{\left\{\ \{z_2\overline{\zeta}_4(s,g(z_2))-\overline{\zeta}_3(s,z_2)\overline{\zeta}_6(s,z_2)\}\ \right\}} \tag{23}$$

where,

$$\begin{aligned}
\overline{\zeta}_1(s,z_1,z_2)&=1-\theta\lambda_1C_1(z_1)\left[\frac{1-\overline{V}(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}\right]\\
\overline{\zeta}_2(s,z_1,z_2)&=\lambda_1C_1(z_1)z_2\left[\frac{1-\overline{B}_3(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}\right]+\theta\lambda_1C_1(z_1)\overline{B}_3(\varphi_2(s,z_2))\left[\frac{1-\overline{V}(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}\right]\\
\overline{\zeta}_3(s,z_2)&=\left[1-\theta+\theta\overline{V}(\varphi_2(s,z_2))\right]\{\overline{\zeta}_2(s,g(z_2))\overline{\zeta}_1(s,g(z_2))\overline{B}_3(\varphi_2(s,z_2))\}\\
\overline{\zeta}_4(s,g(z_2))&=(1-r)\left[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\right]\lambda_1C_1(g(z_2))\left[1-\theta+\theta\overline{V}(\varphi_2(s,z_2))\right]\left[\alpha+\overline{\alpha}\overline{R}(\varphi_1(s,g(z_2)))\right]\\
\overline{\zeta}_5(s,g(z_2))&=\left[1-\theta+\theta\overline{V}(\varphi_2(s,z_2))\right]\left[\overline{R}(\varphi_1(s,g(z_2)))-\overline{R}(\varphi_1(s,z_2))\right]+\overline{R}(\varphi_1(s,z_2))\overline{\zeta}_1(s,g(z_2))\\
\overline{\zeta}_6(s,z)&=(1-r)\{\overline{I}(\varphi(a,s))+\left[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\right][\lambda_2bC_2(z_2)(1-r)+r]\}.
\end{aligned}$$

Theorem 2.1. *The inequality $P_1^{(1)}(1,1)+P_2^{(1)}(1,1)+P^{(2)}(1)=\rho<1$ is a necessary and sufficient condition for the system to be stable, under this condition the marginal PGF of the server's state and orbit size distributions are given by,*

$$\overline{I}(s,z_2)=\overline{I}(0,s,z_2)\left[\frac{1-\overline{I}(\varphi(a,s))}{\varphi(a,s)}\right], \tag{24}$$

$$\overline{P}_i^{(1)}(s,z_1,z_2)=\overline{P}_i^{(1)}(0,s,z_1,z_2)\left[\frac{1-\overline{B}_i(\varphi_1(s,z_1,z_2))}{\varphi_1(s,z_1,z_2)}\right]\ \ i=1,2, \tag{25}$$

$$\overline{P}_0^{(2)}(s,z_2)=\overline{P}_0^{(2)}(0,s,z_2)\left[\frac{1-\overline{B}_3(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}\right], \tag{26}$$

$$\overline{V}(s,z_2)=\overline{V}_0(0,s,z_2)\left[\frac{1-\overline{V}(\varphi_2(s,z_2))}{\varphi_2(s,z_2)}\right], \tag{27}$$

$$\overline{R}(s,z_1,z_2)=\overline{R}(0,s,z_1,z_2)\left[\frac{1-\overline{R}(\varphi_1(s,z_1,z_2))}{\varphi_1(s,z_1,z_2)}\right]. \tag{28}$$

3. Steady State Analysis: Limiting Behaviour

By applying the well-known Tauberian property,

$$\lim_{s\to 0}s\overline{f}(s)=\lim_{t\to\infty}f(t),$$

to the above equations, we obtain the steady- state solutions of this model. In order to determine I_0, we use the normalizing condition $P_i^{(1)}(1,1)+P_0^{(2)}(1)+V(1)+R(1,1)+I(1)+I_0=1;\ i=1,2$. For this, let $P_q(z)$ be the probability generating function of the queue size irrespective of the state of the system. Then adding all the steady state equations, we obtain,

$$P_q(z)=P_i^{(1)}(z_1,z_2)+P_0^{(2)}(z_2)+V(z_2)+R(z_1,z_2)+I(z_2),\ i=1,2. \tag{29}$$

$$P_q(z)=\sum_{i=1}^{4}\frac{\omega_i(z_1,z_2)}{\psi_i(z_1,z_2)}$$

where,

$$\omega_1(z_1,z_2)=P_1^{(1)}(0,z_1,z_2)\{1-\overline{B}_1[q+p\overline{B}_2(\varphi_1(z_1,z_2))]\}$$

$$\omega_2(z_1,z_2) = P_0^{(2)}(0,z_2)\{\zeta_1(g(z_2))\{1-\overline{B_3}(\varphi_2(z_2))[1-\theta+\theta\overline{V}(\varphi_2(z_2))]\}+\theta(1-\overline{V}(\varphi_2(z_2)))\zeta_2(g(z_2))\},$$

$$\omega_3(z_1,z_2) = I_0(0,z_2)\left[\frac{1-\overline{I}(\varphi(a))}{\varphi(a)}\right]\{\varphi_1(z_1,z_2)+\overline{\alpha}\lambda_1 C_1(z_1)(1-\overline{R}(\varphi_1(z_1,z_2)))\varphi_2(z_2)\zeta_1(g(z_2))$$
$$+\lambda_1 C_1(g(z_2))\theta(1-\overline{V}(\varphi_2(z_2)))[\alpha+\overline{\alpha}\overline{R}(\varphi_1(z_1,z_2))]\}$$

$$\omega_4(z_1,z_2) = \overline{\alpha}\lambda_2 b C_2(z_2) I_{0,0}\{\varphi_2(z_2)(1-\overline{R}(\varphi_1(z_1,z_2)))\zeta_1(g(z_2))+\varphi_1(z_1,z_2))\theta(1-\overline{V}(\varphi_2(z_2)))[\overline{R}(\varphi_1(g(z_2)))-\overline{R}(\varphi_1(z_2))]\}$$

$$\psi_1(z_1,z_2) = \varphi_1(z_1,z_2),$$

$$\psi_2(z_1,z_2) = \varphi_2(z_2)\zeta_1(g(z_2)),$$

$$\psi_i(z_1,z_2) = \varphi_1(z_1,z_2)\varphi_2(z_2)\zeta_1(g(z_2)),\ i=3,4.$$

In order to obtain the probability of idle time I_0, we use the normalizing condition, $P_q^{(1)}(1)+I_0=1$. From which we can have,

$$I_0 = \frac{\varphi_1'(1,1)\varphi_1(1)\zeta_1(1)}{Dr} \tag{30}$$
$$Dr = \varphi_1'(1,1)\varphi_1(1)\zeta_1(1)+\omega_1'(1,1)\varphi_2(1)\zeta_1(1)+\omega_2(1,1)\varphi_1'(1,1)+\omega_1'(1)+\omega_3'(1,1)+\omega_4'(1,1).$$

4. The Average Queue Length

The Mean number of customers in the queue and in the orbit under the steady state condition is,

$$L_{q_1} = \frac{d}{dz_1}P_{q_1}(z_1,1)|_{z_1=1},\quad L_{q_2} = \frac{d}{dz_2}P_{q_2}(1,z_2)|_{z_2=1}. \tag{31}$$

then,

$$L_{q_1} = \frac{\psi_1'(z_1,1)\omega_1''(z_1,1)-\psi_1''(z_1,1)\omega_1'(z_1,1)}{2(\psi_1'(z_1,1))^2}+\frac{\psi_3'(1,1)\omega_3''(z_1,1)-\psi_3''(z_1,1)\omega_3'(z_1,1)}{2(\psi_3'(z_1,1))^2}$$
$$+\frac{\psi_4'(z_1,1)\omega_4''(z_1,1)-\psi_4''(z_1,1)\omega_4'(z_1,1)}{2(\psi_4'(z_1,1))^2},$$
$$L_{q_2} = \frac{\psi_1'(1,z_2)\omega_1''(1,z_2)-\psi_1''(1,z_2)\omega_1'(1,z_2)}{2(\psi_1'(1,z_2))^2}+\frac{\psi_2(1,z_2)\omega_2'(1,z_2)-\psi_2'(1,z_2)\omega_2(1,z_2)}{(\psi_2(1,z_2))^2}$$
$$+\frac{\psi_3'(1,z_2)\omega_3''(1,z_2)-\psi_3''(1,z_2)\omega_3'(1,z_2)}{2(\psi_3'(1,z_2))^2}+\frac{\psi_4'(1,z_2)\omega_4''(1,z_2)-\psi_4''(1,z_2)\omega_4'(1,z_2)}{2(\psi_4'(1,z_2))^2},$$

4.1. The Average Waiting Time in the Queue and Orbit

Average waiting time of a customer in the high priority queue is

$$W_{q_1} = \frac{L_{q_1}}{\lambda_1}, \tag{32}$$

Average waiting time of a customer in the low priority orbit is

$$W_{q_2} = \frac{L_{q_2}}{\lambda_2}, \tag{33}$$

where L_{q_1} and L_{q_2} have been found in the above equations.

4.2. Particular Cases

Case 1: $M^X/G/1$ Queueing model:

If there are no high priority customer, no starting failure, no vacation, no reneging, no balking, no immediate feedback. The model under study becomes classical $M^X/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{(1 - \overline{B}(\lambda - \lambda C(z)))I_0}{\overline{B}(\lambda - \lambda C(z)) - z} \tag{34}$$

Case 2: $M/G/1$ Queueing model:

If there are no high priority customer, no starting failure, no vacation, no reneging, no balking, no retrial, no immediate feedback and single arrival. The model under study becomes classical $M/G/1$ queueing system. In this case, the PGF of the busy state is given as,

$$P(z) = \frac{(1 - \overline{B}(\lambda - \lambda z))I_0}{\overline{B}(\lambda - \lambda z) - z} \tag{35}$$

The above two results are coincide with the results of Gross.D and Harris.M (1985).

5. Numerical Results

In this section, we present some numerical examples to study the effect of various parameters. For the purpose of a numerical illustration, we assume that all distribution function like retrial, service of customers, vacation are exponentially distributed. All the parameters values are selected which satisfies its stability condition.

λ_1	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
2.3	0.7647	0.2353	0.2628	1.1716	0.1143	1.9526
2.4	0.7602	0.2398	0.3417	1.1944	0.1424	1.9907
2.5	0.7567	0.2433	0.4221	1.3576	0.1689	2.2626
2.6	0.7540	0.2460	0.5041	1.6643	0.1939	2.7739
2.7	0.7520	0.2480	0.5875	2.1232	0.2176	3.5387
2.8	0.7506	0.2494	0.6724	2.7476	0.2401	4.5793
2.9	0.7496	0.2504	0.7587	3.5555	0.2616	5.9258
3.0	0.7491	0.2509	0.8465	4.5701	0.2822	7.6168

Table 1. Take $(\lambda_2, \mu_1, \mu_2, \mu_3, \eta, \theta, \alpha, \gamma, \beta, \xi, p, b, r) = (0.6, 7, 2, 3, 0.1, 0.25, 0.3, 2, 1, 0.2, 0.6, 0.7, 0.5)$. Effect of λ_1 on various queue characteristics

μ_1	$I_{0,0}$	ρ	L_{q_1}	L_{q_2}	W_{q_1}	W_{q_2}
11.0	0.7668	0.2332	1.6233	1.6744	0.4058	2.0930
11.1	0.7669	0.2331	1.6224	1.4636	0.4056	1.8295
11.2	0.7670	0.2330	1.6215	1.2630	0.4054	1.5788
11.3	0.7671	0.2329	1.6206	1.0721	0.4052	1.3401
11.4	0.7672	0.2328	1.6197	0.8902	0.4049	1.1127
11.5	0.7673	0.2327	1.6189	0.7167	0.4047	0.8959
11.6	0.7674	0.2326	1.6180	0.5512	0.4045	0.6890
11.7	0.7675	0.2325	1.6172	0.3932	0.4043	0.4915
11.8	0.7676	0.2324	1.6164	0.2422	0.4041	0.3028
11.9	0.7677	0.2323	1.6156	0.0979	0.4039	0.1223

Table 2. Take $(\lambda_1, \lambda_2, \mu_2, \mu_3, \eta, \theta, \alpha, \gamma, \beta, \xi, p, b, r) = (4, 0.8, 2, 4, 0.1, 0.25, 0.3, 2, 1, 0.2, 0.6, 0.7, 0.5)$. Effect of μ_1 on various queue characteristics

Table 1 clearly shows that as long as the arrival rate of high priority customers increases the server's idle time decreases. Simultaneously the utilisation factor, average queue length for both high priority and low priority customers are increases.

Table 2 shows that as long as the service rate increases the server's idle time increases and the utilisation factor, average queue length for both high priority and low priority customers are decreases.

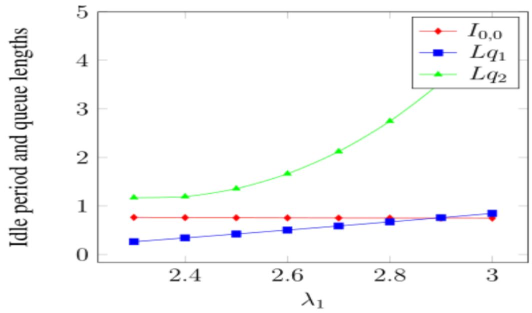

Figure 2. **Average queue sizes Vs High priority arrival rate λ_1.**

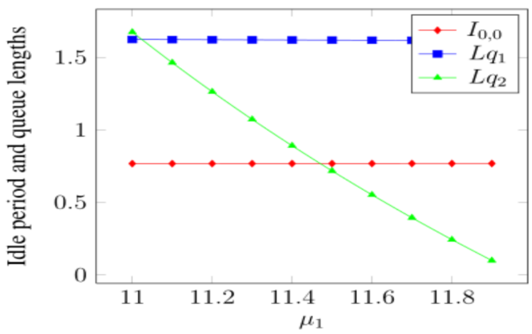

Figure 3. **Average queue sizes Vs service rate μ_1.**

6. Conclusion

In this paper we have analysed a $M^{[X_1]}, M^{[X_2]}/G_1, G_2/1$ retrial queue with feedback and the server subject to starting failures and priority service under modified Bernoulli vacation . In addition, the effect of impatient behaviour of the customer on a service system is studied. The joint distribution of the number of customers in the queue and the number of customers in the orbit are derived. Numerical examples have been carried out to observe the trend of the mean number of customers in the system for varying parametric values. This paper analyzes a single-server retrial queue with constant retrial policy, preemptive repeat priority, orbital search, vacation interruption and repair in order to obtain analytical expressions for various performance measures of interest. The joint steady-state probability generating functions of the server state and the number of customers in the orbit are derived. Numerical examples have been carried out to observe the effects of several parameters on the system.

References

[1] D.R.Cox, *The analysis of non-Markovian stochastic processes by the inclusion of supplementary variables*, Proc. Cambridge Phil. Soc., 51(1955), 433-441.

[2] S.Gao, *A preemptive priority retrial queue with two classes of customers and general retrial times*, Oper Res Int J., 15(2015), 233-251.

[3] D.Gross and C.M.Harris, *Fundamentals of Queueing Theory*, 2nd Edition, Wiley, New York, (1985).

[4] M.Jain, Charu and Bhargava, *Bulk Arrival Retrial Queue with Unreliable Server and Priority Subscribers*, International Journal of Operations Research, 5(2008), 242-259.

[5] K.Kalidass and R.Kasturi, *A two phase service M/G/1 queue with a finite number of immediate Bernoulli feedbacks*, Opsearch, 51(2)(2014), 201-218.

[6] J.C.Ke and F.M.Chang, $M^x/G_1, G_2/1$ *Retrial Queue with Bernoulli Vacation Schedules with Repeated Attempts and Starting Failures*, Applied Mathematical Modeling, 33(2009), 3186-3196.

[7] B.Krishnakumar, S.Pavai Madheswari and A.Vijayakumar, *The M/G/1 retrial queue with feedback and starting failures*, Applied Mathematical Modelling, 26(2002), 1057-1075.

[8] Pakkirisami Rajadurai, Saravanan Yuvarani and Mookkaiya Chandran Saravanarajan, *Performance analysis of preemptive priority retrial queue with immediate Bernoulli feedback under working vacations and vacation interruption*, SJST, 38(2016), 507-520.

[9] D.Sumitha and K.Udaya Chandrika, *Retrial Queuing System with Starting Failure, Single Vacation and Orbital Search*, International Journal of Computer Applications, 40(13)(2012), 29-33.

[10] T.Yang and H.Li, *The M/G/1 Retrial Queue with the Server subjected to Starting Failure*, Queueing System, 16(1994), 83-96.