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Global Cototal Domination in Graphs

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Abstract: A set of vertices D in a graph G is a dominating set, if each vertex of G is dominated by some vertices of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D of a graph G is a global dominating set if D is also a dominating set of $\bar{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. A dominating set D of a graph G is a cototal dominating set if the induced sub graph, $\langle V-D\rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ is the minimum cardinality of a cototal dominating set of G. In this paper we introduce a new concept, the global cototal domination number $\gamma_{gcot}(G)$. A dominating set D of a graph G is a global cototal dominating set if D is both a global dominating set and a cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal domination set of G. We initiate the study of global cototal domination number and present bounds and some exact values of $\gamma_{gcot}(G)$ for some classes of graphs.

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Keywords: Domination number, Global domination number, Total domination number, Cototal domination number, Global cototal domination number.

1. Introduction

All graphs considered in this paper are simple, finite, undirected and connected. For all graph theoretic terminology not defined here, the reader is referred to [7]. Domination is a flourishing branch in graph theory and it has numerous applications to distributed computing, the web graph and ad hoc networks. For a comprehensive introduction to theoretical and applied facts of domination in graphs the reader is directed to the book [8]. Many variants of the domination number have been studied. For instance, a dominating set D of a graph G is a global dominating set if D is also a dominating set of $\bar{G}$. The global domination number $\gamma_g(G)$ is the minimum cardinality of a global dominating set of G. This concept was introduced independently by Brigham and Dutton [2] (the term factor domination number was used) and Sampathkumar [7]. A dominating set D of a graph G is a total dominating set if the induced sub graph $\langle D\rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ is the minimum cardinality of a total dominating set of G. A dominating set D of a graph G is a cototal dominating set if the induced subgraph $\langle V-D\rangle$ has no isolated vertices. The cototal domination number $\gamma_{cot}(G)$ is the minimum cardinality of a cototal dominating set of G. This concept was introduced by Kulli, Janakiram and Iyer in [6]. In this paper, we introduce a new graph parameter the global cototal domination number for a connected graph G.

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2. Main Results

Definition 2.1. *A global cototal dominating set of a graph G is a set D of vertices of G such that D is both global dominating set and cototal dominating set. The global cototal domination number $\gamma_{gcot}(G)$ is the minimum cardinality of a global cototal dominating set of G.*

Theorem 2.2. *Every graph with no isolated vertex has a global cototal dominating set and hence a global cototal domination number.*

Proof. Without loss of generality let $G = (V, E)$ be connected . Then V itself is a global cototal dominating set in G as well as $\bar{G}$, as each vertex is considered to dominate itself. When we remove the vertices of the cototal dominating set D from G, then we get a induced sub graph $\langle V - D \rangle$ which has no isolated vertex. Now we know that G has a global cototal dominating set, we may remove one vertex at a time from V if and only if the resulting subset of V is still a global cototal dominating set. This will give a minimal global cototal dominating set. Among all the global cototal dominating sets, each of the smallest sets has cardinality $\gamma_{gcot}(G)$. Hence forth we consider only graphs with no isolated vertices. □

Theorem 2.3. *For any positive integer $m \geq 2n \geq 2$, $\gamma_{gcot}(k_{m,n}) = 2$.*

Proof. Let G be a complete bipartite graph with partitions V_1 and V_2. Let $u \in V_1$ and $v \in V_2$. Since G is a complete bipartite graph, each vertex in one partition can dominate all vertices in the other partition. Let $D = \{u, v\}$ be the dominating set. Now V-D will be a connected graph with no isolated vertex. Also in $\bar{G}$, the dominating S will dominate all vertices in its own partition. Hence $D = \{u, v\}$ is both global and cototal dominating set in G. Hence $\gamma_{gcot}(K_{m,n}) = 2$. □

Definition 2.4. *A leaf of a tree T is a vertex of degree one, while a support vertex of T is a vertex adjacent to a leaf.*

Theorem 2.5. *If $T \neq K_{1,n}$ is a tree of order $n > 3$, then $\gamma_{gcot}(T) \geq |L|$, where L is the set of leaf vertices of T.*

Proof. Let $T \neq K_{1,n}$ be a tree of order $n > 3$. Let D be any global cototal dominating set with $|D| = \gamma_{gcot}(T)$. Let L be the set of leaf vertices of T. If T contains only leaf and support vertices then $D = L$. Hence $\gamma_{gcot}(T) = |L|$. If T contains vertices other than leaf and support vertices then D must contain all the leaf vertices and at least one vertex other than the leaf vertices. Therefore $|D| > |L|$. Hence $\gamma_{gcot}(T) > |L|$. We calculate the global cototal dominating number in specific families of graphs. □

Proposition 2.6. *For a path P_n on n vertices,*

$$\gamma_{gcot}(P_n) = \begin{cases} \frac{n}{3} + 2, n \equiv 0 \ (mod\ 3); \\ \frac{n+2}{3}, n \equiv 1 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil + 1, n \equiv 2 (mod\ 3). \end{cases}$$

Proof. Let P_n be the path of order n. $V(P_n) = \{v_0, v_1, v_2, v_3, ..., v_{n-1}\}$. Since D is a global cototal dominating set in G, $i + 3 \leq n$ is the least positive integer such that $v_i, v_{i+3} \in D$. D must contain v_0 and v_{n-1}, the end vertex of P_n. If $n \equiv 0 \ (mod\ 3)$ then D contains v_{3i} where $i = 0, 1, ..., \frac{n-3}{3}$ and the vertices v_{n-2} and v_{n-1}. Hence $|D| = \frac{n-3}{3} + 1 + 2 = \frac{n}{3} + 2$. If $n \equiv 1 \ (mod\ 3)$ then D contains v_{3i} where $i = 0, 1, ..., \frac{n-1}{3}$. Thus $|D| = \frac{n-1}{3} + 1 = \frac{n+2}{3}$. If $n \equiv 2 \ (mod\ 3)$ then D has v_{3i} where $i = 0, 1, ..., \frac{n-2}{3}$ and also v_{n-1}. In this case $|D| = \frac{n-2}{3} + 1 + 1 = \frac{n+1}{3} + 1 = \lceil \frac{n}{3} \rceil + 1$. Hence the result follows. □

Proposition 2.7. *For the cycle $C_n, n \geq 6$,*

$$\gamma_{gcot}(C_n) = \begin{cases} \frac{n}{3}, & n \equiv 0 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil, & n \equiv 1 \ (mod\ 3); \\ \lceil \frac{n}{3} \rceil + 1, & n \equiv 2 \ (mod\ 3). \end{cases}$$

Proof. Let $V(C_n) = \{v_0, v_1, v_2, v_3, ..., v_{n-1}\}$ and $E(C_n) = v_i v_{i+1}/i = 0, 1, 2, .., n-1$, subscript modulo n. Consider the sets,

$$D_1 = \left\{v_{3i}/i = 0, 1, 2, ..., \frac{n-3}{3}\right\} \quad \text{when } n \equiv 0 \ (mod\ 3).$$
$$D_2 = \left\{v_{3i}/i = 0, 1, 2, ..., \frac{n-1}{3}\right\} \quad \text{when } n \equiv 1 \ (mod\ 3).$$
$$D_3 = \left\{v_{3i}/i = 0, 1, 2, ..., \frac{n-2}{3}\right\} \cup \{v_{n-1}\} \quad \text{when } n \equiv 2 \ (mod\ 3).$$

The above three sets achieve the global cototal property of C_n in the respective parity conditions. □

Proposition 2.8. *For any complete graph K_n, $\gamma_{gcot}(K_n) = n, n \geq 3$.*

Proof. All the vertices are isolated in the complementary graph of the complete graph. Therefore, the global cototal dominating set must contain all the vertices of K_n. Hence $\gamma_{gcot}(K_n) = n, n \geq 3$. □

Proposition 2.9. *For any star graph $K_{1,n}$, $\gamma_{gcot}(K_{1,n}) = n + 1$, $n \geq 3$.*

Proof. Let $V(G) = v, v_1, v_2, v_3, ..., v_n$, where v is the only vertex of degree n and each v_i is a pendant vertex adjacent to v. The complement of $K_{1,n}$ contains two components K_1 and K_n. Hence $D = V(G)$ is the global cototal dominating set of $K_{1,n}$. Thus $\gamma_{gcot}(K_{1,n}) = n + 1, n \geq 3$. □

Proposition 2.10. *For any wheel W_n, $\gamma_{gcot}(W_n) = \left\{ \begin{array}{ll} 4 & \text{if } n = 3 \\ 3 & \text{otherwise} \end{array} \right\}$*

Proof. Since W_3 is nothing but K_4, from Proposition 2.8 it is obvious that $\gamma_{gcot}(W_3) = 4$. $V(W_n) = \{v_0, u = v_1, v_2, v_3, ..., v_n = v\}$, be the vertex set of the wheel W_n. Let $D = \{v_0, u, v\}$ be the global cototal dominating set of W_n. The complement of W_n contains an isolated vertex. Hence v_0 certainly belongs to the set D. Also the vertices u, v dominate all the other vertices. Hence D is the minimal global cototal dominating set of W_n. Hence $\gamma_{gcot}(W_n) = 3$ if $n \geq 4$. □

Theorem 2.11. *Let $G = T$ be any nontrivial tree. Then $\gamma_{gcot}(G) \geq \Delta(G) + 1$.*

Proof. Let G be any nontrivial tree. If $T = K_{1,n}$, then equality holds. Let D be a global cototal dominating set of T. Clearly D contains L, where L is the set of leaf vertices. Since $\Delta(G) < |L|$ for any tree $T \neq P_n$ or $T \neq K_{1,n}$, the result follows from Theorem 2.5. Next theorem relates the global cototal domination number to the maximum degree and the order of the graph. □

Theorem 2.12. *For any connected graph $G \neq K_n$ with $\delta(G) > 2$, $\gamma_{gcot}(G) \geq \frac{n}{\Delta(G)-1}$.*

Proof. Let t denote the number of edges in G having exactly one vertex in D. Since $\Delta(G) \geq deg(v) \geq \delta(G)$ for all $v \in D$ and each vertex in D is adjacent to at least one member of D, we have $t \leq (\Delta(G) - 2)|D|$, which equals $(\Delta(G) - 2)\gamma_{gcot}(G)$. That is,

$$t \leq (\Delta(G) - 2)\gamma_{gcot}(G) \quad (1)$$

Also, each vertex in $V-D$ is adjacent to at least one vertex of $V-D$. So we have $t \geq |V-D|$ which equals $n-\gamma_{gcot}(G)$. That is,

$$n-\gamma_{gcot}(G) \leq t \tag{2}$$

From (1) and (2), it follows that $n-\gamma_{gcot}(G) \leq t \leq (\Delta(G)-2)\gamma_{gcot}(G)$, which implies $n \leq (\Delta(G)-1)\gamma_{gcot}(G)$. Hence, $\frac{n}{(\Delta(G)-1)} \leq \gamma_{gcot}(G)$. Thus the theorem follows. □

Next theorem gives the upper bound for the global cototal domination number in terms of order and size.

Theorem 2.13. *Let G be a graph of order n and size m, $\delta(G) > 2$. Then $\gamma_{gcot}(G) \leq m - \frac{n}{2}$.*

Proof. Let D be any γ_{gcot}-set of G. Consider $A = \langle V-D\rangle$, $B = \langle D\rangle$. Let n_1 and n_2 be the order of A and B respectively. Let m_1 and m_2 be the sizes of A and B respectively. Thus $m_1 = \frac{1}{2}\sum_{v\in V-S} deg_A(v) \geq \frac{1}{2}(n-\gamma_{gcot}(G))$ and $m_2 = \frac{1}{2}\sum_{v\in S} deg_B(v) \geq \frac{1}{2}\gamma_{gcot}(G)$. Let m_3 denote the number of edges between D and $V-D$. Since D is a γ_{gcot}-set, it is a cototal dominating set and every vertex is adjacent to at least one vertex in D. Thus $m_3 \geq \gamma_{gcot}(G)$. $m = m_1 + m_2 + m_3 \geq \frac{1}{2}(n-\gamma_{gcot}(G)) + \frac{1}{2}\gamma_{gcot}(G) + \gamma_{gcot}(G) = \frac{n}{2} + \gamma_{gcot}(G)$. Therefore, $m \geq \frac{n}{2} + \gamma_{gcot}(G)$. Hence, $\gamma_{gcot}(G) \leq \frac{m-n}{2}$. □

Theorem 2.14. *Let $G \neq C_n$ be any connected graph. Then $\gamma_{gcot}(G) \leq 2m-n+2$. Further equality holds if and only if $G = K_{1,n}$.*

Proof. Obviously $\gamma_{gcot}(G) \leq n = 2(n-1)-n+2$ and since G is connected and $G \neq C_n$, $m \geq n-1$. Thus $\gamma_{gcot}(G) \leq 2m-n+2$. Assume every edge of a tree is incident with a single support vertex, then $m = n-1$ and $\gamma_{gcot}(G) = n = 2m-n+2$. Conversely, assume $\gamma_{gcot}(G) = 2m-n+2$. Then $2m-n+2 \leq n$ which implies that $m \leq n-1$. Let $m = n-1$, then $\gamma_{gcot}(G) = n$. Hence by Proposition 2.6, $G = K_1, n$. □

Theorem 2.15. *Let G be any nontrivial connected graph. If both G and $\bar{G}$ have no isolated vertices then*

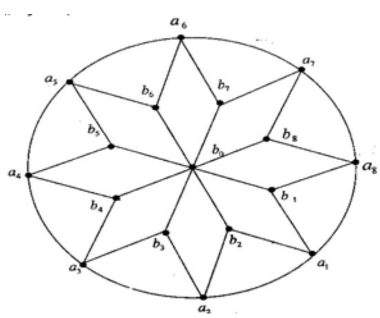
(1). $\gamma_{gcot}(G) + \gamma_{gcot}(\bar{G}) \leq 2n$

(2). $\gamma_{gcot}(G) \cdot \gamma_{gcot}(\bar{G}) \leq n^2$.

3. Lotus Inside Circle [6]

The graph Lotus Inside Circle, denoted by $LIC_n, n \geq 3$, is defined as follows: Let S_n be the star graph with vertices $b_0, b_1, ..., b_n$ whose center is b_0. Let C_n be the cycle of length n whose vertices are $a_1, a_2, ..., a_n$. We join a_i with b_i and b_{i+1} for each $i \geq 1$ and join an with b_1 and b_n.

Example 3.1.



Theorem 3.2. *Forn* ≥ 4, $\gamma_{gcot}(LIC_n) = \begin{cases} \frac{n}{3}+1 & n \equiv 0 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 1 & n \equiv 0 \ (mod\ 3) \end{cases}$

Proof. Let C_n be the cycle of length n whose vertices are $a_1, a_2, ..., a_n$ in the graph LIC_n. Let S_n be the star graph with the vertices $b_0, b_1, b_2, ..., b_n$ with apex vertex b_0 such that each a_i is adjacent to b_i and b_{i+1} taken modulo $n, i = 1, 2, ..., n$.

Case 1: $n \equiv 0$ $(mod\ 3)$. Consider the set $D = \{b_0, a_{3i-2}/i = 1, 2, ..., n/3\}$. From Proposition 2.7, we have $\gamma_{gcot}(C_n) = \frac{n}{3}$ for $n \equiv 0$ $(mod\ 3)$. Therefore $D = \{b_0, a_{3i-2}/i = 1, 2, ..., n/3\}$ is a cototal dominating set for LIC_n. Further, in $\overline{LIC_n}$, b_0 dominates all the vertices of the cycle. Also $a_1 \in D$ dominates all b_i's except b_1 and b_2 which are dominated by $a_{n-2} \in D$. Since D is minimal, we conclude $\gamma_{gcot}(LIC_n) = \frac{n}{3} + 1$.

Case 2: $n \equiv 1$ $(mod\ 3)$. Let $D = \{b_0, a_{3i-2}/i = 1, 2, ..., \frac{n+2}{3}\}$. By Proposition 2.7 we have, $\gamma_{gcot}(C_n) = \lceil \frac{n}{3} \rceil$ for $n \equiv 1$ $(mod\ 3)$. Therefore D is a cototal dominating set forLIC_n. In a similar argument as in Case 1, we can conclude that $\gamma_{gcot}(LIC_n) = \lceil \frac{n}{3} \rceil + 1$.

Case 3: $n \equiv 2$ $(mod\ 3)$. The set $D = \{b_0, a_{3i-2}/i = 1, 2, ..., \frac{n+1}{3}\}$ achieves the result that $\gamma_{gcot}(LIC_n) = \lceil \frac{n}{3} \rceil + 1$. □

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