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K-Horizontal Symmetric Measurable Functions

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Abstract: A non negative measurable function f on a measurable space has a K-Horizontal Symmetric Measurable Function f^{KS} non negative and measurable function such that $\int_E f d\mu = \int_E f^{KS} d\mu$ where E is a measurable set.

Keywords: Measurable function, Non negative measurable function, K-Horizontal Symmetric Measurable Function.

1. Introduction

Wants are unlimited and resources are limited. The life day in and day out is full of puzzles whether the available resources and the desires match the attainability or not. So, linear programming problem is one that provides a way to the best or optimal solution in the feasible region that is determined by the constraints. The desires or conditions are expressed as functions and the resources are the domains of the functions. So, the domain of each function in real time is a finite and non negative axis either as real line or complex line depending on the function whether industrial or scientifically mooted. The overall decision on the domains, functions can be handled at large in the measure theory. That is, whether the situation is measurable in terms of resources and in terms of requirements (desires). There can be numerous measurable sets and numerous measurable functions. Some are non-measurable. In measure theory, one judge which functions are measurable and which are not. With this motive, here is a function that can be checked with its measurability which depends on a measurable function *f*.

Definition 1.1. *A collect $\Im$ of subsets of a set X is called an algebra of sets or a Boolean algebra if*

(i). $A \bigcup B \in \Im$,

(ii). $A \bigcap B \in \Im$ and

(iii). $\tilde{A} \in \Im$ whenever A and B are in $\Im$.

The union intersection properties can be extended to any finite collections using induction.

Definition 1.2. *An algebra $\Im$ of sets is called sigma-algebra denoted by σ-algebra or a Borel field, if every union of a countable collection of sets in $\Im$ is again in $\Im$.*

Note that by virtue of complementation property, the arbitrary intersections will also be in $\Im$.

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Definition 1.3. *The members of* $\Im$ *are called the open sets. A set function* μ *assigned to each* $A \in \Im$ *by* $\mu A = l(A)$ *if* A *is an interval and* $l(A)$ *stands for the length of the interval and* $\mu A = \inf_{A \subseteq \bigcup_{k=1}^{n} I_k} \sum_{k=1}^{n} l(I_k)$, *where* I_k, $1 \leq k \leq n$ *is an interval in* $\Im$.

Definition 1.4. *If the non-negative set function* μ *is defined for every member of* $\Im$ *satisfying*

(i). $A \subseteq B$ *implies* $\mu A \leq \mu B$ *for every* A *and* B *in* $\Im$,

(ii). $\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu A_i$ *for an arbitrary collection* $\{A_i\}$ *in* $\Im$,

(iii). $\mu\phi = 0$,

then μ *is called a measure on* $\Im$.

Note that the inequality (ii) above becomes equality when $\{A_i\}$ is a disjoint collection in $\Im$.

Definition 1.5. $(X, \Im)$ *is the ordered pair such that* $\Im$ *consists of all the subsets of* X *forming a* σ*-Algebra, and* μ *is a measure defined for every member* A *of* $\Im$, A *is a measurable set in* $\Im$. $(X, \Im)$ *is said to be a measurable space.*

Definition 1.6. *A function* f *defined on a measurable space* $(X, \Im)$ *is said to be measurable if* $A = \{x \in X : f(x) \leq \alpha, \alpha \in \mathrm{R}\}$ *is measurable or* $\tilde{A}$ *is measurable.*

Definition 1.7. *A non negative function* f^{KS} *is said to be K-horizontal symmetric function to the non negative measurable function* f *on a set* E *in* $\Im$, *if for each* $f(x)$ *there corresponds* $2K - f(x)$ *which is the symmetric point about the non negative constant function* K *and* $x \in E$.

Definition 1.8. *A function* ϕ *on a set* $E \in \Im$ *defined by* $\phi_i(x) = c_i \cdot \chi_{E_i}$, $E_i \bigcap E_j = \phi$, $1 \leq i, j \leq n$, $\bigcup_{i=1}^{n} E_i = E, c_i \in \mathrm{R}$ *and* $\phi = \sum_{i=1}^{n} \phi_i$ *is called a Simple function.*

Definition 1.9. *A function* f *is an extended real valued function on* X *if for every* $\alpha \in \Re$, *there corresponds* $\{x \in E : f(x) < \alpha\} \subset B_\alpha \subset \{x \in E : f(x) \leq \alpha\}$ *and* f *is defined on* B_α *such that* $E, B_\alpha \in \Im$.

Definition 1.10. *The integral of a simple function* ϕ *upon a measurable set* E *with respect to the measure* μ *is* $\int_E \phi d\mu = \sum_{i=1}^{n} \mu(E \bigcap E_i)$.

Definition 1.11. *An extended real valued* f *that is* $\mu-$*measurable function on* X, *its integral is* $\int_E f d\mu = \sup_{\phi \leq f} \int_E \phi d\mu$ *where* $E \in \Im$ *and* ϕ *is a simple function on* E.

2. Main Results

Lemma 2.1. *There is a sequence of simple functions* $\{\psi_n\}$ *such that* $\int_E f d\mu = \inf_{\psi_n \geq f} \int_E \psi_n d\mu$ *for some measurable function* f.

Lemma 2.2. *If* f *is an extended real valued function on a measurable space* $(X, \Im)$, *then* f^{KS} *is measurable.*

Proof. For arbitrary $\alpha \in \mathrm{R}$, measurability of f, $\{x \in E : f(x) < \alpha\} \subset B_\alpha \subset \{x \in E : f(x) \leq \alpha\}$ such that B_α is measurable and E is an arbitrary member of $\Im$. See that $\{x \in E : cf(x) < c\alpha\} \subset B_\delta \subset \{x \in E : f(x) \leq c\alpha\}$ where $\delta = c\alpha$, $c \in R$ and B_δ is measurable. When $c = -1$, it can be seen that $\{x \in E : -f(x) \geq -\alpha\} \supset B_{-\alpha} \supset \{x \in E : -f(x) > -\alpha\}$ and $B_{-\alpha}$ is measurable and so, by the complementary property of measurability of f, it follows $-f$ is measurable. Further, $\{x \in E : 2K - f(x) < 2K - \alpha\} \subset B_{2K-\alpha} \subset \{x \in E : 2K - f(x) \leq 2K - \alpha\}$ such that $B_{2K-\alpha}$ is measurable leading to $2K - f = f^{KS}$ is measurable. □

Definition 2.3. *A property P is said to hold on almost everywhere of a set E or (a.e.), if*

$$\mu\{x \in E : P(x) \text{ does not hold}\} = 0$$

Definition 2.4. *$(X, \Im)$ is a measurable space and $\{f_n\}$ is a sequence of extended real valued measurable functions said to converge to an extended real valued measurable function f a.e. in measure if to each $\varepsilon > 0$, there corresponds a positive integer N such that $|f_n(x) - f(x)| < \varepsilon$ for all $n > N$ and $x \in E$.*

Lemma 2.5 (Fatou's Lemma)**.** *If $\{f_n\}$ is a sequence of non-negative measurable functions that converge almost everywhere (a.e.) on a set E to a function f, then $\int_E f \le \underline{\lim} \int_E f_n$.*

Lemma 2.6. $\inf \int_E -f_n = -\sup \int_E f_n$ *and* $\sup \int_E -f_n = -\inf \int_E f_n$ *for any non negative sequence of measurable functions* $\{f_n\}$.

Proposition 2.7. *If f is a non negative measurable function on $E \in \Im$ and f^{KS} is its K-horizontal symmetric function on $E \in \Im$, then f^{KS} is measurable such that $\int_E f d\mu = \int_E f^{KS} d\mu$.*

Proof. Consider a partition $\{E_i\}$, $1 \le i \le n$ and $\bigcup_{i=1}^{n} E_i = E$ and $E_i \bigcap E_j = \phi \forall 1 \le i, j \le n$ such that $\phi(x) = 2K - \psi(x)$ for every $x \in E_i$ for some $1 \le i \le n$

$$\begin{aligned}\int_{E_i} \phi d\mu &= \int_{E_i} (2K - \psi) d\mu \\ &= 2K\mu E_i - \int_{E_i} \psi d\mu\end{aligned}$$

Apply supremum on both sides,

$$\begin{aligned}\sup \int_{E_i} \phi d\mu &= \sup \left\{ 2K\mu E_i - \int_{E_i} \psi d\mu \right\} \\ &= 2K\mu E_i - \inf \int_{E_i} \psi d\mu \;\; \text{(Lemma 2.5)}\end{aligned}$$

Apply summation of n partition measurable sets,

$$\begin{aligned}\sup \int_{\bigcup_{i=1}^{n} E_i} \phi d\mu &= \sum_{i=1}^{n} 2K\mu E_i - \inf \int_{\bigcup_{i=1}^{n} E_i} \psi d\mu \\ \sup \int_E \phi d\mu &= 2K\mu E - \inf \int_E \psi d\mu \\ \int_E f d\mu &= 2K\mu E - \int_E f^{KS} d\mu \;\; \text{(Lemma 2.1)}\end{aligned}$$

Sum of the supplementary measures is twice the measure of the constant function

$$\int_E f d\mu = \int_E f^{KS} d\mu$$

Measurability of f^{KS} is established in Lemma 2.2. □

3. Application

Sum of the integrals of the non negative 2-horizontal symmetric measurable functions $f(x) = 2 - \sin x$ and $f^{KS}(x) = 2 + \sin x$ is on $[0, 2\pi]$ is 8π.

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