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An Analysis of Interpolatory polynomials on finite interval

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Abstract: The main object of this paper is to construct a interpolatory polynomial with hermite conditions at end points of interval [-1,1] based on the zeros of the polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ where $P_n^{(k)}(x)$ is the ultraspherical polynomial of degree n .In this paper, we prove existence ,explicit representation and order of convergence of the interpolatory polynomials.

MSC: 41A10, 97N50

Keywords: Pál-type interpolation; Ultraspherical polynomial; Explicit form; Order of convergence

1. Introduction

In 2001, Lenard [4] introduced a $\grave{P}$al-type interpolation polynomials with boundary conditions at end points of interval. She considered two system of real numbers $\{x_i\}_{i=1}^{n-1}$ and $\{x_i^*\}_{i=1}^{n}$ which are the zeros of $P_{n-1}^{(k+1)}(x)$ and $P_n^{(k)}(x)$ respectively, then there exists a unique polynomial $Q_m(x)$ of degree at most m=2n+2k+1 satisfying the interpolation conditions.

$$Q_m(x_i) = y_i, \quad (i = 1, 2, \ldots, n-1) \tag{1}$$

$$Q'_m(x_i^*) = y'_i, \quad (i = 1, 2, \ldots, n) \tag{2}$$

with (Hermite) boundary conditions.

$$Q_m^{(l)}(1) = \alpha_j, \quad (j = 0, 1, \ldots, k) \tag{3}$$

$$Q_m^{(l)}(-1) = \beta_j, \quad (l = 0, 1, \ldots, k+1) \tag{4}$$

where y_i, y'_i, α_j and β_j are arbitrary real numbers, k is a fixed non-negative integer. Later on many authors have considered with above method of interpolation. In Joo and Szili [2] have considered weighted (0,2) interpolation on the roots of Jacobi polynomials. Pal L.G [5] has discussed a general lacunary (0;0,1) interpolation process. In other paper [6] and [7] have discussed pal-type interpolation on the roots of Hermite polynomials. In this paper we study the following (0;0,1) interpolation problem on the interval $[-1, 1]$. Let the set of knots be given by

$$-1 = x_n^* < x_n < x_{n-1}^* < x_{n-1} < \cdots < x_1^* < x_1 < x_0^* = 1, \quad n \geq 1 \tag{5}$$

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Where $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of Ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. On the knots (5) there exist a unique polynomial $R_m(x)$ of degree at most $m = 3n + 2k$ satisfying the interpolatory conditions.

$$R_m(x_i) = y_i, \quad (i = 1, 2, \ldots, n) \tag{6}$$

$$R_m(x_i^*) = y_i^*, \quad (i = 1, 2, \ldots, n-1) \tag{7}$$

$$R_m'(x_i^*) = {y_i^*}', \quad (i = 1, 2, \ldots, n-1) \tag{8}$$

with (Hermite) boundary conditions.

$$R_m^{(l)}(1) = y_1^{(l)}, \quad (l = 0, 1, \ldots, k) \tag{9}$$

$$R_m^{(l)}(-1) = y_{-1}^{(l)}, \quad (l = 0, 1, \ldots, k+1) \tag{10}$$

where y_i ,y_i^*, ${y_i^*}'$, y_1^l and y_{-1}^l are arbitrary real numbers and k is a fixed non-negative integer. Here $P_n^{(k)}(x)$ denotes the Ultraspherical polynomial of degree n with the parameter k.The convergence of this interpolation process was studied by Xie [9] if $f \in C^r[-1,1]$ for $x \in [-1,1]$, then

$$|f(x) - R_{2n+1}(x; f)| = O\left(n^{-r+1}\right) w\left(f^{(r)}; \frac{1}{n}\right) \tag{11}$$

For $k \geq 1$, Lenard [3] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$, then

$$|f(x) - R_m(x; f)| = O\left(n^{k-r+\frac{1}{2}}\right) w\left(f^{(r)}; \frac{1}{n}\right) \tag{12}$$

For $k \geq 0$, Lenard [4] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$, then

$$|f'(x) - R_m'(x; f)| = w\left(f^{(r)}; \frac{1}{n}\right) O\left(n^{k-r+\dfrac{5}{2}}\right) \tag{13}$$

where $w(f^{(r)}, .)$ denotes the modulus of continuity of the r^{th} derivative of the function $f(x)$. If $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip\alpha$, $\alpha > \frac{1}{2}$, then $R_m(x; f)$ and $R_m'(x; f)$ uniformly converges to $f(x)$ and $f'(x)$ respectively on [-1,1].

2. Preliminaries

We shall use the some well known properties and results [8] of the Ultraspherical polynomials.

$$(1 - x^2){P_n^{(k)}}''(x) - 2x(k+1){P_n^{(k)}}'(x) + n(n + 2k + 1)P_n^{(k)}(x) = 0 \tag{14}$$

$${P_n^{(k)}}'(x) = \frac{n + 2k + 1}{2} P_{n-1}^{(k+1)}(x) \tag{15}$$

$$|P_n^{(k)}(x)| = O(n^k), \quad x \in [-1,1] \tag{16}$$

$$(1 - x^2)^{\frac{k}{2}+\frac{1}{4}} |P_n^{(k)}(x)| = O\left(\frac{1}{\sqrt{n}}\right) \tag{17}$$

The fundamental polynomials of Lagrange interpolation are given by

$$l_j(x) = \frac{P_n^{(k)}(x)}{{P_n^{(k)}}'(x_j)(x - x_j)} \tag{18}$$

$$l_j^*(x) = \frac{P_{n-1}^{(k+1)}(x)}{P_{n-1}^{(k+1)'}(x_j^*)(x-x_j^*)} \tag{19}$$

$$l_j(x) = \frac{P_n^{(k)}(x)}{P_n^{(k)'}(x_j)(x-x_j)} = \frac{\tilde{h}_n^{(k)}}{(1-x_j^2)[P_n^{(k)'}(x_j)]^2}\sum_{\nu=0}^{n-1}\frac{1}{h_\nu^{(k)}}P_\nu^{(k)}(x_j)P_\nu^{(k)}(x) \tag{20}$$

Where

$$\tilde{h}_n^{(k)} = \frac{2^{2k}\Gamma\left(2(n+k+1)\right)}{\Gamma\left(n+1\right)\Gamma\left(n+2k+1\right)} \sim C_1 \tag{21}$$

$$h_\nu^{(k)} = \frac{2^{2k+1}}{2\nu+2k+1}\frac{\Gamma\left(2(\nu+k+1)\right)}{\Gamma\left(\nu+1\right)\Gamma\left(\nu+2k+1\right)}\begin{cases}\sim \frac{1}{\nu} & (\nu>0)\\ = C_2 & (\nu=0)\end{cases} \tag{22}$$

where the constants C_1, C_2 depends only α. If $x_1 > x_2 > \cdots > x_n$ are the roots of $P_n^{(k)}(x)$, then the following relations hold [8].

$$(1-x_j^2) \sim \begin{cases}\frac{j^2}{n^2} & (x_j \geq 0)\\ \frac{(n-j)^2}{n^2} & (x_j<0)\end{cases} \tag{23}$$

$$|P_n^{(k)'}(x_j)| \sim \begin{cases}\frac{n^{k+2}}{j^{k+\frac{3}{2}}} & (x_j \geq 0)\\ \frac{n^{k+2}}{(n-j)^{k+\frac{3}{2}}} & (x_j<0)\end{cases} \tag{24}$$

3. Explicit Representation of Interpolatory Polynomials

We shall write $R_m(x)$ satisfying (6), (7), (8), (9) and (10) as

$$R_m(x) = \sum_{j=1}^{n} A_j(x)y_j + \sum_{j=1}^{n-1} B_j(x)y_j^* + \sum_{j=1}^{n-1} C_j(x){y_j^*}' + \sum_{j=0}^{k} D_j(x)y_1^{(l)} + \sum_{j=0}^{k+1} E_j(x)y_{-1}^{(l)} \tag{25}$$

Where $A_j(x)$ and $B_j(x)$ are the fundamental polynomials of first kind and $C_j(x)$ is the fundamental polynomial of second kind. $D_j(x)$ and $E_j(x)$ are the fundamental polynomials which correspond to the boundary conditions each of degree $\leq 3n+2k$, uniquely determined by the following conditions.

For $j = 1, 2, \ldots, n$

$$\begin{cases} A_j(x_i) = \delta_{ji}, & (i = 1,2,\ldots,n)\\ A_j(x_i^*) = 0, & (i=1,2,\ldots,n-1)\\ {A_j}'(x_i^*) = 0, & (i=1,2\ldots,n-1)\\ {A_j}^l(1) = 0, & (l=0,1,\ldots,k)\\ {A_j}^l(-1) = 0, & (l=0,1,\ldots,k+1)\end{cases} \tag{26}$$

For $j = 1, 2, \ldots, n-1$

$$\begin{cases} B_j(x_i) = 0, & (i = 1,2,\ldots,n)\\ B_j(x_i^*) = \delta_{ji}, & (i=1,2,\ldots,n-1)\\ {B_j}'(x_i^*) = 0, & (i=1,2\ldots,n-1)\\ {B_j}^l(1) = 0, & (l=0,1,\ldots,k)\\ {B_j}^l(-1) = 0, & (l=0,1,\ldots,k+1)\end{cases} \tag{27}$$

For $j = 1, 2, \ldots, n-1$

$$\begin{cases} C_j(x_i) = 0, & (i = 1, 2, \ldots, n) \\ C_j(x_i^*) = 0, & (i = 1, 2, \ldots, n-1) \\ {C_j}'(x_i^*) = \delta_{ji}, & (i = 1, 2 \ldots, n-1) \\ {C_j}^l(1) = 0, & (l = 0, 1, \ldots, k) \\ {C_j}^l(-1) = 0, & (l = 0, 1, \ldots, k+1) \end{cases} \tag{28}$$

For $j = 0, 1, \ldots, k$

$$\begin{cases} D_j(x_i) = 0, & (i = 1, 2, \ldots, n) \\ D_j(x_i^*) = 0, & (i = 1, 2, \ldots, n-1) \\ {D_j}'(x_i^*) = 0, & (i = 1, 2 \ldots, n-1) \\ {D_j}^l(1) = \delta_{jl}, & (l = 0, 1, \ldots, k) \\ {D_j}^l(-1) = 0, & (l = 0, 1, \ldots, k+1) \end{cases} \tag{29}$$

For $j = 0, 1, \ldots, k+1$

$$\begin{cases} E_j(x_i) = 0, & (i = 1, 2, \ldots, n) \\ E_j(x_i^*) = 0, & (i = 1, 2, \ldots, n-1) \\ {E_j}'(x_i^*) = 0, & (i = 1, 2 \ldots, n-1) \\ {E_j}^l(1) = 0, & (l = 0, 1, \ldots, k) \\ {E_j}^l(-1) = \delta_{jl}, & (l = 0, 1, \ldots, k+1) \end{cases} \tag{30}$$

We proved the Explicit forms which are given in the following Lemmas.

Lemma 3.1. *The fundamental polynomial $C_j(x)$, for $j = 1, 2, \ldots, n-1$ satisfying the interpolatory conditions (28) are given by*

$$C_j(x) = \frac{(1+x)(1-x^2)^{k+1} P_n^{(k)}(x) P_{n-1}^{(k+1)}(x) l_j^*(x)}{(1+x_j^*)(1-{x_j^*}^2)^{k+1} P_n^{(k)}(x_j^*) {P_{n-1}^{(k+1)}}'(x_j^*)} \tag{31}$$

Lemma 3.2. *The fundamental polynomial $B_j(x)$, for $j = 1, 2, \ldots, n-1$ satisfying the interpolatory conditions (27) are given by*

$$B_j(x) = \frac{(1+x)(1-x^2)^{k+1} P_n^{(k)}(x) \{l_j^*(x)\}^2}{(1+x_j^*)(1-{x_j^*}^2)^{k+1} P_n^{(k)}(x_j^*)} - 2\{{l_j^*}'(x_j^*) - \frac{x_j^*(k+1)}{(1-{x_j^*}^2)}\} C_j(x) \tag{32}$$

Lemma 3.3. *The fundamental polynomial $A_j(x)$, for $j = 1, 2, \ldots, n$ satisfying the interpolatory conditions (26) are given by*

$$A_j(x) = \frac{(1-x^2)^{k+1} [P_{n-1}^{(k+1)}(x)]^2 l_j(x)(1+x)}{(1-x_j^2)^{k+1} [P_{n-1}^{(k+1)}(x_j)]^2 (1+x_j)} \tag{33}$$

Lemma 3.4. *The fundamental polynomial which correspond to the boundary condition $D_j(x)$, for $j = 0, 1, \ldots, k$ satisfying the interpolatory conditions (29) are given by*

$$\begin{aligned} D_j(x) =& (1-x)^j (1+x)^{k+2} \{P_n^{(k)}(x)\}^2 {P_n^{(k)}}'(x) p_j(x) \\ &+ (1+x)(1-x^2)^{k+1} {P_n^{(k)}}'(x) P_n^{(k)}(x) \times \left\{ \frac{{P_n^{(k)}}'(x) q_j(x) - P_n^{(k)}(x) p_j(x)}{(1-x)^{k+1-j}} \right\} \end{aligned} \tag{34}$$

where degree $p_j(x) \leq k-j-1$ and degree $q_j(x) \leq k-j$.

Lemma 3.5. *The fundamental polynomial which correspond to the boundary condition $E_j(x)$, for $j = 0, 1, \ldots, k+1$ satisfying the interpolatory conditions (30) are given by*

For $j = 0, 1 \dots, k$

$$E_j(x) = (1-x)^{k+1}(1+x)^j\{P_n^{(k)}(x)\}^2 P_n^{(k)\prime}(x)\tilde{p_j}(x) + (1-x^2)^{k+1}P_n^{(k)\prime}(x)P_n^{(k)}(x) \times \left\{\frac{P_{n-1}^{(k+1)}(x)\tilde{q_j}(x) - P_n^{(k)}(x)\tilde{p_j}(x)}{(1+x)^{k+1-j}}\right\} \tag{35}$$

where degree $\tilde{p_j}(x) \le k-j$ *and degree* $\tilde{q_j}(x) \le k-j+1$.

For $j = k+1$

$$E_{k+1}(x) = \frac{(1-x^2)^{k+1}P_n^{(k)}(x)\{P_{n-1}^{(k+1)}(x)\}^2}{(k+1)!2^{k+1}P_n^{(k)}(-1)\{P_{n-1}^{(k+1)}(-1)\}^2} \tag{36}$$

By Lemma 3.1, Lemma 3.2, Lemma 3.3, Lemma 3.4 and Lemma 3.5 the polynomial $R_m(x)$ is satisfies the conditions (26)-(30) hence the existence part of theorem is proved.

4. Order of Convergence of the Fundamental Polynomials

Theorem 4.1. *If* $k > 0$, $n \ge 2$, *for the first derivative of the second kind fundamental polynomials on [-1,1] holds.*

$$\sum_{j=1}^{n-1} |C_j'(x)| = O\left(n^{k+\frac{9}{2}}\right) \tag{37}$$

Proof. Differentiating (31), we get

$$\sum_{j=1}^{n-1} |C_j'(x)| = \eta_1 + \eta_2 + \eta_3$$

where

$$\eta_1 = \sum_{j=1}^{n-1} \frac{\{(1-x^2)^{k+1} + 2x(k+1)(1+x)(1-x^2)^k\}|P_n^{(k)}(x)||P_{n-1}^{(k+1)}(x)||l_j^*(x)|}{(1+x_j^*)(1-x_j^{*2})^{k+1}|P_n^{(k)}(x_j^*)||P_{n-1}^{(k+1)\prime}(x_j^*)|}$$

We use the decomposition (19) for $l_j^*(x)$

$$\eta_1 \le \sum_{j=1}^{n-1} \frac{\{(1-x^2)^{k+1} + 2x(k+1)(1+x)(1-x^2)^k\}|P_n^{(k)}(x)||P_{n-1}^{(k+1)}(x)|}{(1+x_j^*)(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_n^{(k)}(x_j^*)||P_{n-1}^{(k+1)\prime}(x_j^*)|^3} \times \tilde{h}_{n-1}^{(k+1)} \times \left\{\gamma_1 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{k+1}}(1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)}(x)|\right\}$$

where γ_1 is a constant independent of x. By using (23) and (24), we get

$$\frac{1}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^3} = O\left(n-1\right)^{\frac{-3}{2}} \tag{38}$$

Using (16), (17), (22), (23) and (38), we obtain

$$\eta_1 = O(n^{k+\frac{5}{2}})$$

$$\eta_2 = \sum_{j=1}^{n-1} \frac{(1+x)(1-x^2)^{k+1}\{|P_n^{(k)\prime}(x)||P_{n-1}^{(k+1)}(x)| + |P_n^{(k)}(x)||P_{n-1}^{(k+1)\prime}(x)|\}|l_j^*(x)|}{(1+x_j^*)(1-x_j^{*2})^{k+1}|P_n^{(k)}(x_j^*)||P_{n-1}^{(k+1)\prime}(x_j^*)|}$$

$$\eta_2 \le \sum_{j=1}^{n-1} \frac{(1+x)(1-x^2)^{k+1}\{\frac{(n+2k+1)}{2}|P_{n-1}^{(k+1)}(x)|^2 + \frac{(n+2k+2)}{2}|P_n^{(k)}(x)||P_{n-2}^{(k+2)}(x)|\}}{(1+x_j^*)(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^3|P_n^{(k)}(x_j^*)|} \times \tilde{h}_{n-1}^{(k+1)} \times \left\{\gamma_2 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{k+1}}(1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)}(x)|\right\}$$

where γ_2 is a constant independent of x. Using (16), (17), (22), (23) and (38), we get

$$\eta_2 = O\left(n^{k+\frac{9}{2}}\right)$$

$$\eta_3 = \sum_{j=1}^{n-1} \frac{(1+x)(1-x^2)^{k+1}|P_n^{(k)}(x)||P_{n-1}^{(k+1)}(x)||l_j^{*\prime}(x)|}{(1+x_j^*)(1-x_j^{*2})^{k+1}|P_n^{(k)}(x_j^*)||P_{n-1}^{(k+1)\prime}(x_j^*)|} \tag{39}$$

$$\eta_3 \le \sum_{j=1}^{n-1} \frac{(1+x)(1-x^2)^{k+1}|P_n^{(k)}(x)||P_{n-1}^{(k+1)}(x)|}{(1+x_j^*)(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^3|P_n^{(k)}(x_j^*)|} \times \tilde{h}_{n-1}^{(k+1)}$$
$$\times \left\{ \gamma_3 + \sum_{\nu=1}^{n-2} \frac{1}{h_\nu^{k+1}} (1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}} |P_\nu^{(k+1)}(x_j^*)||P_\nu^{(k+1)\prime}(x)| \right\} \tag{40}$$

where γ_3 is a constant independent of x. Using (15), (16), (17), (22), (23) and (38), we obtain

$$\eta_3 = O\left(n^{k+\frac{7}{2}}\right)$$

Hence the theorem is proved. □

Theorem 4.2. *If $k > 0$, $n \ge 2$, for the first derivative of the first kind fundamental polynomials on [-1,1] holds.*

$$\sum_{j=1}^{n-1}(1-x_j^{*2})|B_j'(x)| = O\left(n^{2k+7}\right) \tag{41}$$

Proof. Differentiating (32), we get

$$\sum_{j=1}^{n-1}(1-x_j^{*2})|B_j'(x)| = \zeta_1+\zeta_2+\zeta_3 \tag{42}$$

where

$$\zeta_1 = \sum_{j=1}^{n-1} \frac{[(1+x)(1-x^2)|P_n^{(k)\prime}(x)| + \{2x(k+1)(1+x)+(1-x^2)\}|P_n^{(k)}(x)|](1-x^2)^k}{(1+x_j^*)(1-x_j^{*2})^k|P_n^{(k)}(x_j^*)|} \times |l_j^*(x)|^2 \tag{43}$$

We use the decomposition (20) for $l_j^*(x)$ and using (15) then we get

$$\zeta_1 \le \sum_{j=1}^{n-1} \frac{[(1+x)(1-x^2)^{\frac{(n+2k+1)}{2}}|P_{n-1}^{(k+1)}(x)| + \{2x(k+1)(1+x)+(1-x^2)\}|P_n^{(k)}(x)|](1-x^2)^k}{(1+x_j^*)(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^4|P_n^{(k)}(x_j^*)|}$$
$$\times \{\tilde{h}_{n-1}^{(k+1)}\}^2 \left\{ \gamma_4 + \sum_{\nu=1}^{n-2}\sum_{\nu=1}^{n-2} \frac{1}{\{h_\nu^{(k+1)}\}^2}(1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k+1)}(x_j^*)|^2|P_\nu^{(k+1)}(x)|^2 \right\}$$

where γ_4 is a constant independent of x. By using (23) and (24) then it holds

$$\frac{1}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^4} = O\,(n-1)^{-2} \tag{44}$$

Using (16), (17), (22), (23) and (44), we have

$$\zeta_1 = O\left(n^{2k+6}\right)$$

$$\zeta_2 = \sum_{j=1}^{n-1} \frac{2(1+x)(1-x^2)^{k+1}|P_n^{(k)}(x)||l_j^*(x)||l_j^{*\prime}(x)|}{(1+x_j^*)(1-x_j^{*2})^k|P_n^{(k)}(x_j^*)|}$$

$$\zeta_2 \le \sum_{j=1}^{n-1} \frac{2(1+x)(1-x^2)^{k+1}|P_n^{(k)}(x)| \times \{\tilde{h}_{n-1}^{(k+1)}\}^2}{(1+x_j^*)(1-x_j^{*2})^{\frac{3k}{2}+\frac{9}{4}}|P_{n-1}^{(k+1)\prime}(x_j^*)|^4|P_n^{(k)}(x_j^*)|}$$
$$\times \left\{ \gamma_5 + \sum_{\nu=1}^{n-2}\sum_{\nu=1}^{n-2} \frac{1}{\{h_\nu^{(k+1)}\}^2}(1-x_j^{*2})^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k+1)}(x_j^*)|^2|P_\nu^{(k+1)}(x)||P_\nu^{(k+1)\prime}(x)| \right\}$$

where γ_5 is a constant independent of x. Using (15), (16), (17), (22), (23) and (44), we get

$$\zeta_2 = O\left(n^{2k+7}\right)$$

$$\zeta_3=\sum_{j=1}^{n-1}2\{|l_j^{*\prime}(x_j^*)|(1-x_j^{*2})+x_j^*(k+1)\}|C_j{}'(x)|$$

Differentiating (19), it holds

$$l_j^{*\prime}(x_j^*)=\frac{P_{n-1}^{(k+1)''}(x_j^*)}{2P_{n-1}^{(k+1)'}(x_j^*)} \tag{45}$$

By using (15), (16) and (45), we get

$$|l_j^{*\prime}(x_j^*)|=O(n^2) \tag{46}$$

Using (23), (37) and (46), we obtain

$$\zeta_3=O\left(n^{k+\frac{13}{2}}\right)$$

Hence the theorem is proved. □

Theorem 4.3. *If $k>0$, $n\geq 2$, for the first derivative of the first kind fundamental polynomials on [-1,1] holds.*

$$\sum_{j=1}^{n}(1-x_j{}^2)|A_j'(x)|=O\left(n^{2k+5}\right) \tag{47}$$

Proof. Differentiating (33), we get

$$\sum_{j=1}^{n}(1-x_j{}^2)|A_j'(x)|=\xi_1+\xi_2+\xi_3 \tag{48}$$

where

$$\xi_1=\sum_{j=1}^{n}\frac{(1+x)(1-x^2)^{k+1}\{P_{n-1}^{(k+1)}(x)\}^2|l_j{}'(x)|}{(1-x_j{}^2)^k(1+x_j)|P_{n-1}^{(k+1)}(x_j)|^2}$$

We use the decomposition (20) for $l_j(x)$

$$\xi_1\leq\sum_{j=1}^{n}\frac{(1-x^2)^{k+1}|P_{n-1}^{(k+1)}(x)|^2(n+2k+1)^2(1+x)\times\tilde{h}_n^{(k)}}{4(1+x_j)\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)'}(x_j)|\}^4}\times\left\{\gamma_6+\sum_{\nu=1}^{n-1}\frac{1}{h_\nu^{(k)}}(1-x_j^2)^k|P_\nu^{(k)}(x_j)||P_\nu^{(k)'}(x)|\right\}$$

where γ_6 is a constant independent of x. Using (23), (24), it holds

$$\frac{1}{\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)'}(x_j)|\}^4}=O\left(\frac{1}{n^2}\right) \tag{49}$$

By using (16), (17), (22), (23) and (49), we get

$$\begin{aligned}
\xi_1&=O(n^{2k+5})\\
\xi_2&=\sum_{j=1}^{n}\frac{2(1+x)(1-x^2)^{k+1}|P_{n-1}^{(k+1)}(x)||P_{n-1}^{(k+1)'}(x)||l_j(x)|}{(1-x_j{}^2)^k(1+x_j)|P_{n-1}^{(k+1)}(x_j)|^2}\\
\xi_2&\leq\sum_{j=1}^{n}\frac{2(1+x)(n+2k+1)^2(1-x^2)^{k+1}|P_{n-1}^{(k+1)}(x)||P_{n-1}^{(k+1)'}(x)|\times\tilde{h}_n^{(k)}}{4(1+x_j)\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)'}(x_j)|\}^4}\\
&\times\left\{\gamma_7+\sum_{\nu=1}^{n-1}\frac{1}{h_\nu^{(k)}}(1-x_j^2)^k|P_\nu^{(k)}(x_j)||P_\nu^{(k)}(x)|\right\}
\end{aligned}$$

where γ_7 is a constant independent of x. Using (15) and (16) then it holds

$$|P_{n-1}^{(k+1)'}(x)|=O(n^{k+3}) \tag{50}$$

By using (16), (17), (22), (23), (49) and (50), we get

$$\xi_2 = O(n^{2k+5})$$

$$\xi_3 = \sum_{j=1}^{n} \frac{\{(1-x^2)^{k+1} + 2x(k+1)(1+x)(1-x^2)^k\}|P_{n-1}^{(k+1)}(x)|^2|l_j(x)|}{(1-x_j{}^2)^k(1+x_j)|P_{n-1}^{(k+1)}(x_j)|^2}$$

$$\xi_3 \le \sum_{j=1}^{n} \frac{(n+2k+1)^2\{(1-x^2)^{k+1} + 2x(k+1)(1+x)(1-x^2)^k\}|P_{n-1}^{(k+1)}(x)|^2}{4(1+x_j)\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)'}(x_j)|\}^4} \times \tilde{h}_n^{(k)}$$

$$\times \left\{\gamma_8 + \sum_{\nu=1}^{n-1} \frac{1}{h_\nu^{(k)}}(1-x_j^2)^k|P_\nu^{(k)}(x_j)||P_\nu^{(k)}(x)|\right\}$$

where γ_8 is a constant independent of x. By using (16), (17), (22), (23) and (49) then we obtain

$$\xi_3 = O(n^{2k+3})$$

Hence the theorem is proved. □

Theorem 4.4. *Let $k \ge 0$ be a fixed integer m=3n+2k and let $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the Ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively if $f \in C^r[-1,1]$ $(r \ge k+1, n \ge 2r-k+2)$ then the interpolational polynomial*

$$R_m(x;f) = \sum_{i=1}^{n} f(x_i)A_i(x) + \sum_{i=1}^{n-1} f(x_i^*)B_i(x) + \sum_{i=1}^{n-1} f'(x_i^*)C_i(x) + \sum_{j=0}^{k} f^{(j)}(1)D_j(x) + \sum_{j=0}^{k+1} f^{(j)}(-1)E_j(x)$$

with the fundamental polynomials given in (31)-(36) satisfies for $x \in [-1,1]$

$$|f'(x) - R'_m(x;f)| = w\left(f^{(r)};\frac{1}{n}\right)O\left(n^{2k-r+7}\right) \tag{51}$$

Proof. For $k=0$ we refer to (11), proved by Xie and Zhou [9]. Let $f \in C^r[-1,1]$, by the theorem of Gopengauz [1] for every $m \ge 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $j = 0,\ldots,r$

$$|f^{(j)}(x) - p_m^{(j)}(x)| \le M_{r,j}\left(\frac{\sqrt{1-x^2}}{m}\right)^{r-j} w\left(f^{(r)};\frac{\sqrt{1-x^2}}{m}\right)$$

where $w(f^{(r)};.)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,j}$ depend only on r and j. Furthermore,

$$f^{(j)}(\pm 1) = p_m^{(j)}(\pm 1) \quad (j = 0,\ldots,r)$$

By the uniqueness of the interpolational polynomials $R_m(x;f)$ it is clear that $R_m(x;p_m) = p_m(x)$. Hence for $x \in [-1,1]$

$$|f'(x) - R'_m(x;f)| \le |f'(x) - p'_m(x)| + |R'_m(x;p_m) - R'_m(x;f)|$$

$$\le |f'(x) - p'_m(x)| + \sum_{j=1}^{n} |f(x_j) - p_m(x_j)||A'_j(x)| + \sum_{j=1}^{n-1} |f(x_j^*) - p_m(x_j^*)||B'_j(x)|$$

$$+ \sum_{j=1}^{n-1} |f'(x_j^*) - p'_m(x_j^*)||C'_j(x)|$$

$$\le M_{r,0}\frac{1}{n^r}w\left(f^{(r)};\frac{1}{n}\right)\sum_{j=1}^{n}(1-x_j^2)|A'_j(x)| + M_{r,0}\frac{1}{n^r}w\left(f^{(r)};\frac{1}{n}\right)\sum_{j=1}^{n-1}\left(1-x_j^{*2}\right)|B'_j(x)|$$

$$+ M_{r,1}\frac{1}{n^{r-1}}w\left(f^{(r)};\frac{1}{n}\right)\{1 + \sum_{j=1}^{n-1}|C'_j(x)|\}$$

Now applying the estimates (37), (41) and (47) we have

$$\begin{aligned}|f'(x)-R'_m(x;f)| &\leq O(1)\frac{1}{n^r}w\left(f^{(r)};\frac{1}{n}\right)n^{2k+5}+O(1)\frac{1}{n^r}w\left(f^{(r)};\frac{1}{n}\right)n^{2k+7}+O(1)\frac{1}{n^{r-1}}w\left(f^{(r)};\frac{1}{n}\right)\left(1+n^{k+\frac{9}{2}}\right)\\ &=O(1)\,n^{2k-r+7}w\left(f^{(r)};\frac{1}{n}\right)\end{aligned}$$

which is the statement of the theorem. □

By using Main Theorem and (12) we can state the following convergence theorem.

Theorem 4.5. *Let $k \geq 0$ be a fixed integer, $m = 3n+2k$, $n \geq k+4$, let $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{k+1}(x)$ respectively. If $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip\alpha$, $\alpha > \frac{1}{2}$, then $R_m(x;f)$ and $R'_m(x;f)$ uniformly converge to $f(x)$ and $f'(x)$, respectively on [-1,1] as $n \to \infty$.*

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