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Transitivity of the Direct Product of the Alternating Group Acting on the Cartesian Product of Three Sets

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Abstract: The transitivity of the action of the direct product of the alternating Group on Cartesian product of three sets is investigated. In this paper, we show that the group action is transitive.

MSC: 05E10.

Keywords: Direct Product, Alternating Group, Cartesian product, Transitive group action.

1. Introduction

In this paper we consider the Alternating groups (A_n, X_1), (A_n, X_2) and (A_n, X_3), where the sets X_1, X_2 and X_3 are disjoint and of cardinality n. So the direct product $A_n \times A_n \times A_n$ acts on the Cartesian product $X_1 \times X_2 \times X_3$ by the rule

$$(x_1, x_2, x_3)(g_1, g_2, g_3) = (x_1g_1, x_2g_2, x_3g_3) \;\; \forall \;\; x_i \in X_i, g_i \in A_i$$

We shall investigate the transitivity of $A_2 \times A_2 \times A_2$, $A_3 \times A_3 \times A_3$, $A_4 \times A_4 \times A_4$ and $A_5 \times A_5 \times A_5$ before giving the results for $A_n \times A_n \times A_n$.

1.1. Notation and Preliminary Results

Definition 1.1. *Let G act on a set X. Then X is partitioned into disjoint equivalence classes called orbits or transitivity classes of the action. For each $x \in X$ the orbit containing x is called the orbit of x and is denoted by $Orb_G x$. Thus $Orb_G x = \{gx \,|g \in G\}$.*

Definition 1.2. *The action of a group G on the set X is said to be transitive if for each pair of points $x, y \in X$, there exists $g \in G$ such that $gx = y$; in other words, if the action has only one orbit. A group which is not transitive is called intransitive.*

Definition 1.3. *Let G act on a set X and let $x \in X$. The stabilizer of x in G is denoted by $Stab_G x$ is given by $Stab_G x = \{g \in G\,|gx = x\}$. $Stab_G x$ forms a subgroup of G called the Isotropy group of x. It is also denoted by G_x.*

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Definition 1.4. *Let G act on a set X. The set of elements of X fixed by $g \in G$ is called the fixed point set of G and is denoted by $Fix(g)$. Thus $Fix(g) = \{x \in X \,|gx = x\}$.*

Definition 1.5 ([2, 3, 8])**.** *Let G be a finite group acting on a set X. The number of orbits in X under G is given by $\frac{1}{|G|}\sum_{g\in G} |fix(g)|$.*

Theorem 1.6 (Orbit-Stabilizer Theorem [2, 3, 8])**.** *Let G be a group acting on a finite set X and $x \in X$. Then $|Orb_G x| = |G : G_x|$, the index of G_x in G.*

Definition 1.7. *Suppose G is a group acting transitively on a set X and let G_x be the stabilizer in G of a point $x \in X$. The orbits $\Delta_0 = x, \Delta_1, \Delta_2, \ldots, \Delta_{r-1}$ of G_x on X are known as suborbits of G. The rank of G in this case is r. The sizes $n_i = |\Delta_i|\,(i = 0, 1, ..., r-1)$ often called the 'lengths' of suborbits are known as the subdegrees of G. It can be shown that both r and the cardinalities of the suborbits. Δ_i $(i = 0, 1, \ldots, r-1)$ are independent of the choices of $x \in X$.*

Definition 1.8 ([1])**.** *Let (G_1, X_1) and (G_2, X_2) be permutation groups. The direct product $G_1 \times G_2$ acts on the disjoint union $X_1 \cup X_2$ by the rule*

$$x(g_1, g_2) = \begin{cases} xg_1; & \text{if } x \in X_1 \\ xg_2; & \text{if } x \in X_2 \end{cases}$$

and on the Cartesian product $X_1 \times X_2$ by the rule $(x_1, x_2)(g_1, g_2) = (x_1g_1, x_2g_2)$.

2. Main Results

Lemma 2.1. *If (A_2, X), (A_2, Y) and (A_2, Z) are alternating groups with $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and $Z = \{z_1, z_2\}$, then the action of $A_2 \times A_2 \times A_2$ on $X \times Y \times Z$ is transitive.*

Proof. Let $G = A_2 \times A_2 \times A_2$ and $K = X \times Y \times Z$, then the elements of G are (e_1, e_2, e_3) where each e_i represents the identity from each alternating group and K has 3 elements. The stabilizer of an element (x_1, y_1, z_1), $stab_G(x_1, y_1, z_1)$ is (e_1, e_2, e_3). Hence by Theorem 1.6.

$$\begin{aligned} |Orb_G(x_1, y_1, z_1)| &= |G : stab_G(x_1, y_1, z_1)| \\ &= \frac{|G|}{|stab_G(x_1, y_1, z_1)|} \\ &= 8 \\ &= |X \times Y \times Z| \end{aligned}$$

Therefore the action is transitive since there is only 1 orbit. □

Lemma 2.2. *If (A_3, X), (A_3, Y) and (A_3, Z) are alternating groups with $X_1 = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $Z = \{z_1, z_2, z_3\}$, then $A_3 \times A_3 \times A_3$ acts transitively on $X \times Y \times Z$.*

Proof. The elements of (A_3, X), (A_3, Y) and (A_3, Z) are $(e_1, (x_1x_2x_3), (x_1x_3x_2))$, $(e_2, (y_1y_2y_3), (y_1y_3y_2))$, and $(e_3, (z_1z_2z_3), (z_1z_3z_2))$ respectively and therefore, $G = A_3 \times A_3 \times A_3$ has 27 elements from the direct product, and if $K = X \times Y \times Z$, then K has 27 elements each of which is an ordered triple, moreover, $stab_G(x_1, y_1, z_1) = (e_1, e_2, e_3)$. Using Theorem 1.6.

$$|Orb_G(x_1, y_1, z_1)| = |G : stab_G(x_1, y_1, z_1)|$$

$$= \frac{|G|}{|stab_G(x_1, y_1, z_1)|}$$

$$= 27$$

$$= |X \times Y \times Z|$$

Therefore the action is transitive. □

Lemma 2.3. *If (A_4, X), (A_4, Y) and (A_4, Z) are alternating groups with $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3, y_4\}$ and $Z = \{z_1, z_2, z_3, z_4\}$, then $A_4 \times A_4 \times A_4$ acts transitively on $X \times Y \times Z$.*

Proof. Let $G = A_4 \times A_4 \times A_4$ and $K = X \times Y \times Z$. If (x_1, y_1, z_1) represents an arbitrary element from K, then $stab_G(x_1, y_1, z_1)$is given in the Table 1.

Type of ordered triple of permutations fixing (x_1, y_1, z_1)	Number of Permutations
(e_1, e_2, e_3)	1
$(e_1, e_2, (z_2z_3z_4))$	2
$(e_1, (y_2y_4y_3), e_3)$	2
$((x_2x_3x_4), e_2, e_3)$	2
$(e_1, (y_2y_3y_4), (z_2z_3z_4))$	4
$((x_2x_3x_4), e_2, (z_2z_3z_4))$	4
$((x_2x_3x_4), (y_2y_3y_4), e_3)$	4
$((x_2x_3x_4), (y_2y_3y_4), (z_2z_4z_3))$	8
TOTAL	27

Table 1. **Elements of the stabilizer of** (x_1, y_1, z_1)

Therefore

$$|Orb_G(x_1, y_1, z_1)| = |G : stab_G(x_1, y_1, z_1)|$$

$$= \frac{|G|}{|stab_G(x_1, y_1, z_1)|}$$

$$= \frac{12^3}{27}$$

$$= 64$$

$$= |X \times Y \times Z|$$

Therefore the action is transitive. □

Lemma 2.4. *If (A_5, X), (A_5, Y) and (A_5, Z) are alternating groups with $X = \{x_1, x_2, x_3, x_4, x_5\}$, $Y = \{y_1, y_2, y_3, y_4, y_5\}$ and $Z = \{z_1, z_2, z_3, z_4, z_5\}$, then $A_5 \times A_5 \times A_5$ acts transitively on $X \times Y \times Z$.*

Proof. Let $G = A_5 \times A_5 \times A_5$ and $K = X \times Y \times Z$. If (x_1, y_1, z_1) represents an arbitrary element from K, then $stab_G(x_1, y_1, z_1)$ is given on the Table 2.

Type of ordered triple of permutations fixing (x_1, y_1, z_1)	Number of Permutations
(e_1, e_2, e_3)	1
$(e_1, e_2, (abc))$	8
$(e_1, e_2, (ab)(cd))$	3
$(e_1, (abc), e_3)$	8
$(e_1, (ab)(cd), e_3)$	3

Type of ordered triple of permutations fixing (x_1, y_1, z_1)	Number of Permutations
$((abc), e_2, e_3)$	8
$((ab)(cd), e_2, e_3)$	3
$(e_1, (abc), (def))$	64
$(e_1, (abc), (de)(fg))$	24
$(e_1, (ab)(cd), (efg))$	24
$(e_1, (ab)(cd), (ef)(gh))$	9
$((abc), e_2, (def))$	64
$((abc), e_2, (de)(fg))$	24
$((ab)(cd), e_2, (efg))$	24
$((ab)(cd), e_2, (ef)(gh))$	9
$((abc), (def), e_3)$	64
$((abc), (de)(fg), e_3)$	24
$((ab)(cd), (efg), e_3)$	24
$((ab)(cd), (ef)(gh), e_3)$	9
$((abc), (def), (ghi))$	512
$((abc), (def), (gh)(ij))$	192
$((abc), (de)(fg), (hij))$	192
$((ab)(cd), (efg), (hij))$	192
$((abc), (de)(fg), (hi)(jk))$	72
$((ab)(cd), (efg), (hi)(jk))$	72
$((ab)(cd), (ef)(gh), (ijk))$	72
$((ab)(cd), (ef)(gh), (ij)(kl))$	27
TOTAL	**1728**

Table 2. **Elements of the stabilizer of** (x_1, y_1, z_1)

Therefore

$$\begin{aligned}|Orb_G(x_1, y_1, z_1)| &= |G : stab_G(x_1, y_1, z_1)| \\ &= \frac{|G|}{|stab_G(x_1, y_1, z_1)|} \\ &= \frac{60^3}{1728} \\ &= 125 \\ &= |X \times Y \times Z|\end{aligned}$$

Therefore the action is transitive. □

Theorem 2.5. *If (A_n, X), (A_n, Y) and (A_n, Z) are alternating groups with $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$, and if $n \geq 4$, then the action of $A_n \times A_n \times A_n$ on $X \times Y \times Z$ is transitive.*

Proof. Let $G = A_n \times A_n \times A_n$ and $K = X \times Y \times Z$, then $stab_G(x_1, y_1, z_1)$ is isomorphic to $A_{n-1} \times A_{n-1} \times A_{n-1}$ where the permuting elements are from the sets $X - \{x_1\}$, $Y - \{y_1\}$ and $Z - \{z_1\}$. Using Theorem 1.6.,

$$\begin{aligned}|Orb_G(x_1, y_1, z_1)| &= |G : stab_G(x_1, y_1, z_1)| \\ &= \frac{|G|}{|stab_G(x_1, y_1, z_1)|} \\ &= \frac{\frac{n!}{2} \times \frac{n!}{2} \times \frac{n!}{2}}{\frac{(n-1)!}{2} \times \frac{(n-1)!}{2} \times \frac{(n-1)!}{2}} \\ &= n^3 \\ &= |X \times Y \times Z|\end{aligned}$$

Hence the action is transitive. □

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