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Post Optimality Analysis in Fuzzy Multi Objective Linear Programming

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Abstract: Models of linear programming problems with fuzzy constraints are well known in the current literature. In this paper, the post optimality analysis (POA) of a given fuzzy efficient solution in a general fuzzy multi-objective linear programming (FMOLP) problem is considered. The POA determines the largest sensitivity region of changes in the input data the fuzzy efficient solution of FMOLP problem remains unchanged. For this reason, three cases are considered separately: changing in one objective function coefficient (the change in one of objective functions' coefficient), objective function removal and objective function addition. Each case is illustrated by a numerical example.

Keywords: Fuzzy multi objective linear programming, Fuzzy efficiency, Pareto optimality, Post optimality analysis.

1. Introduction

The Fuzzy set theory has been applied to many disciplines such as control theory and management sciences, mathematical modeling and industrial applications. The concept of fuzzy mathematical programming in general level was first proposed by Tanaka [18] which was based on Fuzzy theorem developed by Bellman and Zadeh [1]. The first formulation of fuzzy linear programming (FLP) has been introduced by Zimmermann [20]. Zadeh and Kacprzyk [?] presented the concept of fuzzy logic for knowledge-based systems which include the use of fuzzy relations. Afterward, many authors considered various types of the FLP problems were investigated led to several approaches for solving these problems [4, 13, 15, 16]. In addition, Piegat [?] presented a new definition of the fuzzy set. Based on this definition, the number of arithmetic operations would dramatically decrease compare to the results produced by the existing fuzzy arithmetic.

After finding the optimal solution to the given LP model, the POA can be exploited to analyze the consequences of imposing changes in problem's parameters. One of the first papers on the POA in multi-objective linear programming was an article by Hansen [8] in which the authors analyzed the sensitivity of problems parameters after scalarization.

Many papers on the POA [2, 10] are focused on showing how the optimal solution of the linear programming changes when the problem's data are changed. For this case, they construct methods on how to find the breakpoints and the rate of changes (shadow price) of the objective function. Gal and Wolf [6] presented a survey of the literature considering the sensitivity of input data in multi-objective function (MOLP) problems. A geometric interpretation of tolerance analysis is given by Borges and Antunes [3]. Such an approach was used by Hladik [9] in the case of the tolerance approach and Sitarz

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[17] in the case of standard sensitivity approach. Jones [22] presented a weight sensitivity algorithm that can be used to investigate a portion of weight space of interest to the decision maker in a goal or multiple objective programming.

However, there are only a few papers to deal with post optimality analysis in published works on fuzzy linear programming. The concept of sensitivity analysis in fuzzy number linear programming (FNLP) problems by applying fuzzy simplex algorithms and using the general linear ranking functions on fuzzy numbers was considered by Ebrahimnejad [5].

This paper has a contribution to post optimality analysis in fuzzy multi-objective linear programming problems (FMOLP). We consider the POA for selected coefficients of (one of the) objective functions in FMOLP problems. By using the approach of Jimenez and Bilbao [11] we obtain an interval for the selected coefficient where the fuzzy efficient solution remains unchanged. Finally, we examine whether by removing or adding an objective function into the initial problem the given fuzzy efficient point remains fuzzy efficient or not?

This paper is organized as follows: In section 2, we review some basic concepts and results of fuzzy numbers. In addition, FMOLP problem is introduced in this section. The Post Optimality Analysis of one objective's coefficient is discussed in section 3. Also, the effect of removing and adding an objective function on the efficiency of the solutions is expressed in this section.

2. Preliminaries

In this section, we review some preliminaries on fuzzy numbers and their ordering, FMOLP and its solution approach. These concepts are to be used in the next sections.

2.1. Fuzzy Number and its Ordering

The uncertainty Management is an important issue in the design of expert systems because a lot of information in the knowledge base of a typical expert system is imprecise, incomplete or not totally reliable. An approach to the uncertainty Management used in fuzzy environment is the logic underlying approximately or equivalently. A feature of fuzzy environment which is a great importance to the uncertainty Management is providing a systematic framework for dealing with uncertain linguistic variables (e.g. many, few, almost all, infrequently, about 0.8,...) by introducing fuzzy quantifiers and fuzzy numbers. Here, we recall the notions of fuzzy numbers and their comparison.

Definition 2.1 (Fuzzy Number). *A fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$ is going to be an LR flat fuzzy number if its membership function is given by:*

$$\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right), & x \leq m, \\ R\left(\frac{x-n}{\beta}\right), & x \geq n, \\ 1, & m \leq x \leq n, \end{cases} \tag{1}$$

where $\alpha > 0$ and $\beta > 0$.

If $m = n$ then $\tilde{A} = (m, n, \alpha, \beta)$ will be converted into $\tilde{A} = (m, \alpha, \beta)$ and is going to be an LR fuzzy number. L and R are called reference functions, which are continuous, non-increasing functions defining left and right shapes of $\mu_{\tilde{A}}(x)$ respectively (and $L(0) = R(0) = 1$). Two special cases are triangular and trapezoidal fuzzy number, for which $L(x) = R(x) = \max\{0, 1-x\}$, are linear functions.

Definition 2.2 (Ranking Function). *A ranking function $R : F(\mathfrak{R}) \to \mathfrak{R}$ maps each fuzzy number into the real line, where* *a natural order exists. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, then*

$$\begin{aligned} &\tilde{a} \preceq_R \tilde{b} \text{ if and only if } R(\tilde{a}) \leq R(\tilde{b}), \\ &\tilde{a} \prec_R \tilde{b} \text{ if and only if } R(\tilde{a}) < R(\tilde{b}), \\ &\tilde{a} \simeq_R \tilde{b} \text{ if and only if } R(\tilde{a}) = R(\tilde{b}). \end{aligned} \tag{2}$$

Several rankings are introduced by some authors. In this paper we use the following ranking proposed by Yager [19]:

$$R(\tilde{a}) = \frac{1}{2}\int_0^1 (\inf \tilde{a}_\lambda + \sup \tilde{a}_\lambda) d\lambda, \tag{3}$$

where $\tilde{a}_\lambda$ is the λ-cut of $\tilde{a}$. For trapezoidal fuzzy number $\tilde{A} = (m, n, \alpha, \beta)$, $R(\tilde{A})$ reduces to:

$$R(\tilde{A}) = \frac{1}{2}(m+n) + \frac{1}{4}(\beta - \alpha). \tag{4}$$

2.2. Multi Objective Linear Programming

Consider the following multi objective linear programming problem with k objective functions

$$\max \ z(x) = (z_1(x) = c_1 x, \ z_2(x) = c_2 x, \ldots, z_k(x) = c_k x) \tag{5}$$

such that $x \in X$, where the feasible set $X = \{x \in \Re^n : Ax \leq b, x \geq 0\}$, $c_i = (c_{i1}, \ldots, c_{in}) \in \Re^n$, $i = 1, \ldots, k$, $b = (b_1, \ldots, b_m) \in \Re^m$ and $A = [a_{ij}]_{m*n} \in \Re^{m*n}$. The similarity of a multi-objective programming problem and a single programming problem is clear; the only difference is that instead of just one objective function in a single programming problem, there are some objective functions in a multi-objective programming problem. If the notion of optimality is used to an objective function and multi-objective optimization problem, then we have the following definition.

Definition 2.3. *$x^* \in X$ is said to be a complete optimal solution to the problem (5) if it optimizes all objective functions simultaneously; which means, for all $x^* \in X$ we have $z_i(x) \leq z_i(x^*)$.*

Generally, access to a complete optimal solution is not possible, a new concept which called an efficient solution, is introduced.

Definition 2.4. *$x^* \in X$ is said to be a efficient (Pareto Optimal) solution to the problem (5) if and only if there is no another $x \in S$ where $z_i(x) \leq z_i(x^*)$ for all $i = 1, 2, \ldots, k$ and $z_i(x) < z_i(x^*)$. For at least one $l = 1, 2, \ldots, k$ holds. Otherwise, x^* is called inefficient.*

2.3. Fuzzy Multi Objective Linear Programming

Let a FMOLP problem with k objective functions $\tilde{z}_i(x) = c_i^T x$, $i = 1, 2, \ldots, k$ be

$$P: \max \ \tilde{z}(x) = (\tilde{z}_1(x), \tilde{z}_2(x), \ldots, \tilde{z}_k(x)) \tag{6}$$

such that $x \in S$, where the feasible set $S = \left\{x \in \Re^n : \tilde{A}x \preceq_R \tilde{b}, x \geq 0\right\}$, R is an arbitrary ranking function, $\tilde{c}_i = (\tilde{c}_{i1}, \ldots, \tilde{c}_{in}) \in (F(\Re))^n$, $i = 1, \ldots, k$, $\tilde{b} = (\tilde{b}_1, \ldots, \tilde{b}_m) \in (F(\Re))^m$ and $\tilde{A} = [\tilde{a}_{ij}]_{m*n} \in (F(\Re))^{m*n}$.

Definition 2.5 (Fuzzy Efficient Solution)**.** *$x^* \in S$ is said to be a fuzzy efficient solution to the problem (6) if and only if there does not exist another $x \in S$ for which $\mu_{\tilde{z}_i}(x^*) \leq \mu_{\tilde{z}_i}(x)$ for all $i = 1, 2, \ldots, k$ and $\mu_{\tilde{z}_l}(x^*) \neq \mu_{\tilde{z}_l}(x)$ for at least one $l = 1, 2, \ldots, k$ holds.*

In the other words, $x^* \in S$ is a fuzzy efficient solution if and only if

$$\forall\, x \in S;\ \mu_{\tilde{z}_i}(x^*) \leq \mu_{\tilde{z}_i}(x)\ :\ i = 1, 2, \ldots, k \quad \mu_{\tilde{z}_i}(x^*) = \mu_{\tilde{z}_i}(x)\ :\ i = 1, 2, \ldots, k. \tag{7}$$

Several authors have proposed procedures in order to achieve fuzzy efficient solutions. Here, we refer to the two-phase approach proposed by Guua and Wu [7]. A FMOLP problem, using the fuzzy decision (max-min) of Bellman and Zadeh with introducing the auxiliary variable λ adopts the following problem:

$$\begin{aligned} &\max && \lambda \\ &\text{subject to} && 1 \geq \mu_{\tilde{Z}_i}(x) \geq \lambda \geq 0\ : i = 1, \ldots, k, \\ & && x \in S \end{aligned} \tag{8}$$

The Guua and Wu algorithm is as follows:

Step 1: Solve problem (8). Let the optimal solution be given by (x^*, λ^*). If the optimal solution is unique, then x^* is a fuzzy efficient solution, then, Stop.

Step 2: Solve the second phase model, i.e.,

$$\begin{aligned} &\max && \sum_{i=1}^{k} \lambda_i \\ &\text{subject to} && 1 \geq \mu_{Z_i}(x) \geq \lambda_i \geq \mu_{Z_i}(x^*)\ : i = 1, \ldots, k, \\ & && x \in S \end{aligned} \tag{9}$$

Let the optimal solution be given by (x^{**}, λ^{**}). In this case, x^{**} is a fuzzy efficient solution.

3. Post Optimality

The attainment of the optimal solution to a linear programming problem is often desirable to study the effect of discrete changes in the different coefficients of the problem on the current optimal solution. One way to accomplish this is to solve the problem anew, but this may be computationally inefficient. The changes in the linear programming problem usually studied by post optimality analysis include:

- Tightness of the constraints, that is, changes in the right-hand side of the constraints.
- Coefficients of the objectives function.
- Technological coefficients of decision variables.
- Addition of new variables to the problem.
- Addition of new constraint(s).
- Addition of new constraint(s).
- Removal of new constraint(s).

In this section, we consider the post optimality analysis for three cases: changing in one objective function coefficient (the change in one of objective functions' coefficient), objective function removal and objective function addition.

3.1. Convexity

There is the element-wise analysis approach to the sensitivity analysis presented below. Let x^* be a fuzzy efficient solution of problem (6). We wish to determine the region for the parameter $\tilde{t}$, where x^* remains a fuzzy efficient solution of the following problem:

$$
\begin{aligned}
P\left(\tilde{t}\right): \ & \max && \tilde{z}\left(x\right) = \left(\tilde{z}_1\left(x\right), \tilde{z}_2\left(x\right), \ldots, \tilde{z}_{r-1}\left(x\right), \tilde{z}_{r+1}\left(x\right), \ldots, \tilde{z}_k\left(x\right)\right), \\
& \max && \tilde{z}_r = \sum_{\substack{j=1 \\ j\neq p}}^{j=n} \tilde{c}_{rj}x_j + \tilde{t}x_p \\
& \text{subject to} && x \in S = \left\{x \in \Re^n \ : \ \tilde{A}x \preceq_R \tilde{b}, x \geq 0\right\}.
\end{aligned} \tag{10}
$$

The next theorem shows that the set of such $\tilde{t}$ is convex. In the proof of the theorem we use a linear membership function with a given ranking function R [15]:

$$
\mu_{\tilde{z}_i}\left(x\right) = 1 - \frac{R\left(\tilde{\overline{z}}_i\right) - R\left(\sum_j \tilde{c}_{ij}x_j\right)}{q_i} = 1 - \frac{\overline{z}_i - \sum_j c_{ij}x_j}{q_i}, \tag{11}
$$

where $\tilde{\overline{z}}_i$ is an aspiration level for i^{th} objective function and q_i be a tolerance for $\tilde{z}_i$, subjectively chooses by Decision Maker (DM). However, this can be easily extended to a more general shape of membership functions.

Theorem 3.1. *Let x^* be a fuzzy efficient solution of problem (6). Then the set of all $\tilde{t}_\lambda$ where x^* is a fuzzy efficient solution in problem (10) is convex.*

Proof. Let x^* be a fuzzy efficient solution for problem $P(\tilde{t}_0)$ and $P(\tilde{t}_1)$. Assume $\tilde{t}_\lambda = \lambda\tilde{t}_0 + (1-\lambda)\,\tilde{t}_1, \lambda \in [0,1]$ holds. We must show that x^* is a fuzzy efficient solution for problem $P(\tilde{t}_\lambda)$. Let there exists $x \in S$ such that:

$$
\mu_{\tilde{Z}_i}^{\tilde{t}_\lambda}\left(x^*\right) = 1 - \frac{\overline{Z}_i - \sum\limits_{j;j\neq p} c_{ij}x_j^* - \left(\lambda t_0 + \left(1-\lambda\right)t_1\right)x_p^*}{q_i} \leq 1 - \frac{\overline{Z}_i - \sum_{j;j\neq p} c_{ij}x_j - \left(\lambda t_0 + \left(1-\lambda\right)t_1\right)x_p}{q_i} = \mu_{\tilde{Z}_i}^{\tilde{t}_\lambda}\left(x\right), \tag{12}
$$

for all $i = 1, 2, \ldots, k$. Therefore:

$$
\sum_{j;j\neq p} c_{ij}x_j^* + (\lambda t_0 + \left(1-\lambda\right)t_1)x_p^* \leq \sum_{j;j\neq p} c_{ij}x_j + (\lambda t_0 + \left(1-\lambda\right)t_1)x_p, \ i = 1, 2, \ldots, k. \tag{13}
$$

Since,

$$
\sum_{j;j\neq p} c_{ij}x_j^* = \lambda \sum_{j;j\neq p} c_{ij}x_j^* + \left(1-\lambda\right)\sum_{j;j\neq p} c_{ij}x_j^*, \ i = 1, 2, \ldots, k \tag{14}
$$

$$
\sum_{j;j\neq p} c_{ij}x_j = \lambda \sum_{j;j\neq p} c_{ij}x_j + \left(1-\lambda\right)\sum_{j;j\neq p} c_{ij}x_j, \ i = 1, 2, \ldots, k \tag{15}
$$

hence:

$$
\lambda\left(\sum_{j;j\neq p} c_{ij}x_j^* + t_0x_p^*\right) + (1-\lambda)\left(\sum_{j;j\neq p} c_{ij}x_j^* + t_1x_p^*\right) \leq \lambda\left(\sum_{j;j\neq p} c_{ij}x_j + t_0x_p\right) + (1-\lambda)\left(\sum_{j;j\neq p} c_{ij}x_j + t_1x_p\right), \ i = 1, 2, \ldots, k \tag{16}
$$

Therefore

$$
\lambda\left(\sum_{j;j\neq p} c_{ij}x_j^* + t_0x_p^*\right) \leq \lambda\left(\sum_{j;j\neq p} c_{ij}x_j + t_0x_p\right), i = 1, 2, \ldots, k \tag{17}
$$

or

$$(1-\lambda)\left(\sum_{j;j\neq p} c_{ij}x_j^* + t_1 x_p^*\right) \leq (1-\lambda)\left(\sum_{j;j\neq p} c_{ij}x_j + t_1 x_p\right), i=1,2,\ldots,k. \tag{18}$$

Set

$$A_i = \sum_{j;j\neq p} c_{ij}x_j^* + t_0 x_p^*,$$

$$B_i = \sum_{j;j\neq p} c_{ij}x_j + t_0 x_p,$$

$$C_i = \sum_{j;j\neq p} c_{ij}x_j^* + t_1 x_p^*,$$

$$D_i = \sum_{j;j\neq p} c_{ij}x_j + t_1 x_p, i=1,\ldots,k.$$

Therefore inequalities (16) imply: $A_i \leq C_i$ or $B_i \leq D_i$, $i=1,\ldots,k$. For $1\leq i\leq k$ we have the following procedure: If $A_i \leq C_i$, then $\mu_{\tilde{Z}_i}^{\tilde{t}_0}(x^*) \leq \mu_{\tilde{Z}_i}^{\tilde{t}_0}(x)$. Since x^* is fuzzy efficient for $P(\tilde{t}_0)$, (7) implies $\mu_{\tilde{Z}_i}^{\tilde{t}_0}(x^*) = \mu_{\tilde{Z}_i}^{\tilde{t}_0}(x)$, and hence $A_i = C_i$. Now, by (16) we must have $B_i \leq D_i$, and the similar way shows that $B_i = D_i$. Thus $A_i + B_i = C_i + D_i$. Assuming that $B_i \leq D_i$ leads to the same result. Therefore $\mu_{\tilde{Z}_i}^{\tilde{t}_\lambda}(x^*) = \mu_{\tilde{Z}_i}^{\tilde{t}_\lambda}(x)$, and hence x^* is a fuzzy efficient solution for $P(\tilde{t}_\lambda)$ by (7). □

3.2. The tolerance of objective coefficient

In two-phase approach proposed by Guua and Wu (1999), if one of the objectives to be fulfilled entirely, the fuzzy efficient solution may not be Pareto Optimal [22]. In this section, we recall an algorithm where the founded fuzzy efficient solution remains Pareto Optimal. Moreover, we obtain an interval as a tolerance for the selected coefficient of an objective function such that the fuzzy efficient and Pareto optimal solution does not change. First, we need to consider the following problem:

$$\begin{array}{ll} \max & \lambda \\ \text{subject to} & 1 \geq \mu_{\tilde{Z}_i}(x) = 1 - \frac{b_i - c_i^T x}{p_i} \geq \lambda \geq 0, \quad i=1,2,\ \ldots,k, \\ & x \in S \end{array} \tag{19}$$

Let the optimal solution be given by (x^*, λ^*). Now see Algorithm 2.

Step 1: If the optimal solution is unique then:

(a). If $\mu_{\tilde{Z}_i}(x^*) < 1$ for all $i=1,\ldots,k$, then x^* is fuzzy efficient and Pareto optimal [11]. Solving the following inequality for t_p, gives a tolerance for c_{wp}:

$$1 \geq 1 - \frac{b_w - \left(c_{w1}x_1^* + \cdots + t_p x_p^* + \cdots + c_{wn}x_n^*\right)}{p_w} \geq \lambda^*. \tag{20}$$

Stop.

(b). If $\mu_{\tilde{Z}_i}(x^*) = 1$ for some $1 \leq i \leq k$, then x^* is fuzzy efficient but it may not be Pareto optimal [11]. Go to Step 3.

If there exist multiple optimal solutions: go to Step 2.

Step 2: Let (x^{**}, λ^{**}) be the optimal solution of the following problem:

$$\begin{array}{ll} \max & \sum_{i=1}^{k} \lambda_i \\ \text{subject to} & 1 \geq \mu_{Z_i}(x) \geq \lambda_i \geq \mu_{Z_i}(x^*), \quad i=1,\ldots,k, \\ & x \in S \end{array} \tag{21}$$

(a). If $\mu_{\tilde{Z}_i}(x^{**}) < 1$ for all $i = 1, \ldots, k$, then x^{**} is fuzzy efficient and Pareto optimal [11]. Solving the following inequality for t_p, gives a tolerance for c_{wp}:

$$1 \geq 1 - \frac{b_w - \left(c_{w1}x_1^{**} + \cdots + t_p x_p^{**} + \cdots + c_{wn}x_n^{**}\right)}{p_w} \geq \lambda^{**}. \tag{22}$$

Stop.

(b). If $\mu_{\tilde{Z}_i}(x^{**}) = 1$ for some $1 \leq i \leq k$, however x^{**} is fuzzy efficient but it may not be Pareto optimal [11].

Step 3: Solve the following problem:

$$\begin{array}{ll} \max & \sum\limits_{s=1}^{p} n_s \\ \text{subject to} & Z_s(x) - n_s = Z_s(x^{**}) \quad s \in I_1, \\ & \mu_{Z_r}(x) = \mu_{Z_r}(x^{**}) \quad r \in I_2, \\ & x \in S;\ n_s \geq 0, \end{array} \tag{23}$$

where $I_1 = \{s : \mu_{Z_{\tilde{s}}}(x^{**}) = 1\}$ and $I_2 = \{r : \mu_{Z_{\tilde{r}}}(x^{**}) < 1\}$. The optimal solution, x^o, is fuzzy efficient and Pareto optimal.

Set $\lambda^o = 1 - \frac{b_w - c_w^T x^o}{p_w}$ and solving the following inequality for t_p in order to obtain a tolerance for c_{wp}:

$$1 \geq 1 - \frac{b_w - \left(c_{w1}x_1^{o} + \cdots + t_p x_p^{o} + \cdots + c_{wn}x_n^{o}\right)}{p_w} \geq \lambda^{o}. \tag{24}$$

Here, the following example illustrates the usefulness of the proposed approach.

Example 3.2. *Consider the following problem:*

$$\begin{array}{ll} \max & \tilde{z}_1 = \left(5, 7\frac{1}{2}, \frac{1}{2}\right)x_1 + \left(2, 6, \frac{1}{2}, \frac{1}{2}\right)x_2 + \left(1, 3, \frac{1}{2}, \frac{1}{2}\right)x_3 \\ \max & \tilde{z}_2 = \left(2, 4, \frac{1}{2}, \frac{1}{2}\right)x_1 + \left(3, 7, \frac{1}{2}, \frac{1}{2}\right)x_2 - \left(1, 3, \frac{1}{2}, \frac{1}{2}\right)x_3 \\ \max & \tilde{z}_3 = \left(1, 3, \frac{1}{2}, \frac{1}{2}\right)x_1 + \left(3, 4, \frac{1}{2}, \frac{1}{2}\right)x_2 + \left(3, 5, \frac{1}{2}, \frac{1}{2}\right)x_3 \\ \textit{subject to} & \left(3, 7, \frac{1}{2}, \frac{1}{2}\right)x_1 + \left(2, 4, \frac{1}{2}, \frac{1}{2}\right)x_2 + \left(4, 8, \frac{1}{2}, \frac{1}{2}\right)x_3 \preceq_R \left(10, 14, \frac{1}{2}, \frac{1}{2}\right), \\ & \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)x_1 + \left(\frac{1}{2}, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}\right)x_2 \preceq_R \left(1, 5, \frac{1}{2}, \frac{1}{2}\right), \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{25}$$

Now, we apply the Yager ranking function for the fuzzy coefficients. Thus, the problem reduces to:

$$\begin{array}{ll} \max & z_1 = 6x_1 + 4x_2 + 2x_3 \\ \max & z_2 = 3x_1 + 5x_2 - 2x_3 \\ \max & z_3 = 2x_1 + 3.5x_2 + 4x_3 \\ \text{subject to} & 5x_1 + 3x_2 + 6x_3 \leq 12, \\ & x_1 + x_2 \leq 3, \\ & x_1, x_2, x_3 \geq 0. \end{array} \tag{26}$$

Let the DM's aspiration levels for the objectives are 8, 10 and 4, respectively, and their tolerance threshold are 10, 8 and 7, respectively. For the sake of simplicity, we work with linear membership functions [12, 21]. The max-min problem leads to:

$$\begin{aligned} &\max && \lambda \\ &\text{subject to} && \lambda \leq 1-\tfrac{8-(6x_1+4x_2+2x_3)}{10} \leq 1, \\ &&& \lambda \leq 1-\tfrac{10-(3x_1+5x_2-2x_3)}{8} \leq 1, \\ &&& \lambda \leq 1-\tfrac{4-(2x_1+3.5x_2+4x_3)}{7} \leq 1, \\ &&& 5x_1+3x_2+6x_3 \leq 12, \\ &&& x_1+x_2 \leq 3, \\ &&& x_1,x_2,x_3 \geq 0, \\ &&& 0 \leq \lambda \leq 1. \end{aligned} \tag{27}$$

The optimal solution is $x^* = (x_1^*, x_2^*, x_3^*) = (0.92, 0.61, 0)$, $\lambda^* = 0.48$, $z_1(x^*) = 7.96$, $z_2(x^*) = 5.81$, $z_3(x^*) = 3.97$, $\mu_{z_1}(x^*) = 0.98$, $\mu_{z_2}(x^*) = \lambda^* = 0.48$, $\mu_{z_3}(x^*) = 0.99$. The optimal solution is unique and all the satisfaction degrees are strictly less than 1, hence x^* is fuzzy efficient and Pareto optimal.

In order to obtain an interval as a tolerance for c_{11} such that x^* remains fuzzy efficient and Pareto optimal, use the derived (x^*, λ^*) in inequality (20):

$$0.48 \leq 1-\frac{8-(0.92t+4\times 0.61)}{10} \leq 1 \Rightarrow t \in [0.39, 6.04]. \tag{28}$$

Solving the following interval programming problem shows that for all values $c_{11} \in [0.39, 6.04]$ the fuzzy efficient and Pareto optimal solution for both problems (26) and the following problem are the same:

$$\begin{aligned} &\max && z_1 = [0.39, 6.04]x_1 + 4x_2 + 2x_3 \\ &\max && z_2 = 3x_1 + 5x_2 - 2x_3 \\ &\max && z_3 = 2x_1 + 3.5x_2 + 4x_3 \\ &\text{subject to} && 5x_1+3x_2+6x_3 \leq 12, \\ &&& x_1+x_2 \leq 3, \\ &&& x_1,x_2,x_3 \geq 0. \end{aligned} \tag{29}$$

The max-min approach leads to:

$$\begin{aligned} &\max && \lambda \\ &\text{subject to} && \lambda \leq 1-\tfrac{8-([0.39,6.04]x_1+4x_2+2x_3)}{10} \leq 1, \\ &&& \lambda \leq 1-\tfrac{10-(3x_1+5x_2-2x_3)}{8} \leq 1, \\ &&& \lambda \leq 1-\tfrac{4-(2x_1+3.5x_2+4x_3)}{7} \leq 1, \\ &&& 5x_1+3x_2+6x_3 \leq 12, \\ &&& x_1+x_2 \leq 3, \\ &&& x_1,x_2,x_3 \geq 0, \\ &&& 0 \leq \lambda \leq 1. \end{aligned} \tag{30}$$

The optimal solution is $x_1^* = 0.92$, $x_2^* = 0.61$, $x_3^* = 0$ with $\lambda^* = 0.48$, that is the same solution for problem (27).

In order to obtain intervals as tolerances for c_{11} and c_{22} such that x^* remains fuzzy efficient and Pareto optimal, use the derived (x^*, λ^*) in inequality (20):

$$\begin{cases} 0.48 \leq 1-\frac{8-(0.92t_1+4\times 0.61)}{10} \leq 1 \Rightarrow t_1 \in [0.39, 6.04], \\ 0.48 \leq 1-\frac{10-(3\times 0.92+0.61t_2)}{7} \leq 1 \Rightarrow t_2 \in [4.99, 11.87]. \end{cases} \tag{31}$$

By substituting these two intervals and solving analogous problems, we have:

$$\begin{aligned}
\max \quad & \lambda \\
\text{subject to} \quad & -6.04x_1 - 4x_2 - 2x_3 + 10\lambda \leq 2, \\
& 6.04x_1 + 4x_2 + 2x_3 \leq 8, \\
& 3x_1 + 11.87x_2 - 2x_3 - 8\lambda \geq 2, \\
& 3x_1 + 11.87x_2 - 2x_3 \leq 10, \\
& -2x_1 - 3.5x_2 - 4x_3 + 7\lambda \leq 3, \\
& 2x_1 + 3.5x_2 + 4x_3 \leq 4, \\
& 5x_1 + 3x_2 + 6x_3 \leq 12, \\
& x_1 + x_2 \leq 3, \\
& x_1, x_2, x_3 \geq 0, \\
& 0 \leq \lambda \leq 1.
\end{aligned} \tag{32}$$

The optimal solution is $x_1^* = 0.92$, $x_2^* = 0.61$ and $x_3^* = 0$ with $\lambda^* = 0.48$, that is the same solution for problem (27).

3.3. Objective Function Removal

After removing s^{th} $(1 \leq s \leq k)$ objective function from problem (6), we have the following problem:

$$\begin{aligned}
\max \quad & \tilde{z}_i = \sum \tilde{c}_{ij} x_j, \ i = 1, 2, \ldots k; \ i \neq s \\
\text{subject to} \quad & x \in S = \left\{ x \in \mathfrak{R}^n : \tilde{A}x \preceq_R \tilde{b}, \ x \geq 0 \right\}.
\end{aligned} \tag{33}$$

Now, we want to verify if the fuzzy efficient solution x^* in problem (6) remains a fuzzy efficient solution in problem (33). The next theorem is considering this matter.

In the proof of the next theorem we use the linear membership function (11).

Theorem 3.3. *Let x^* be a fuzzy efficient solution of problem (6). If x^* is optimal for the following problem:*

$$\begin{aligned}
\min \quad & \tilde{z}_s = \textstyle\sum_j \tilde{c}_{sj} x_j \\
\textit{subject to} \quad & x \in S' = \left\{ x \in \mathfrak{R}^n : \tilde{A}x \preceq_R \tilde{b}, \tilde{c}x \succsim_R \tilde{c}x^*, x \geq 0 \right\}.
\end{aligned} \tag{34}$$

Then x^* is a fuzzy efficient solution of problem (33).

Proof. Let (by contradiction) x^* isn't a fuzzy efficient solution for problem (34). Therefore:

$$\begin{aligned}
\exists\, x \in S' : \ \mu_{\tilde{Z}_i}(x^*) = 1 - \frac{\overline{Z}_i - \sum_j c_{ij} x_j^*}{q_i} \leq 1 - \frac{\overline{Z}_i - \sum_j c_{ij} x_j}{q_i} = \mu_{\tilde{Z}_i}(x) \ : i = 1, \ldots, k, \ i \neq s, \\
\mu_{\tilde{Z}_l}(x^*) = 1 - \frac{\overline{Z}_l - \sum_j c_{lj} x_j^*}{q_l} < 1 - \frac{\overline{Z}_l - \sum_j c_{lj} x_j}{q_l} = \mu_{\tilde{Z}_l}(x)
\end{aligned} \tag{35}$$

for some $1 \leq l \leq k$, $l \neq s$. Since x^* is an optimal solution for problem (34), thus we obtain:

$$\tilde{c}_s x^* \preceq_R \tilde{c}_s x \Rightarrow R(\tilde{c}_s x^*) \leq R(\tilde{c}_s x) \Rightarrow c_s x^* \leq c_s x. \tag{36}$$

Therefore, we have:

$$\mu_{\tilde{Z}_s}(x^*) = 1 - \frac{\overline{Z}_s - \sum_j c_{sj} x_j^*}{q_s} \leq 1 - \frac{\overline{Z}_s - \sum_j c_{sj} x_j}{q_s} = \mu_{\tilde{Z}_s}(x) \Rightarrow \mu_{\tilde{Z}_s}(x^*) \leq \mu_{\tilde{Z}_s}(x). \tag{37}$$

But inequalities (35) and (37) lead to:

$$\begin{aligned}&\mu_{\tilde{Z}_i}(x^*) = 1 - \frac{\overline{Z}_i - \sum_j c_{ij}x_j^*}{q_i} \le 1 - \frac{\overline{Z}_i - \sum_j c_{ij}x_j}{q_i} = \mu_{\tilde{Z}_i}(x) : i = 1, \ldots, k, \\ &\bigwedge \mu_{\tilde{Z}_l}(x^*) < \mu_{\tilde{Z}_l}(x) \quad \text{for some } 1 \le l \le k.\end{aligned} \tag{38}$$

It means x^* is not a fuzzy efficient solution for problem (6). This is to contradict with the hypothesis of theorem. □

Example 3.4. *Consider the following problem:*

$$\begin{aligned}
\max \quad & \tilde{z}_1 = -\left(5, 9, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\max \quad & \tilde{z}_2 = -\left(8, 10, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(3, 5, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\max \quad & \tilde{z}_3 = \left(5, 11, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 - \left(1, 3, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\textit{subject to} \quad & \left(8, 12, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(20, 30, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(13, 17, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& x_1, x_2 \ge 0.
\end{aligned} \tag{39}$$

By using the Yager ranking function for the fuzzy coefficients the following problem will be obtained:

$$\begin{aligned}
\max \quad & z_1 = -7x_1 + x_2 \\
\max \quad & z_2 = -9x_1 + 4x_2 \\
\max \quad & z_3 = 8x_1 - 2x_2 \\
\text{subject to} \quad & 10x_1 + 25x_2 \le 1, \\
& 15x_1 + 15x_2 \le 15, \\
& x_1, x_2 \ge 0.
\end{aligned} \tag{40}$$

Let the DM's aspiration levels for the objectives are 5, 7 and 6, respectively, and their tolerance threshold are 10, 15 and 20, respectively. For the sake of simplicity, we work with linear membership functions. For finding a fuzzy efficient solution according to the first step of the two-phase approach (1999) we must solve the following problem:

$$\begin{aligned}
\max \quad & \lambda \\
\text{subject to} \quad & \lambda \le 1 - \tfrac{5-(-7x_1+x_2)}{10} \le 1, \\
& \lambda \le 1 - \tfrac{7-(-9x_1+4x_2)}{15} \le 1, \\
& \lambda \le 1 - \tfrac{6-(8x_1-2x_2)}{20} \le 1, \\
& 10x_1 + 25x_2 \le 1, \\
& 15x_1 + 15x_2 \le 15, \\
& x_1, x_2 \ge 0, \\
& 0 \le \lambda \le 1.
\end{aligned} \tag{41}$$

The optimal solution is $x^* = (x_1^*, x_2^*) = (0, 0.04)$, $\lambda^* = 0.5$. Since the optimal solution is unique, according to the first step of the two-phase approach, x^* is a fuzzy efficient solution of problem (39). Now, consider the following problem:

$$\begin{aligned}
\min \quad & \tilde{z} = \left(5, 11, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 - \left(1, 3, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\text{subject to} \quad & \left(8, 12, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(20, 30, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(13, 17, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& x_1, x_2 \ge 0.
\end{aligned} \tag{42}$$

By using the Yager ranking function for the fuzzy coefficients, we have the following:

$$\begin{aligned}
\min \quad & z = 8x_1 - 2x_2 \\
\text{subject to} \quad & 10x_1 + 25x_2 \leq 1 \\
& 15x_1 + 15x_2 \leq 15, \\
& x_1, x_2 \geq 0.
\end{aligned} \tag{43}$$

The optimal solution of problem (43) is $x^* = (x_1^*, x_2^*) = (0, 0.04)$. Therefore, according to theorem (2), x^* is a fuzzy efficient solution of the following problem that is arisen by removing the third objective function from problem (39):

$$\begin{aligned}
\max \quad & \tilde{z}_1 = -\left(5, 9, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\max \quad & \tilde{z}_2 = -\left(8, 10, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(3, 5, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \\
\text{subject to} \quad & \left(8, 12, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(20, 30, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_1 + \left(10, 20, \tfrac{1}{2}, \tfrac{1}{2}\right) x_2 \preceq_R \left(13, 17, \tfrac{1}{2}, \tfrac{1}{2}\right), \\
& x_1, x_2 \geq 0.
\end{aligned} \tag{44}$$

3.4. Objective Function Addition

Consider the new following problem arising from problem (6) by adding a new objective function:

$$\begin{aligned}
\max \quad & \tilde{z}_i = \sum \tilde{c}_{ij} x_j, \quad i = 1, 2, \ldots, k \\
\max \quad & \tilde{z}_{k+1} = \sum \tilde{d}_j x_j \\
\text{subject to} \quad & x \in S = \left\{ x \in \mathfrak{R}^n : \tilde{A}x \preceq_R \tilde{b},\ x \geq 0 \right\}.
\end{aligned} \tag{45}$$

Now, we want to verify if the fuzzy efficient solution, x^*, in problem (6) remains a fuzzy efficient solution for problem (45).

Theorem 3.5. *Let x^* be a fuzzy efficient solution of problem (6). Then, x^* is a fuzzy efficient solution of problem (45) if and only if x^* is an optimal solution of the following problem:*

$$\begin{aligned}
\max \quad & \tilde{z}_{k+1} = \sum \tilde{d}_j x_j \\
\textit{subject to} \quad & x \in S^{''} = \left\{ x \in \mathfrak{R}^n \ : \ \tilde{A}x \preceq_R \tilde{b}, \tilde{c}x \cong_R \tilde{c}x^*,\ x \geq 0 \right\}.
\end{aligned} \tag{46}$$

Proof. If x^* is an optimal solution for problem (46), obviously, x^* is fuzzy efficient for problem (45).
Conversely, let x^* be a fuzzy efficient solution for problem (6), but it is not optimal for problem (46). Thus, there exists $x \in S^{''}$ such that $\sum_j d_j x_j^* < \sum_j d_j x_j$. Therefore

$$\mu_{\tilde{Z}_{k+1}}(x^*) = 1 - \frac{\overline{z}_{k+1} - \sum_j d_j x_j^*}{q_{k+1}} < 1 - \frac{\overline{z}_{k+1} - \sum_j d_j x_j}{q_{k+1}} = \mu_{\tilde{Z}_{k+1}}(x). \tag{47}$$

Since x^* is a fuzzy efficient solution for problem (6), by (7) we must have $\mu_{\tilde{Z}_i}(x^*) = \mu_{\tilde{Z}_i}(x)$, $i = 1, \ldots, k$. $x \in S^{''}$ and inequality (47) imply:

$$\begin{aligned}
\mu_{\tilde{Z}_i}(x^*) = \mu_{\tilde{Z}_i}(x) \ : \ i = 1, \ldots, k, \\
\bigwedge \mu_{\tilde{Z}_{k+1}}(x^*) < \mu_{\tilde{Z}_{k+1}}(x),
\end{aligned} \tag{48}$$

which is a contradiction with fuzzy efficiency x^* for problem (45). $\square$

Example 3.6. *Consider problem* (39) *with its fuzzy efficient solution as* $x^* = (x_1^*, x_2^*) = (0, 0.04)$. *Now, consider the following problem:*

$$\begin{aligned} \max \quad & \tilde{z} = -\left(1,3,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \\ \textit{subject to} \quad & \left(8,12,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(20,30,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \preceq_R \left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \\ & \left(10,20,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(10,20,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \preceq_R \left(13,17,\tfrac{1}{2},\tfrac{1}{2}\right), \\ & x_1, x_2 \geq 0. \end{aligned} \tag{49}$$

By using the Yager ranking function for the fuzzy coefficients we will have the following problem:

$$\begin{aligned} \max \quad & z = -2x_1 + x_2 \\ \text{subject to} \quad & 10x_1 + 25x_2 \leq 1, \\ & 15x_1 + 15x_2 \leq 15, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{50}$$

The optimal solution is $x^* = (x_1^*, x_2^*) = (0, 0.04)$. Therefore, according to theorem (3), it can be concluded that x^* is a fuzzy efficient solution of the following problem that is arisen with adding a new objective function to problem (39):

$$\begin{aligned} \max \quad & \tilde{z}_1 = -\left(5,9,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \\ \max \quad & \tilde{z}_2 = -\left(8,10,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(3,5,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \\ \max \quad & \tilde{z}_3 = \left(5,11,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 - \left(1,3,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \\ \max \quad & \tilde{z}_4 = -\left(1,3,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \\ \text{subject to} \quad & \left(8,12,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(20,30,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \preceq_R \left(\tfrac{1}{2},\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}\right), \\ & \left(10,20,\tfrac{1}{2},\tfrac{1}{2}\right)x_1 + \left(10,20,\tfrac{1}{2},\tfrac{1}{2}\right)x_2 \preceq_R \left(13,17,\tfrac{1}{2},\tfrac{1}{2}\right), \\ & x_1, x_2 \geq 0. \end{aligned} \tag{51}$$

4. Conclusion

The post optimality analysis of fuzzy efficient solutions was presented in this paper. Three cases were considered: changing in one objective function coefficient, objective function removal and objective function addition. The convexity property of the set of parameters for which a given fuzzy efficient solution remains fuzzy efficient is proved. In the case that one objective function coefficient changes, a computational procedure was presented to obtain post optimality results. An algorithm based on the two-phase approach was used for obtaining the fuzzy efficient and Pareto optimal solution for the fuzzy multi-objective linear programming problem. The approach was also easy to implement, and we believe it could be incorporated in linear optimization software to enrich the post optimality analysis and to give more insight on the fuzzy efficient solutions. In addition, in this paper the post optimality analysis was examined for two cases, objective function removal and objective function addition. In each of these cases, we presented a theory which let us analyze the sensitivity of a given fuzzy efficient solution.

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