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Laguerre Wavelet-Galerkin Method for the Numerical Solution of One Dimensional Partial Differential Equations

S. C. Shiralashetti[1,*], L. M. Angadi[2] and S. Kumbinarasaiah[1]

1 Department of Mathematics, Karnatak University, Dharwad, Karnataka, India.

2 Department of Mathematics, Government First Grade College, Chikodi, Karnataka, India.

Abstract: In this paper, we proposed Laguerre wavelet-Galerkin method for the numerical solution of one dimensional partial differential equations. Here, we used Laguerre wavelets as a weight functions that are assumed basis elements which allow us to obtain the numerical solutions of the differential equations. The obtained numerical results are compared with the classical finite difference method and exact solution. Numerical test problems are considered to demonstrate the applicability and validity of the purposed technique.

Keywords: Galerkin method, Laguerre wavelet basis, differential equations, Finite difference method.

1. Introduction

Differential equations occur frequently in the fields of engineering and science such as gas dynamics, nuclear physics, atomic structures and chemical reactions. In most cases, we do not always find the exact solutions for these equations via analytical methods. In this case, it is very meaningful to give the high precision numerical solutions for this kind of problem by numerical methods. Recently, some of the numerical methods are used for the numerical solutions of differential equations. For example, Haar wavelet collocation method [1], Legendre wavelet collocation method [2], Series solution [3] etc.

The subject of wavelets has received much attention because of the comprehensive mathematical power and the good application potential of wavelets in many interesting physical problems. Wavelet functions have generated significant interest from both theoretical and applied research over the last few years. The name wavelet comes from the requirement that they should integrate to zero, waving above and below x-axis. The concepts for understanding wavelets were provided recently by Meyer, Mallat, Daubechies and many others. Since then, the number of applications where wavelets have been used has exploded. Many different types of wavelet functions have been presented over the past few years [4].

In wavelet theory, the contribution of orthogonal bases of compactly supported wavelet by Daubechies (1988) and multiresolution analysis based fast wavelet transform algorithm by Belkin (1991), wavelet based approximation of ordinary differential equations gained momentum in attractive way. Special interest has been dedicated to the construction of compactly supported smooth wavelet bases. As we have noted earlier that, spectral bases are infinitely differentiable but have global support. On the other side, bases functions used in finite-element methods have small compact support but poor continuity properties. Already we know that, spectral methods have good spectral localization but poor spatial localization, while

* *E-mail: shiralashettisc@gmail.com*

finite element methods have good spatial localization, but poor spectral localization. Wavelet bases execute to combine the advantages of both spectral and finite element bases. We can look forward to numerical methods based on wavelet bases to be capable to attain good spatial and spectral resolutions. Representation of a smooth function in terms of a series expansion using orthogonal polynomials is a fundamental idea in approximation theory and forms the basis of spectral methods of solution of differential equations with functional arguments. An approach to study differential equations is the use of wavelet function bases in place of other conventional piecewise polynomial trial functions in finite element type methods. Because of its implementation simplicity, the Galerkin method is considered the most widely used in applied mathematics [5, 6]. The advantage of wavelet-Galerkin method over finite difference or finite element method has lead to tremendous applications in science and engineering. To a certain extent, the wavelet technique is a strong competitor to the finite element method. Although the wavelet method provided an efficient alternative technique for solving differential equations numerically.

In this paper, we developed Laguerre wavelet-Galerkin method for the numerical solution of differential equations. This method is based on expanding the solution by Laguerre wavelets with unknown coefficients. The properties of Laguerre wavelets together with the Galerkin method are utilized to evaluate the unknown coefficients and then a numerical solution of the differential equation is obtained. The organization of the paper is as follows. Preliminaries of Laguerre wavelets are given section 2. Some results on Laguerre wavelets are given in section 3, Section 4 deals with Laguerre wavelet-Galerkin method for the solution of differential equations. Numerical Experiment is given in section 5. Finally, conclusions of the proposed work are discussed in section 6.

2. Preliminaries of Laguerre Wavelets

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, we have the following family of continuous wavelets [7]:

$$\psi_{a,b}(x) = |a|^{\frac{-1}{2}}\, \psi\left(\frac{x-b}{a}\right), \quad a, b \in R;\ a \neq 0$$

If we restrict the parameters a and b to discrete values as

$$a = a_0^{-k},\ b = nb_0 a_0^{-k},\ a_0 > 1,\ b_0 > 0$$

we have the following family of discrete wavelets

$$\psi_{k,n}(x) = |a_0|^{\frac{1}{2}}\, \psi\left(a_0^k x - nb_0\right),\ k, n \in Z$$

Where $\psi_{k,n}$ form a wavelet bases for a, b. In particular, when $a_0 = 2$ and $b_0 = 1$, then $\psi_{k,n}(x)$ forms an orthonormal basis. The Laguerre wavelets $\psi_{k,n}(x) = \psi(k, n, m, x)$ involve four arguments, k is assumed any positive integer, m is the degree of the Laguerre polynomials and it is the Normalized time. They are defined on the interval $[0, \infty)$ as

$$\psi_{k,n}(x) = \begin{cases} 2^{\frac{1}{2}}\bar{L}_m\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases} \tag{1}$$

Where

$$\bar{L}(x) = \frac{1}{m!} L_m(x) \tag{2}$$

$m = 0, 1, 2, 3, \ldots, M-1$. In equation (2) the coefficients are used for orthonormality. Here $L_m(x)$ are the Laguerre polynomials of degree m with respect to the weight function $W(x) = 1$ on the interval $[0, \infty)$ and satisfy the following recursive formula $L_0(x) = 1$, $L_1(x) = 1 - x$,

$$L_{m+2}(x) = \frac{(2m+3-x) L_{m+1}(x) - (m+1) L_m(x)}{m+2}, \ m = 0, 1, 2, 3, \ldots$$

For $k = 1$ and $n = 1$ in (1) and (2), then the Laguerre wavelets are given by

$$\begin{aligned}
\psi_{1,0}(x) &= \sqrt{2} \\
\psi_{1,1}(x) &= 2\sqrt{2}x(1-x) \\
\psi_{1,2}(x) &= \frac{\sqrt{2}}{4}(4x^2 - 12x + 7) \\
\psi_{1,3}(x) &= \frac{\sqrt{2}}{18}(-4x^3 + 24x^2 - 39x + 17) \\
\psi_{1,4}(x) &= \frac{\sqrt{2}}{24}\left(\frac{2}{3}x^4 - \frac{20}{3}x^3 + 21x^2 - \frac{73}{3}x + \frac{209}{24}\right) \\
\psi_{1,5}(x) &= \frac{\sqrt{2}}{120}\left(\frac{4}{15}x^5 + 4x^4 - \frac{62}{3}x^3 + \frac{136}{3}x^2 - \frac{167}{4}x + \frac{773}{60}\right) \quad \text{and so on.}
\end{aligned}$$

3. Results on Laguerre Wavelets

Theorem 3.1 ([7]). *The series solution $u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m} \psi_{n,m}(x)$ defined in Equation (1) using Laguerre wavelet method is converges to $u(x)$.*

Proof. $L^2(R)$ be the infinite dimensional Hilbert space and $\psi_{k,n}(x)$ defined as

$$\psi_{k,n}(x) = \begin{cases} 2^{\frac{1}{2}} \bar{L}_m\left(2^k x - 2n + 1\right), & \frac{n-1}{2^{k-1}} \le x < \frac{n}{2^{k-1}} \\ 0, & \text{Otherwise} \end{cases}$$

forms orthonormal basis. Let $u(x) = \sum_{i=0}^{M-1} C_{n,i} \psi_{n,i}(x)$, where $C_{n,i} = \langle u(x), \psi_{n,i}(x) \rangle$ for fixed n. Let us define the sequence of partial sums S_n of $\{S_n\}$, let S_n and S_m are the partial sums with $n \ge m$. Now we have to prove S_n is Cauchy sequence in Hilbert space. Choose $S_n = \sum_{i=0}^{n} C_{n,i} \psi_{n,i}(x)$. Now

$$\langle u(x), S_n \rangle = \left\langle u(x), \sum_{i=0}^{n} C_{n,i} \psi_{n,i}(x) \right\rangle = \sum_{i=m+1}^{n} |C_{n,i}|^2$$

We claim that

$$\left\| \sum_{i=m+1}^{n} C_{n,i} \psi_{n,i} \right\|^2 = \sum_{i=m+1}^{n} |C_{n,i}|^2 \, n > m$$

$$\|S_n - S_m\|^2 = \sum_{i=m+1}^{n} |C_{n,i}|^2$$

Now

$$\left\| \sum_{i=m+1}^{n} C_{n,i} \psi_{n,i}(x) \right\|^2 = \left\langle \sum_{i=m+1}^{n} C_{n,i} \psi_{n,i}(x), \sum_{i=m+1}^{n} C_{n,i} \psi_{n,i}(x) \right\rangle$$

By Bessel's Inequality, Since $\sum_{i=m+1}^{n} |C_{n,i}|^2 \le \|u(x)\|^2$. Therefore $\sum_{i=1}^{n} |C_{n,i}|^2$ is bounded and convergent. Hence $\left\| \sum_{i=m+1}^{n} C_{n,i}\psi_{n,i}(x) \right\|^2 \to 0$ as $m, n \to \infty$. This implies $\left\| \sum_{i=m+1}^{n} C_{n,i}\psi_{n,i}(x) \right\|^2 \to 0$. Therefore $\{S_n\}$ is a Cauchy sequence and it converges to K (say). We assert that $u(x) = K$. Now

$$\langle K - u(x), \psi_{n,i}(x)\rangle = \langle K, \psi_{n,i}(x)\rangle - \langle u(x), \psi_{n,i}(x)\rangle$$

$$\langle K - u(x), \psi_{n,i}(x)\rangle = \langle K, \psi_{n,i}(x)\rangle - \left\langle \lim_{n\to\infty} S_n, \psi_{n,i}(x)\right\rangle = 0$$

$$\langle K - u(x), \psi_{n,i}(x)\rangle = 0$$

Hence $u(x) = K$ and $\sum_{i=0}^{n} C_{n,i}\psi_{n,i}(x)$ converges to u (x) as $n \to \infty$ and proved. □

Theorem 3.2. *Laguerre wavelets $\{\psi_{i,j}\}$ are Uniformly Continuous on interval I.*

Corollary 3.3. *Laguerre wavelets $\{\psi_{i,j}\}$ are Uniformly Continuous on interval I then it is continuous.*

Theorem 3.4. *If $\psi_{i,j} : I \to R$ is Uniformly Continuous on subset I of R and $\{x_n\}$ is a Cauchy sequence in I then $\{\psi_{i,j}(x_n)\}$ is Cauchy sequence in R (where $\psi_{i,j}$ are Laguerre wavelets).*

Theorem 3.5. *Suppose that $u(x) = C^m[0,1]$ and $c^T\psi$ is the approximate solution using Laguerre wavelets. Then the error bound would be given by [7]*

$$\|E(x)\| \le \left\| \frac{2}{m!4^m 2^{m(k-1)}} \max_{x\in[0,1]} |u^m(x)| \right\|.$$

4. Method of Solution

Consider the one dimensional partial differential equation is of the form,

$$\frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u = f(x) \tag{3}$$

With boundary conditions

$$u(0) = a, \; u(1) = b \tag{4}$$

Where α, β are may be constant or either a functions of x or functions of u and $f(x)$ be a continuous function. Write the equation (3) as

$$R(x) = \frac{\partial^2 u}{\partial x^2} + \alpha\frac{\partial u}{\partial x} + \beta u - f(x) \tag{5}$$

where $R(x)$ is the residual of the Equation (3). When $R(x) = 0$ for the exact solution, $u(x)$ only which will satisfy the boundary conditions. Consider the trail series solution of the differential equation (3), $u(x)$ defined over $[0, 1)$ can be expanded as a Laguerre wavelet, satisfying the given boundary conditions which is involving unknown parameter as follows,

$$u(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=1}^{M} c_{i,j}\psi_{i,j}(x) \tag{6}$$

where $c'_{i,j}s$ are unknown coefficients to be determined. Accuracy in the solution is increased by choosing higher degree Laguerre wavelet polynomials. Differentiating Equation (6) twice with respect to x and substitute the values of in $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial x}$, u equation (5). To find $c'_{i,j}s$ we choose weight functions as assumed bases elements and integrate on boundary values together with the residual to zero [8].

$$i.e., \int_0^1 \psi_{1,j}(x) R(x)\, dx = 0, \; j = 1, 2, \ldots, n$$

then we obtain a system of linear equations, on solving this system, we get unknown parameters. Then substitute these unknowns in the trail solution, numerical solution of Equation (3) is obtained.

5. Numerical Experiment

Problem 5.1. *First, consider the second order equation [9],*

$$\frac{\partial^2 u}{\partial x^2} + u = x^2,\ 0 \le x \le 1 \tag{7}$$

With boundary conditions:

$$u\,(0) = 0,\ \ u\,(1) = 0 \tag{8}$$

The implementation of the Equation (7) as per the method explained in section 3 is as follows:

The residual of Equation (7) can be written as:

$$R\,(x) = \frac{\partial^2 u}{\partial x^2} + u - x^2 \tag{9}$$

Now choosing the weight function $w\,(x) = x(1-x)$ for Laguerre wavelet bases to satisfy the given boundary conditions (8), i.e. $\Psi\,(x) = w\,(x) \times \psi\,(x)$

$$\begin{aligned}
\Psi_{1,1}(x) &= \psi_{1,1}(x) \times x(1-x) = 2\sqrt{2}x(1-x)\\
\Psi_{1,2}(x) &= \psi_{1,2}(x) \times x(1-x) = \frac{\sqrt{2}}{4}(4x^2 - 12x + 7)x(1-x)\\
\Psi_{1,3}(x) &= \psi_{1,3}(x) \times x(1-x) = \frac{\sqrt{2}}{18}(-4x^3 + 24x^2 - 39x + 17)x(1-x)
\end{aligned}$$

Assuming the trail solution of (7) for $k = 1$ and $m = 3$ is given by

$$u(x) = c_{1,1}\Psi_{1,1}\,(x) + c_{1,2}\Psi_{1,2}\,(x) + c_{1,3}\Psi_{1,3}\,(x) \tag{10}$$

Then the Equation (10) becomes

$$u(x) = c_{1,1}2\sqrt{2}(x - x^2) + c_{1,2}\frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 19x^2 + 7x) + c_{1,3}\frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 63x^3 - 56x^2 + 17x) \tag{11}$$

Differentiating Equation (11) twice w.r.t. x we get,

$$\frac{\partial^2 u}{\partial x^2} = c_{1,1}2\sqrt{2}(-2) + c_{1,2}\frac{\sqrt{2}}{4}(-48x^2 + 96x - 38) + c_{1,3}\frac{\sqrt{2}}{18}(80x^3 - 336x^2 + 378x - 112) \tag{12}$$

Using Equation (11) and (12), then Equation (9) becomes,

$$\begin{aligned}
R\,(x) &= c_{1,1}2\sqrt{2}(-2) + c_{1,2}\frac{\sqrt{2}}{4}(-48x^2 + 96x - 38) + c_{1,3}\frac{\sqrt{2}}{18}(80x^3 - 336x^2 + 378x - 112)\\
&\quad + \left(\begin{array}{l} c_{1,1}2\sqrt{2}(x - x^2) + c_{1,2}\frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 19x^2 + 7x) +\\ c_{1,3}\frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 63x^3 - 56x^2 + 17x)\end{array}\right) - x^2\\
\Rightarrow R\,(x) &= c_{1,1}2\sqrt{2}(x - x^2 - 2) + c_{1,2}\frac{\sqrt{2}}{4}(-4x^4 + 16x^3 - 67x^2 + 103x - 38)\\
&\quad + c_{1,3}\frac{\sqrt{2}}{18}(4x^5 - 28x^4 + 143x^3 - 392x^2 + 395x - 112) - x^2
\end{aligned} \tag{13}$$

This is the residual of Equation (7). The "weight functions" are the same as the bases functions. Then by the weighted Galerkin method, we consider the following:

$$\int_0^1 \Psi_{1,j}(x)\, R(x)\, dx = 0; j = 1, 2, 3 \tag{14}$$

For $j = 1, 2, 3$ in Equation (14), i.e. $\int_0^1 \Psi_{1,1}(x)\, R(x)\, dx = 0$, $\int_0^1 \Psi_{1,2}(x)\, R(x)\, dx = 0$, $\int_0^1 \Psi_{1,3}(x)\, R(x)\, dx = 0$

$$\Rightarrow \left(\frac{12}{5}\right) c_{1,1} + \left(\frac{139}{210}\right) c_{1,2} + \left(\frac{11}{42}\right) c_{1,3} + \frac{\sqrt{2}}{10} = 0 \tag{15}$$

$$\left(\frac{139}{210}\right) c_{1,1} + \left(\frac{589}{1008}\right) c_{1,2} + \left(\frac{12857}{45360}\right) c_{1,3} + \frac{19\sqrt{2}}{1680} = 0 \tag{16}$$

$$\left(\frac{11}{42}\right) c_{1,1} + \left(\frac{12857}{45360}\right) c_{1,2} + \left(\frac{26623}{187110}\right) c_{1,3} + \frac{\sqrt{2}}{360} = 0 \tag{17}$$

We have three equations (15), (16) and (17) with three unknown coefficients i.e. $c_{1,1}$, $c_{1,2}$ and $c_{1,3}$. Solving this by suitable method, we obtain the values of $c_{1,1} = -0.0827$, $c_{1,2} = 0.1736$ and $c_{1,3} = -0.2213$. Substituting these values in Equation (11), we get the numerical solution; the absolute errors are presented in table 1 and comparison with exact solution of Equation (7) is $u(x) = \frac{\sin(x)+2\sin(1-x)}{\sin(1)} + x^2 - 2$ in figure 1.

x	Exact solution	Absolute error by present method at $k = 1$				Absolute error by FDM
		$M = 3$	$M = 5$	$M = 6$	$M = 7$	
0.1	-0.0095547	1.8591e-04	3.0020e-05	3.4395e-07	5.4859e-08	7.3600e-05
0.2	-0.0188974	1.9932e-04	1.9425e-05	8.4005e-07	8.4005e-10	1.2990e-04
0.3	-0.0276349	3.4432e-04	2.1394e-05	6.3998e-07	8.1039e-08	1.6870e-04
0.4	-0.0351804	3.2147e-04	3.2782e-05	1.1330e-06	1.3677e-08	1.9020e-04
0.5	-0.0407591	1.8409e-05	1.4091e-05	1.3556e-07	8.8366e-08	1.9480e-04
0.6	-0.0434159	4.6402e-05	1.5586e-05	1.2558e-06	1.8372e-09	1.8310e-04
0.7	-0.0420253	1.5404e-05	1.0029e-05	4.7666e-07	8.8241e-08	1.5460e-04
0.8	-0.0353022	1.1753e-04	2.5687e-05	1.0746e-06	1.2934e-08	1.1590e-04
0.9	-0.0218150	2.2751e-04	1.4782e-05	2.7686e-07	5.8824e-08	6.3100e-05

Table 1. **Comparison of the absolute error for the Problem 5.1**

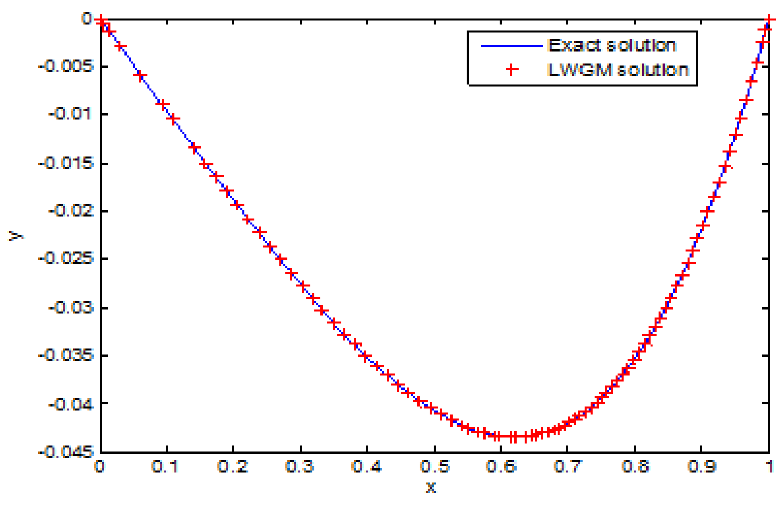

Figure 1. **Comparison of numerical and exact solutions of the Problem 5.1**

Problem 5.2. *Consider [10]*

$$\frac{\partial^2 u}{\partial x^2} - u = x - 1,\ 0 \le x \le 1 \tag{18}$$

With boundary conditions:

$$u(0) = 0,\ u(1) = 0 \tag{19}$$

Which has the exact solution

$$u(x) = -\frac{1}{1-e^2}e^x + \frac{e^2}{1-e^2}e^{-x} - x + 1.$$

By applying the method explained in the section 3, we obtain the constants $c_{1,1} = 0.0617$, $c_{1,2} = 0.1028$ and $c_{1,3} = -0.0884$. Substituting these values in Equation (11), we get the numerical solution; the absolute errors are presented in table 2 and comparison with exact solution of Equation (18) in figure 2.

x	Exact solution	Absolute error by present method at $k=1$				Absolute error by FDM	Absolute error by WGM [10]
		$M=3$	$M=5$	$M=6$	$M=7$		
0.1	0.0265183	5.6173e-05	1.5870e-06	2.1653e-07	3.3629e-08	2.0600e-05	1.1169e-03
0.2	0.0442945	3.5255e-05	7.6746e-06	5.3406e-07	5.2781e-10	3.4100e-05	1.0556e-03
0.3	0.0545074	9.3776e-05	4.5703e-06	4.0268e-07	4.9668e-08	4.1700e-05	5.4474e-05
0.4	0.0582599	6.5725e-05	5.1430e-06	7.1460e-07	8.2258e-09	4.4200e-05	1.5723e-03
0.5	0.0565906	2.0088e-05	9.3789e-06	8.6651e-07	5.4068e-08	4.2700e-05	3.4459e-03
0.6	0.0504834	9.6090e-05	3.2062e-06	7.9136e-07	1.1475e-09	3.7900e-05	5.1333e-03
0.7	0.0408782	9.9122e-05	6.9267e-06	2.9611e-07	5.4063e-08	3.0600e-05	6.1570e-03
0.8	0.0286795	1.3171e-05	8.3266e-06	6.8098e-07	8.0976e-09	2.1400e-05	6.0045e-03
0.9	0.0147663	8.8254e-05	3.3572e-06	1.7047e-07	3.6099e-08	1.1000e-05	4.1374e-03

Table 2. **Comparison of the absolute error for the Problem 5.2**

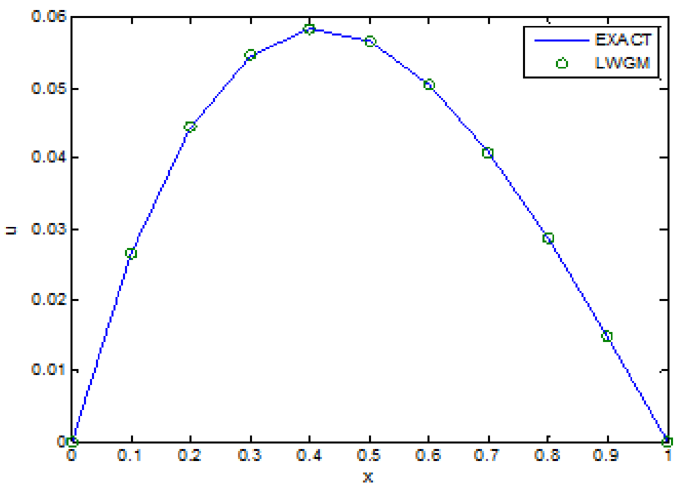

Figure 2. **Comparison of numerical and exact solutions of the Problem 5.2**

Problem 5.3. *Next, consider [11]*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} = -1,\ 0 \le x \le 1 \tag{20}$$

With boundary conditions:

$$u(0) = 0,\ u(1) = 0 \tag{21}$$

Which has the exact solution $u(x) = x - \left(\frac{e^x - 1}{e - 1}\right)$.

By applying the method explained in the section 3, we obtain the constants $c_{1,1} = 0.1954$, $c_{1,2} = -0.1467$ and $c_{1,3} = 0.1742$. Substituting these values in Equation (11), we get the numerical solution; the absolute errors are presented in table 3 and comparison with exact solution of Equation (20) in figure 3.

x	Exact solution	Absolute error by present method for $k = 1$ and				Absolute error by FDM
		$M = 3$	$M = 5$	$M = 6$	$M = 7$	
0.1	0.038793	1.0905e-04	9.3842e-05	3.9531e-07	6.5877e-08	1.5380e-03
0.2	0.071149	5.0048e-05	1.2226e-05	1.0677e-06	3.8889e-09	2.9140e-03
0.3	0.096390	1.5728e-04	5.0008e-06	7.2338e-07	9.7717e-08	4.0770e-03
0.4	0.113769	1.1343e-04	1.3045e-05	1.3896e-06	1.1982e-08	4.9700e-03
0.5	0.122459	3.9041e-05	1.0008-05	8.9067e-08	1.0446e-07	5.5260e-03
0.6	0.121546	1.7899e-04	7.9935e-06	1.4694e-06	2.0470e-09	5.6650e-03
0.7	0.110020	1.9479e-04	3.9923e-06	6.1416e-07	1.0237e-07	5.2960e-03
0.8	0.086764	5.0958e-05	8.9944e-06	1.2224e-06	1.1917e-08	4.3130e-03
0.9	0.050545	1.3099e-04	5.9956e-06	3.4909e-07	3.4909e-07	2.5950e-03

Table 3. **Comparison of the absolute error for the Problem 5.3**

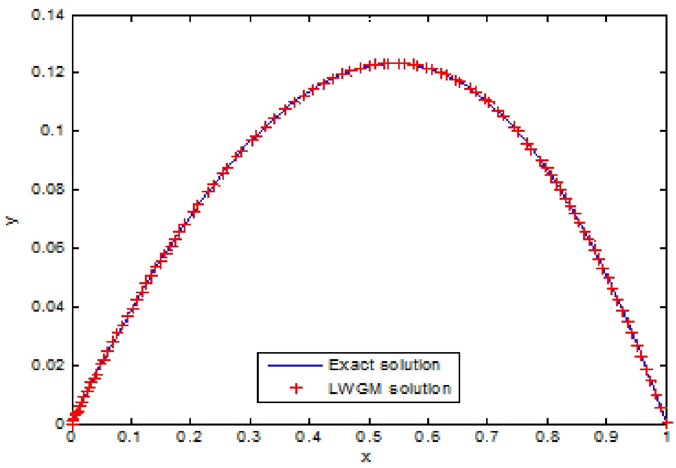

Figure 3. **Comparison of numerical and exact solutions of the Problem 5.3**

Problem 5.4. *Now, consider the singular boundary value problem [12],*

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + u = x^2 - x^3 - 9x + 4, \ \ 0 \leq x \leq 1 \tag{22}$$

With boundary conditions:

$$u(0) = 0, \ \ u(1) = 0 \tag{23}$$

As in the previous example, we obtained the numerical solution and are presented in comparison with exact solution $u(x) = x^2 - x^3$ in table 4 and figure 4.

x	Exact solution	Numerical solution		
		LWGM		FDM
		For $k = 1$ and $m = 3$	For $k = 1$ and $m = 5$	
0.1	0.009000	0.010673	0.009482	-0.014709
0.2	0.032000	0.033159	0.032067	-0.013726
0.3	0.063000	0.063290	0.063049	-0.002584

x	Exact solution	Numerical solution		
		LWGM		FDM
		For $k = 1$ and $m = 3$	For $k = 1$ and $m = 5$	
0.4	0.096000	0.095881	0.096028	0.015387
0.5	0.125000	0.125034	0.125018	0.036564
0.6	0.144000	0.144429	0.144061	0.056572
0.7	0.147000	0.147623	0.147039	0.070066
0.8	0.128000	0.128350	0.128037	0.070568
0.9	0.081000	0.080816	0.081075	0.050294

Table 4. **Comparison of numerical solution and exact solution of the Problem 5.4**

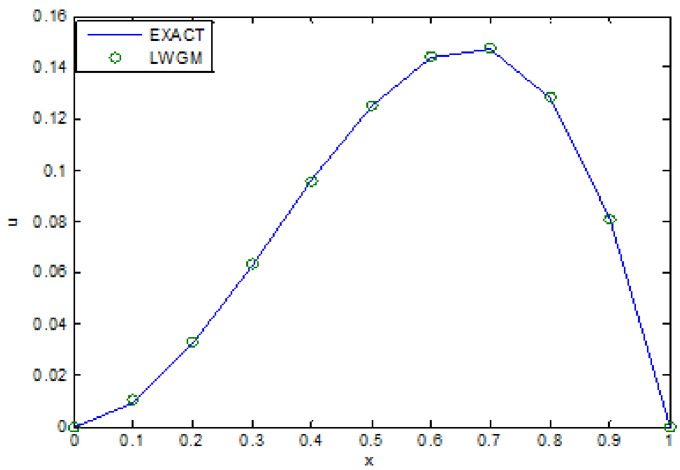

Figure 4. **Comparison of numerical and exact solution of the Problem 5.4**

Problem 5.5. *Finally, consider the non linear singular boundary value problem [13],*

$$\frac{\partial^2 u}{\partial x^2} - u^2 = 2\pi^2 \cos(2\pi x) - \sin^4(2\pi x), \;\; 0 \le x \le 1 \tag{24}$$

With boundary conditions:

$$u(0) = 0, \;\; u(1) = 0 \tag{25}$$

As in the previous example, we obtained the numerical solution and are presented in comparison with exact solution $u(x) = \sin^2(\pi x)$ in table 5 and figure 5.

x	Exact solution	Numerical solution by present method (For $k = 1$ and $m = 3$)
0.1	0.095491502812526	0.077123753683308
0.2	0.345491502812526	0.371584280913036
0.3	0.654508497187474	0.689818949681055
0.4	0.904508497187474	0.910317324117415
0.5	1.000000000000000	0.973879745067861
0.6	0.904508497187474	0.873875910671372
0.7	0.654508497187474	0.646503456937673
0.8	0.345491502812526	0.361046538324776
0.9	0.095491502812526	0.110134408316507

Table 5. **Comparison of numerical solution and exact solution of the Problem 5.5**

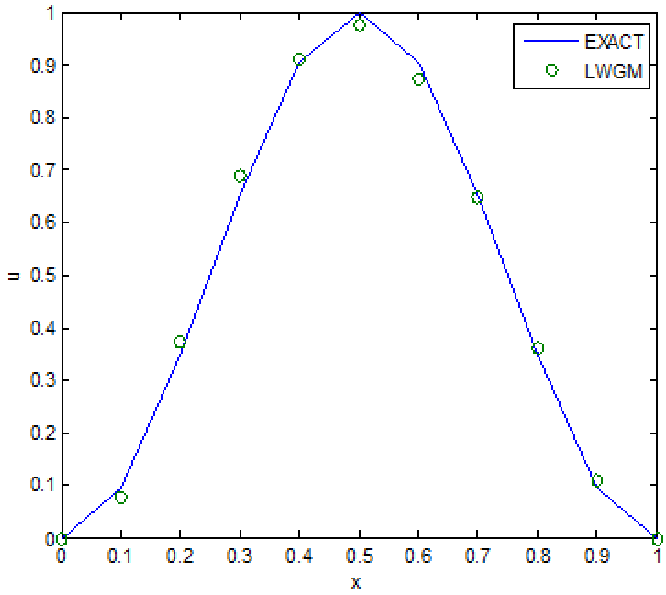

Figure 5. **Comparison of numerical and exact solution of the Problem 5.5**

6. Conclusion

In this paper, we proposed the Wavelet-Galerkin method for the numerical solution of differential equations using Laguerre wavelets. From the tables and figures which show that the numerical solutions obtained by the proposed method are better than FDM and nearer to the exact solution. As increasing the values of M we get more accuracy in the numerical solution which represented in the above tables. Hence the Wavelet-Galerkin method using Laguerre wavelets is effective for solving differential equations. By small change in present method we can apply for higher order also.

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