Int. J. Math. And Appl., 6(1–E)(2018), 983–987

ISSN: 2347-1557

Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Commutative Results for Rings

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Abstract: In this paper, we provided two commutativity theorems are : If R is a semi prime ring and there exist a fixed positive integer $m > 1$ such that either (i) $[[a, b]^m - [a^m, b^m], a] = 0$ or $[(a \circ b)^m - (a^m \circ b^m), a] = 0$, then R is commutative ring. (ii) For all a, b in R there exists a positive integer $m = m(a, b) > 1$ such that $(ab)^m = ba$, then R is commutative ring.

Keywords: Commutativity ring, associative ring, Center.

1. Introduction

In this paper we focus on certain basic definitions of some algebraic concepts which we consider essential for proving the commutative rings. A descriptive survey of the work done on the commutativity of associative rings by Herstein, Bell, Johnsen, Qutcalt, Yaqub, Quadri and Abu-Khuzam is also given. Throughout this paper, R will be semi prime ring with a center $Z(R)$. For any a, b in R as usual, $[a, b] = ab - ba$ and $(a \circ b) = ab + ba$, are the well known Lie and Jordan products respectively. In this paper, the authors together with M.A.Khan [6] replaced the associative product of the ring R by the above defined non-associative products in the identity $(ab)^2 = a^2b^2$ [8] and investigated the commutativity of associative structure. The result is as follows: "If R is 2-torsion free ring with unity 1 in which either $[a, b]^2 = [a^2, b^2]$ or $(a \circ b)^2 = a^2 \circ b^2$, then R is commutative". Now, we extend this result further and prove commutative results.

2. Preliminaries

Definition 2.1 (Associative Ring). *An associative ring R, sometimes called a ring in short, is an algebraic system with two binary operations addition "+" and multiplication "." such that*

(1). The elements of R form an abelian group under "+" and a semi group under "."

(2). Multiplication "." is distributive on the right as well as on the left over addition "+", that is $(x + y)z = xz + yz$, $z(x + y) = zx + zy$ for all x, y ,z in R.

Definition 2.2 (Non-Associative Ring). *A non-associative ring R is an additive abelian group in which a multiplication is defined, which is distributive over addition on the left as well as on the right, that is*

$$(x + y)z = xz + yz, z(x + y) = zx + zy \text{ for all } x, y, z \in R.$$

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A non-associative ring is different from an associative ring in which the full associative law of multiplication is no longer assumed to be associative. That is, it is not necessarily to be associative. It strongly implies that the associative law of multiplication has not been done away with and it has merely weakened. examples for non-associative rings are alternative rings, Lie rings and Jordon rings. In 1930 alternative ring are defined by Artin and Max Zorn.

Definition 2.3 (Alternative Ring)**.** *An alternative ring R is a ring in which $(xx)\,y = x\,(xy)$, $y\,(xx) = (yx)\,x$ for all $x, y \in R$. These equations are said to be the left and right alternative laws respectively.*

Definition 2.4 (Lie Ring)**.** *A Lie ring R is a ring in which the multiplication is anti-commutative, that is, $x^2 = 0$ (implying $xy = -yx$) and the Jacobi identity $(xy)\,z + (yz)\,x + (zx)\,y = 0$ for all $x, y, z \in R$ is satisfied.*

Definition 2.5 (Flexible Ring)**.** *If in a ring R, the identity $(x.y.z) = 0$, that is, $(xy)\,x = x(yx)$ for all x,y in R holds, then R is called flexible ring. Alternative, commutative, anti commutative and there by Jordan and Lie rings are flexible.*

Definition 2.6 (Jordan Ring)**.** *A Jordan ring R is a ring in which products are commutative, that $xy = yx$ and is satisfy the Jordan identity $(xy)\,x^2 = x\,(yx)^2$ for all x, y in R.*

Definition 2.7 (Associator)**.** *The associator (x, y, z) is defined by $(x, y, z) = (xy)z - x(yz)$ for all x, y, z, in a ring. Associator is a key role in the study of non-associative rings. It can be consider as a measure of the non-associativity of a ring. This definition given by Max Zorn defined that a finite alternative division ring is associative. In terms of associators, a ring is called left alternative if $(x, x, y) = 0$ right alternative if $(y, x, x) = 0$, for all $x, y \in R$ and alternative if both the conditions hold.*

Definition 2.8 (Commutative Ring)**.** *If the multiplication in a ring R is such that $xy = yx$ for all $x, y \in R$ in this case R is called a commutative ring.*

A non-commutative ring differs from commutative ring in that the multiplication is not assumed to be commutative. That is, we need not assume $xy = yx$ for all $x, y \in R$ as an axiom. It does not mean that elements $x, y \in R$ such that $xy \neq yx$. The ring of 2×2 matrices over rationales and the ring of real quaternion due to Hamilton are the examples of non-commutative rings.

Definition 2.9 (Commutator)**.** *The commutator (x,y) is defined by $(x, y) = xy - yx$ for all x,y in ring.*

Definition 2.10 (Nucleus)**.** *By the Nucleus N of ring R, we mean the set of all elements n in R such that* $(n, R, R) = (R, n, R) = (R.R.n) = 0$.

Definition 2.11 (Center of a Ring)**.** *Center $Z(R)$ of a ring is defined as follows:*

$$Z(R) = \{x \in R/xy - yx, \;\; \textit{for every } y \in R\}$$

Definition 2.12 (Characteristic of a Ring)**.** *If there exists a positive integer n such that $na = 0$ for every element a of a ring R, the smallest positive integer is called characteristic of R (usually written as char R). If no such positive integer exists, R is said to have characteristic zero. Obviously if char R is not equal to m, then $ma = 0$ for some $a \in R$ implies that* $a = 0$.

Definition 2.13 (Divisibility)**.** *We define a ring R to be m-divisible (m, a set of natural numbers) if $mx = 0$ implies $x = 0$ for all x in R.*

Definition 2.14 (Division Ring). *A ring R is said to be a division ring if its non-zero elements form a group with respect to multiplication.*

Definition 2.15 (Assosymmetric Ring). *An assosymmetric ring R, in which $(x, y, z) = (p(x)), (p(y)), (p(z))$ where P is any permutation of x, y, z in R.*

Definition 2.16 (Prime Ring). *A ring R is prime if A and B are ideals of R such that $AB = 0$ then either $A = 0$ or $B = 0$.*

Definition 2.17 (Semi Prime Ring). *A ring R is semi prime if for any ideal A or R, $A^2 = 0$ implies $A = 0$. These rings are also referred to as rings free from trivial ideals.*

Definition 2.18 (Nilpotent Ring). *A ring is called nilpotent if there is a fixed positive integer "t" such that every product involving "t" elements is zero.*

Definition 2.19 (Standard Ring). *A ring is defined to be standard in case the following two identities hold $(wx, y, z) + (xz, y, w) + (wz, y, x) = 0$ and $(x, y, z) + (z, x, y) - (x, z, y) = 0$.*

Definition 2.20 (Primitive Ring). *A ring is defined primitive in case it possesses a regular maximal right ideal which contains no two sided ideal of the ring other than the zero ideal.*

Definition 2.21 (Weakly Standard Ring). *A ring R is defined to be weakly standard if R is flexible and if the following two identities hold in R. $((w, x)y, z) = 0$ and $(w, (x, y), z) = 0$.*

Definition 2.22 (Accessible Ring). *A ring R is called accessible in case it satisfies the identities:*

(1). $(x, y, z) + (z, x, y) - (x, z, y) = 0$

(2). $((w, x), y, z) = 0$ *for all w, x,y and z in R.*

Definition 2.23 (Periodic Element). *An element x belongs to R is called a periodic element if there exist distinct m, n belongs to Z^+ such that $x^n = x^m$.*

Definition 2.24 (Periodic Ring). *A ring is called a periodic ring if every x in R, there exists distinct positive integers $m = m(x)$, $n = n(x)$ such that $x^m = x^n$. Due to Chacron R is periodic if and only for each $x \in R$, there exists a positive integers $k = k(x)$ and a polynomial $f(\lambda) = f_x(\lambda)$ with integer co-efficient such that $x^k = x^{k+1} f(x)$.*

Definition 2.25 (S-Unital Ring). *A ring R is called a left (respectively right) s-unital ring if $x \in Rx$ (respectably $x \in xR$) for each $x \in R$. Further R is called a s-unital if it is both left as well as s-unital, i.e., if $x \in xR \cap Rx$, for each $x \in R$.*

Definition 2.26 (Torsion-Free Ring). *A Ring R is said to be m-torsion free if $mx = 0$ implies $x = 0$ for all x in R.*

Definition 2.27 (Reduced Ring). *A ring is called reduced if $N = \{0\}$, where N is the set of nilpotent elements of R.*

Definition 2.28 (Simple Ring). *A ring R is said to be simple whenever A is an ideal of R, then either $A = R$ or $A = 0$.*

Definition 2.29 (Semi-Simple Ring). *A ring R is semi simple in case the radical (i,e., the maximal ideal consisting of all nilpotent elements) is the zero ideal. Obviously a simple ring is prime, which in turn is free from trivial ideals.*

Definition 2.30 (Potent Ring). *An element x of R is called potent if $x^n = x$ for some positive integer $n = n(x) > 1$.*

3. Main Results

Theorem 3.1. *Let R be a semi prime ring and $m > 1$ be a fixed positive integer. If R satisfies any one of the following identities:*

(1). $[[a,b]^m - [a^m, b^m], a] = 0$,

(2). $[[a,b]^m - [a^m, b^m], b] = 0$,

(3). $[[(a \circ b)^m - [a^m \circ b^m], a] = 0$,

(4). $[[(a \circ b)^m - [a^m \circ b^m], b] = 0$ *for all a, b in R,then in every case R is a commutative ring.*

We know that the Boolean ring satisfying $a^2 = a$, is necessarily commutative and so in such rings $(ab)^2 = ab$ and $(ab)^2 = ba$. But there exits non Boolean rings satisfying $(ab)^2 = ab$ or $(ab)^2 = ba$. Very recently Sercoid and Mac Hale [7] have proved the commutativity of rings with $(ab)^2 = ab$ and $(ab)^{m(a,b)} = ab$. In [6] the commutativity of rings satisfying $(ab)^2 = ba$ has been investigated. Now we can generalize the mentioned result [6] as follows:

Theorem 3.2. *If R be a ring such that $(ab)^m = ba$ for all a,b in R, where $m = m(a,b) > 1$ is an integer. Then R is a commutative ring.*

Proof of Theorem 3.1. If R is a semi prime ring satisfying the hypothesis of the theorem, then it is isomorphic to a subdirectsum of prime rings R_i each of which has a homomorphic image of R satisfies the hypothesis placed on R. Thus we may assume that the ring R is prime satisfying any one of the identities $(1) - (4)$. By posner's theorem [2], the central quotient of R is central simple algebra over a field.

Case 1: Let the ground field be finite. Then, the center $Z(R)$ of R is a finite integral domain and R is equal to it's central quotient of R. Hence R is a matrix ring $M_r(F)$ for some $r \geq 1$ and some field F.

Case 2: Let the ground field be infinite and $p(a,b) = 0$ be the polynomial identity for R. We write, $p = p_0 + p_1 + \cdots + p_n$, where p_i is the homogeneous polynomial in a, b, then $p_0 = p_1 = \cdots = p_m = 0$ for all a, b in R, since the center of R is infinite. Thus $p_0 = p_1 = \cdots = p_m = 0$ is also valid in the central quotient of R. Thus $p = p_0 + p_1 + \cdots + p_m = 0$ is satisfied by elements in $A \otimes_F L$, where A is a central quotient of R, and $F = Z(A)$, L any field extension of ?. In particular taking L to be a splitting field of A, $A \otimes_F L = M_r(L)$. Thus $p = 0$ is satisfied by elements in $M_r(L)$. Now, we claim that in every case $r = 1$. Let e_{ij}, $1 \leq i, j \leq r$, be the matrix in $M_2(F)$ with 1 on the position (i,j) and with zeros elsewhere.

(i). If R satisfies (1) or (2), then $p(e_{11}, e_{11} + e_{12}) \neq 0$.

(ii). If R satisfies (3) or (4), then again $p(e_{11}, e_{12} + e_{22}) \neq 0$.

Thus in every case we get a contradiction and hence $r = 1$, since $r = 1$, the central quotient is contained in the respective ground field and R itself is commutative. The ring of 3×3 strictly upper triangular matrices over a ring provides an example to show that the above theorem is not valid for arbitrary rings. A ring R without a proper nil ideal is necessarily semi prime. Hence the following Corollary is a special case of our theorem. But it may be somewhat interesting to give an easy direct proof of the same. □

Corollary 3.3. *If R is a ring without proper nil ideals and $m > 1$ is a fixed positive integer. R satisfies one of the polynomial identities (1)-(4), then R must be commutative.*

Proof. Each of the conditions $(1)-(4)$ is a polynomial identity $p(a,b)=0$, where $p(a,b)$ is a polynomial in two non commutative variables with rational integral coefficients at least one of which is equal to 1. Moreover, none of the rings $M_2(GF(p))$ (p a prime) satisfies this identity. In fact, for $e_{11} \in M_2(GF(p))$, we have $p(e_{11}, e_{11}+e_{12}) \neq 0$ in the case (1) or (2), and $p(e_{11}, e_{11}+e_{22}) \neq 0$. in the case (3) or (4). Hence the ring R is a commutative ring by using theorem of T.Kezlan [1]. □

Proof of Theorem 3.2. R satisfies the condition $(ab)^m = ba$ for all a, b in R and $m = m(a,b) > 1$. Clearly $ab = 0$ implies $ba = 0$. It follows that for any nilpotent element a in R, (ab) is nilpotent for all b in R. Thus the nilpotent elements of R annihilate R on both sides, and are therefore, central. Now for a in R, there exists $n = n(a)$ such that $a^{2n} = a^2$ and $2n \neq 2$. Thus R is a periodic ring with central nilpotent elements. Hence, commutativity of ring R follows by the theorem of Herstein [3]. □

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