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Solving Multi-Objective Mathematical Programming Problems in Fuzzy Approach

C. Loganathan[1] and M. Lalitha[2,*]

1 Department of Mathematics, Maharaja Arts and Science College, Coimbatore, Tamilnadu, India.

2 Department of Mathematics, Kongu Arts & Science College, Erode, Tamilnadu, India.

Abstract: In this paper we compare the solution of multi objective nonlinear programming problem with this solution obtained in Zimmermann's method. Zimmermann used membership function to solve the multi objective nonlinear programming problems. We have used α-cut to solve the multi objective nonlinear programming problems.

Keywords: Fuzzy nonlinear programming, Triangular fuzzy number, optimal solution, multi objective,α- cut, Fuzzy number, Fuzzy set theory.

1. Introduction

Recent developments in multiobjective programming by Geoffrion. In a multi objective nonlinear programming problem applied to real life model the data can rarely be determined exactly with certainty and precision. The concept of maximizing decision was initially proposed by Bellman and Zadeh [2], zadeh [1, 3, 4]. By adopting this concept o fuzzy sets was applied in mathematical programs firstly by Zimmermann [5]. Many practical problems cannot be represented by nonlinear programming model. Therefore, attempts were made to develop more general mathematical programming methods and many significant advances have been made in the area of multi objective nonlinear programming. Several authors in the literature have studied the fuzzy multi objective quadratic programming problems. The fuzzy nonlinear programming problem is not just an alternative or even a superior way of analyzing a given problem, it's useful in solving problems in which difficult or impossible to use due to the inherent qualitative imprecise or subjective nature of the problem formulation or to have an accurate solution Mont and Wolfe [7, 8] show interesting results with convex functions and related scalar objective programs . Zimmermann [9, 10] first applied fuzzy programming to multi objective linear programming problems by using the concept given by Bellman and Zadeh [2]. Zimmermann [9, 10] first classified fuzzy mathematical programming (FMP) method into twodifferent models namely symmetric and non symmetric models. Leung [12] classified (FMP) into the following four categories:

(1). A fuzzy objective and fuzzy constraints

(2). A fuzzy objective and precise constraints

* *E-mail: lali_math@rediffmail.com*

(3). A precise objective and fuzzy constraints

(4). Robust programming (one of the possibilistic programming

Here we present a fuzzy programming approach to some crisp multi objective decision making (MODM) problems. A mathematical model for MODM problem can be stated as
Find $X = (x_1 x_2 \ldots x_n)^T$. So as to

$$\text{Maximize (Minimize)} \;\; [f_1(x), f_2(x), \ldots, f_k(x)], \quad k = 1, 2, \ldots, K \tag{1}$$

Subject to

$$g_i(x) \; (\leq, \; =, \; \geq) \; b_i, \;\; i = 1, 2, \ldots, m \tag{2}$$

$$x_j \geq 0, \qquad\qquad j = 1, 2, \ldots, n \tag{3}$$

Where $f_j(x)$, $j \in J$ are the benefit maximization objectives, $f_i(x)$, $i \in I$ are the cost minimization objectives and linear (or) non-linear $I \cup J = 1, 2, \ldots, K$. It is noted that all functions $f_k(x)$, $k = 1, 2, \ldots, K$ and $g_i(x)$, $i = 1, 2, \ldots, m$ may be linear or non-linear. If all the objective functions are maximization type, then the problem is known as vector maximization problem. If all are of minimization, then it is known as vector minimization problem.

2. Multi-Objective Non-Linear Programming Model (Vector Maximum Problem)

A mathematical model can be stated as:
Find $X = (x_1 x_2 \ldots x_n)^T$. So as to

$$\text{Maximize} \;\; Z_k(x) = \sum_{j=1}^{n} c_j^k x_j^{\alpha_j}, \quad k = 1, 2, \ldots, K \tag{4}$$

Subject to

$$\sum_{j=1}^{n} a_{ij} x_j \; (\leq, \; =, \; \geq) \; b_i, \;\; i = 1, 2, \ldots, m \tag{5}$$

$$x_j \geq 0, \qquad\qquad j = 1, 2, \ldots, n \tag{6}$$

It is assumed that the objective functions are crisp but the objectives are conflicting in nature. It is also assumed that the problem is feasible and there exits an optimal compromise solution. We apply fuzzy programming approach to find an optimal compromise solution. The steps of the method are as follows:

2.1. Zimmerman'S Method

Step 1: Solve the multi-objective non-linear programming problem by using any non- linear programming algorithm, considering only one of the objectives at a time and ignoring all others. Repeat the process K times for K different objective functions [6]. Let $x^{(1)}, x^{(2)}, \ldots, x^{(k)}$ be the ideal solutions for the respective objective functions.

Step 2: Using all the above ideal solutions in Step 1, construct a pay-off matrix of size K by K. Then from the pay-off matrix estimate the lower bound (L_k) and the upper bound (U_k) for the kth objective function Z_k as:

$$L_k \leq Z_k \leq U_k, \quad k = 1, 2, \ldots, K$$

Step 3: Define a fuzzy non-linear membership function $(\mu(Z_k(x))$ [9, 11, 13] for the kth objective function Z_k, $k = 1, 2, \ldots, K$, [12].

$$\mu(Z_k(x) = \begin{cases} 0 & \text{if } Z_k \leq L_k \\ 1 - \frac{(U_k - Z_K)}{(U_k - L_k)} & \text{if } L_k \leq Z_k \leq U_k \\ 1 & \text{if } Z_k \leq U_k \end{cases} \tag{7}$$

Step 4: Use the above membership functions to formulate a crisp model by introducing an augmented variable λ

$$\text{Maximize}: \ 1 - \frac{(U_k - Z_K)}{(U_k - L_k)}, \ k = 1, 2, \ldots, K \tag{8}$$

$$\text{Minimize}: \ \frac{(U_k - Z_K)}{(U_k - L_k)}, \qquad k = 1, 2, \ldots, K \tag{9}$$

Subject to (5) and (6) can be further simplified as:

$$\text{Minimize}: \quad \lambda \tag{10}$$

Subject to

$$\sum_{j=1}^{n} c_j^k x_i + (U_k - L_k)\lambda \geq U_k, \qquad k = 1, 2, \ldots, K \tag{11}$$

$$\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq) \ b_i, \ \ i = 1, 2, \ldots, m \tag{12}$$

$$\lambda \geq 0, \ \ x_j \geq 0, \qquad j = 1, 2, \ldots, n \tag{13}$$

Step 5: Solve the crisp model (as stated in equation (10). (13)) by an NLP algorithm and find the optimal compromise solution X^*. Evaluate all the objective functions at the optimal compromise solution X^*.

3. Numerical Example

A numerical example with two objective functions, three constraints and two variables is considered to illustrate the solution procedure.

Find $X = (x_1, x_2)^T$. So as to

$$\text{Maximize } = \begin{cases} Z_1 = 2x_1 + 3x_2 - 2x_1^2 \\ Z_2 = 2x_1 + 3x_2 - 4x_1^2 \end{cases}$$

Subject to

$$x_1 + 4x_2 \leq 4$$

$$x_1 + x_2 \leq 2$$

For the first objective function the ideal solution is obtained as:

$$X^{(1)} = \begin{pmatrix} x_1^{(1)} = 0.3 \\ x_2^{(1)} = 0.921 \end{pmatrix}$$

and

$$Z_1 = 3.183$$

For the second objective function the ideal solution is obtained as:

$$X^{(2)} = \begin{pmatrix} x_1^{(2)} = 0.1 \\ x_2^{(2)} = 0.9839 \end{pmatrix}$$

and

$$Z_2 = 3.112$$

A pay-off matrix is formulated as:

	Z_1	Z_2
$x^{(1)}$	3.183	3
$x^{(2)}$	3.132	3.112

From the pay-off matrix, lower bound and the upper bound are estimated as:

$$3.132 \leq Z_1 \leq 3.183$$
$$3 \leq Z_2 \leq 3.112$$

Using the membership functions as defined in equation (7) and introducing and augmented variable a crisp model is formulated as:

$$\text{Minimize}: \quad \lambda$$

Subject to

$$2x_1 + 3x_2 - 2x_1^2 + 0.1\lambda \geq 3.183$$
$$2x_1 + 3x_2 - 4x_1^2 + 0.112\lambda \geq 3.112$$
$$x_1 + 4x_2 \leq 4$$
$$x_1 + x_2 \leq 2$$
$$x_1, x_2, \lambda \geq 0$$

Finally, the crisp model is solved to find the optimal compromise solution as:

$$X^* = \begin{pmatrix} x_1 = 0.62 \\ x_2 = 0.8 \\ \lambda = 0.327 \end{pmatrix}$$

The values of the objective function at X* is obtained as:

$$Z_1^* = 2.8712, Z_2^* = 2.1024$$

If we consider a vector minimum NLP problem, then the same fuzzy programming method can be used. However, one should redefine the membership functions. Other steps remain unchanged.

4. Modified Method (Main Result)

A mathematical model of multi objective non- linear programming problem in real plane is [14, 15].

Find $X = (x_1 x_2 \dots x_n)^T$. So as to

$$\text{Maximize } Z_k(x) = \sum_{j=1}^{n} c_j^k x_j^{\alpha_j},\ k = 1, 2, \dots, K \tag{14}$$

Subject to

$$\begin{aligned} &\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq)\ b_i, \quad i = 1, 2, \dots, m, \\ &x_j \geq 0, \quad j = 1, 2, \dots, n \end{aligned} \tag{15}$$

In our result we assume the variable set $X = (x_1 x_2 \dots x_n)^T$ and right hand side vector b_i of constraints are in fuzzy domain. Let x_i's and b_i's be fuzzy triangular numbers and the matrix A is in crisp form whose elements are real numbers.

4.1. Algorithm

Step 1: Define the membership function corresponding to X as

$$\mu_{x_i}(x) = \begin{cases} 0 & \text{if } x_i \preceq \underline{x_i} \\ 1 - \frac{\overline{x_i} - x_i}{\overline{x_i} - \underline{x_i}} & \text{if } \underline{x_i} \preceq x_i \preceq \overline{x_i} \\ 1 & \text{if } x_i \succeq \overline{x_i} \end{cases} \tag{16}$$

Step 2: Use α-cut to make the fuzzy system to crisp and use general method to solve the system. For any $\alpha \in [0, 1]$

$$\frac{\overline{x_i} - x_i}{\overline{x_i} - \underline{x_i}} = \alpha \tag{17}$$

If $x_i = (1 - \alpha)\overline{x_i} + \alpha \underline{x_i}$.

Step 3: Using this α-cut we change the multi objective non-linear programming problem as:

$$\text{Maximize / Minimize } \ Z_k(x) = \sum_{j=1}^{n} c_j^k ((1 - \alpha)\overline{x_j} + \alpha x_j),\ 0 \leq \alpha \leq 1,\ k = 1, 2, \dots, m \tag{18}$$

Subject to

$$\sum_{j=1}^{n} a_{ij} ((1 - \alpha)\overline{x_j} + \alpha x_j)\ (=, \geq)\ (1 - \alpha)\overline{b_i} + \alpha b_i,\ i = 1, 2, \dots, m \tag{19}$$

And $x_j \geq 0$, $j = 1, 2, \dots, n$.

Step 4: By putting $\alpha = 0$ and $\alpha = 1$ obtain the lower bound (x_i) and upper bound $(\overline{x_i})$ of the optimal solution of that MONLPP (4).

Step 5: Obtain the optimal solution of the MOLPP by average of the lower and upper bound of the solution. Also by taking the average of the point we get the point where optimal solution will exist. Repeat this process k-time for k different objective functions [17].

5. Numerical Example

A numerical example with two objective functions, two variables and two constraints is considered [16].

Find $X = (\tilde{x}_1, \tilde{x}_2)^T$. So as to

$$\text{Maximize} \quad = \begin{cases} Z_1 = 2\tilde{x}_1 + 3\tilde{x}_2 - 2\tilde{x}_1^2 \\ Z_2 = 2\tilde{x}_1 + 3\tilde{x}_2 - 4\tilde{x}_1^2 \end{cases}$$

Subject to

$$\tilde{x}_1 + 4\tilde{x}_2 \leq \tilde{4}$$
$$\tilde{x}_1 + \tilde{x}_2 \leq \tilde{2}$$
$$\tilde{x}_1, \tilde{x}_2 \geq 0$$

We use fuzzy triangular numbers:

$$\tilde{4} = [2, 4, 6]$$
$$\tilde{2} = [1, 2, 4]$$

Using the equation, the MOLPP becomes

$$\text{Maximize} \quad \{Z_1 = (1-\alpha)2\overline{x_1} + 2\underline{x_1} + (1-\alpha)3\overline{x_2} + \alpha 3\underline{x_2} - [(1-\alpha)2\overline{x_1^2} + \alpha 2\underline{x_1^2}]$$
$$\text{Maximize} \quad \{Z_2 = (1-\alpha)2\overline{x_1} + 2\underline{x_1} + (1-\alpha)3\overline{x_2} + \alpha 3\underline{x_2} - [(1-\alpha)4\overline{x_1^2} + \alpha 4\underline{x_1^2}]$$

Subject to

$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)4\overline{x_2} + \alpha 4\underline{x_2} \leq (1-\alpha)6 + 2\alpha$$
$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)\overline{x_2} + \alpha\underline{x_2} \leq (1-\alpha) + \alpha$$
$$\overline{x_1}, \underline{x_1}, \overline{x_2}, \underline{x_2} \geq 0$$

We solve by taking the first objective function

$$Maximize \{Z_1 = (1-\alpha)2\overline{x_1} + 2\underline{x_1} + (1-\alpha)3\overline{x_2} + \alpha 3\underline{x_2} - [(1-\alpha)2\overline{x_1^2} + \alpha 2\underline{x_1^2}]$$

$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)4\overline{x_2} + \alpha 4\underline{x_2} \leq 6 - 4\alpha$$
$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)\overline{x_2} + \alpha\underline{x_2} \leq 4 - 3\alpha$$
$$\overline{x_1}, \underline{x_1}, \overline{x_2}, \underline{x_2} \geq 0$$

For α= 0 the MOLPP becomes,

$$Z_1 = 2\tilde{x}_1 + 3\tilde{x}_2 - 2\tilde{x}_1^2$$

Subject to

$$\tilde{x}_1 + 4\tilde{x}_2 \leq 6$$

$$\tilde{x}_1 + \tilde{x}_2 \le 4$$
$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

and the optimal solution is at $(0.3125, 1.421)$ with $Z_1 = 4.6927$. For $\alpha = 1$ the MONLPP becomes,

$$Z_1 = 2\tilde{x}_1 + 3\tilde{x}_2 - 2\tilde{x}_1^2$$

Subject to

$$\tilde{x}_1 + 4\tilde{x}_2 \le 2$$
$$\tilde{x}_1 + \tilde{x}_2 \le 1$$
$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

and the optimal solution is at $(0.3125, 0.4219)$ with $Z_1 = 1.6954$. Hence the optimal solution of MONLPP occur $\left(\frac{0.3125+0.3125}{2}, \frac{1.421+0.4219}{2}\right) = (0.3125, 0.9213)$ with optimal objective an value $\frac{4.6927+1.6954}{2} = 3.19405$. For the second objective function the MONLPP is

$$Maximize \left\{ Z_2 = (1-\alpha)2\overline{x_1} + 2\underline{x_1} + (1-\alpha)3\overline{x_2} + \alpha 3\underline{x_2} - [(1-\alpha)4\overline{x_1^2} + \alpha 4\underline{x_1^2}] \right.$$

Subject to

$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)4\overline{x_2} + \alpha 4\underline{x_2} \le 6 - 4\alpha$$
$$(1-\alpha)\overline{x_1} + \alpha\underline{x_1} - [(1-\alpha)\overline{x_2} + \alpha\underline{x_2} \le 4 - 3\alpha$$
$$\overline{x_1}, \underline{x_1}, \overline{x_2}, \underline{x_2} \ge 0$$

For α= 0 the MOLPP becomes,

$$Z_2 = 2\tilde{x}_1 + 3\tilde{x}_2 - 4\tilde{x}_1^2$$

Subject to

$$\tilde{x}_1 + 4\tilde{x}_2 \le 6$$
$$\tilde{x}_1 + \tilde{x}_2 \le 4$$
$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

and the optimal solution is at $(0.1, 1.483)$ with $Z_2 = 4.609$. For α= 1 the MONLPP becomes,

$$Z_2 = 2\tilde{x}_1 + 3\tilde{x}_2 - 4\tilde{x}_1^2$$

$$\tilde{x}_1 + 4\tilde{x}_2 \le 2$$
$$\tilde{x}_1 + \tilde{x}_2 \le 1$$
$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

And the optimal solution is at $(0.1, 0.489)$ with $Z_2 = 1.627$. Hence the optimal solution of MONLPP occur $\left(\frac{0.1+0.1}{2}, \frac{1.483+0.489}{2}\right) = (0.1, 0.986)$ with optimal objective an value $\frac{4.609+1.627}{2} = 3.118$.

6. Conclusion

The modified method defined by the authors in this paper using α- cut and fuzzy triangular numbers extends the solution to an interval on the real line and hence generalizes Zimmermann's method. Zimmermann's method is guarantees stable and a crisp fixed solution to multi-objective mathematical programming problems. We conclude that a number of fuzzy optimal solutions are possible on the considered interval.

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