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Generalized Hyers-Ulam Stability for Ladder Network and Fibonacci Sequence

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Abstract: We enumerate the Hyers-Ulam-Rassias stability of the homogeneous linear difference equations of Ladder network and Fibonacci sequence with initial conditions by using Z-Transforms.

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1. Introduction

S.M. Ulam [17] posed a question about the stability of functional equations in 1940. In the following year the question was affirmatively answered by D.H. Hyers [7] for Cauchy additive functional equation in Banach spaces. Since then several mathematicians have been extensively examined the stability problems for other functional equations in various directions (see [4, 12, 16, 18]). A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations, integral equations and difference equations. Now a days some authors are very interested in proving the Hyers-Ulam stability of difference equations of linear and non-linear recurrences (see [3, 9, 10, 13, 14]).

In this paper, we prove the Hyers-Ulam-Rassias stability of linear difference equation of Ladder network and Fibonacci sequence of the form

$$i(l+2) - 3\,i(l+1) + i(l) = 0 \tag{1}$$

$$f(s+2) - f(s+1) - f(s) = 0 \tag{2}$$

with initial conditions

$$i(0) = 0 \text{ and } i(1) = 1 \tag{3}$$

$$f(0) = 0 \text{ and } f(1) = 1 \tag{4}$$

by using Z-Transforms method.

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2. Preliminaries

Now, we will give the definition of Hyer-Ulam-Rassias stability of the homogeneous linear difference equations (1) and (2) with (3) and (4) respectively.

Definition 2.1. *We say that the homogeneous linear difference equation (1) has the Hyers-Ulam-Rassias stability with (3), if for every $\epsilon > 0$ there exists a positive constant $\boldsymbol{L}$ such that $i(l)$ be a function satisfies the inequality*

$$|i(l+2) - 3\ i(l+1) + i(l)| \leq \epsilon\ \theta(l),$$

with (3) then there exists a function $j(l)$ satisfying (1) with $j(0) = 0$ and $j(1) = 1$ such that $|i(l) - j(l)| \leq \boldsymbol{L}(\epsilon)\ \theta(l)$.

Definition 2.2. *Let the homogeneous linear difference equation (2) has the Hyers-Ulam-Rassias stability, if for every $\epsilon > 0$ there exists a positive constant $\boldsymbol{L}$ such that $f(s)$ be a function satisfying*

$$|f(s+2) - f(s+1) + f(s)| \leq \epsilon\ \theta(s),$$

with (4) then there exists a function $g(s)$ satisfying (2) with $g(0) = 0$ and $g(1) = 1$ such that $|f(s) - g(s)| \leq \boldsymbol{L}(\epsilon)\ \theta(s)$.

3. Hyers-Ulam-Rassias Stability of Ladder Network (1)

Theorem 3.1. *For every $\epsilon > 0$ and $\theta : (0, \infty) \to (0, \infty)$ be a function, there exists a positive constant $\boldsymbol{L}$ such that a function $i : (0, \infty) \to \mathbb{F}$ satisfies the inequality*

$$|i(l+2) - 3\ i(l+1) + i(l)| \leq \epsilon\ \theta(l), \tag{5}$$

for each value of l with initial condition (3), then there exists a solution function $j : (0, \infty) \to \mathbb{F}$ of the difference equation (1) with $j(0) = 0$ and $j(1) = 1$ such that $|i(l) - j(l)| \leq \boldsymbol{L}(\epsilon)\ \theta(l)$, for all $l > 0$.

Proof. If we define a function $u : (0, \infty) \to \mathbb{F}$ such that $u(l) = i(l+2) - 3\ i(l+1) + i(l)$, for all $l > 0$. Also, in view of (5), we have $|u(l)| \leq \epsilon\ \theta(l)$. Now, taking Z-Transforms to $u(l)$, we get

$$Z\left[u(l)\right] = U(z) = \left(z^2 - 3z + 1\right) I(z) - z\left(z - 3\right)\ i(0) - z\ i(1). \tag{6}$$

In view of (6), a function $i_0 : (0, \infty) \to \mathbb{F}$ is a solution of (1) if and only if

$$\left(z^2 - 3z + 1\right) I(z) - z\left(z - 3\right)\ i_0(0) - z\ i_0(1) = 0. \tag{7}$$

Using the initial conditions (3) in (6), we have

$$U(z) = \left(z^2 - 3z + 1\right) I(z) - z. \tag{8}$$

Since $z^2 - 3z + 1 = (z - s)\ (z - t)$, where $s = \dfrac{3 + \sqrt{5}}{2}$ and $t = \dfrac{3 - \sqrt{5}}{2}$. Then (8) becomes,

$$I(z) = \frac{U(z)}{(z-s)\ (z-t)} + \frac{z}{(z-s)\ (z-t)}. \tag{9}$$

Now, we define a function $j(l) = \frac{s^l - t^l}{s - t}\ i(1)$, then applying Z-Transforms to $j(l)$, we get

$$Z\left[j(l)\right] = J(z) = \frac{z\ i(1)}{(z-s)\ (z-t)}.$$

Hence

$$(z-s)\ (z-t)\ J(z) - z(z-3)\ i(0) - z\ i(1) = 0. \tag{10}$$

Since we have $j(0) = i(0) = 0$ and $j(1) = i(1) = 1$. Now,

$$\begin{aligned} Z\left[j(l+2) - 3\ j(l+1) + j(l)\right] &= Z\left[j(l+2)\right] - 3\ Z\left[j(l+1)\right] + Z\left[j(l)\right] \\ &= \left(z^2 - 3z + 1\right)\ J(z) - z\left(z-3\right)\ j(0) - z\ j(1) \\ Z\left[j(l+2) - 3\ j(l+1) + j(l)\right] &= 0. \qquad \text{[from (10)]} \end{aligned}$$

Since Z is one-to-one operator, it holds that $j(l+2) - 3\ j(l+1) + j(l) = 0$. Therefore, $j(l)$ is a solution of (1). Then we have

$$Z\left[i(l)\right] - Z\left[j(l)\right] = I(z) - J(z) = \frac{U(z)}{(z-s)\ (z-t)} = Z\left[r(l) * u(l)\right],$$

where $r(l) = \frac{1}{z}\left\{\frac{s^l - t^l}{s-t}\right\}$. Since Z-Transforms is linear and one-to-one, we have $i(l) - j(l) = (r(l) * u(l))$. Now, taking modulus on both sides, we get

$$|i(l) - j(l)| = |r(l) * u(l)| = \left|\sum_{l=-\infty}^{\infty} r\left(l-k\right)\ u(l)\right| \leq \sum_{l=-\infty}^{\infty} \left|r\left(l-k\right)\right|\ \left|u(l)\right| \leq \mathbf{L}(\epsilon)\ \theta(l)$$

for every $l > 0$, where $\mathbf{L} = \sum_{l=-\infty}^{\infty} \left|r\left(l-k\right)\right| = \sum_{l=-\infty}^{\infty} \left|\frac{1}{z}\left\{\frac{s^{l-k} - t^{l-k}}{s-t}\right\}\right|$ exists for each value of l. Then by the virtue of Definition 2.1, the linear difference equation (1) has the Hyers-Ulam-Rassias stability. □

4. Hyers-Ulam-Rassias stability of Fibonacci sequence (2)

Theorem 4.1. *For every $\epsilon > 0$ and $\theta : (0,\infty) \to (0,\infty)$ be a function, there exists a positive constant $\boldsymbol{L}$ such that a function $f : (0,\infty) \to \mathbb{F}$ satisfies the inequality*

$$|f(s+2) - f(s+1) + f(s)| \leq \epsilon\ \theta(s), \tag{11}$$

for each value of s with initial conditions (4), then there exists a solution function $g : (0,\infty) \to \mathbb{F}$ of the difference equation (2) with $g(0) = 0$ and $g(1) = 1$ such that $|f(s) - g(s)| \leq \boldsymbol{L}(\epsilon)\ \theta(s)$, for all $s > 0$.

Proof. If we define a function $u : (0,\infty) \to \mathbb{F}$ such that $u(s) = f(s+2) - f(s+1) + f(s)$, for all $s > 0$. Also, in view of (11), we have $|u(s)| \leq \epsilon\ \theta(s)$. Now, taking Z-Transforms to $u(s)$, we get

$$Z\left[u(s)\right] = U(z) = \left(z^2 - z + 1\right) F(z) - z\left(z-1\right) f(0) - zf(1) \tag{12}$$

In view of (12), a function $f_0 : (0,\infty) \to \mathbb{F}$ is a solution of (2) if and only if

$$\left(z^2 - z + 1\right) F(z) - z\left(z-1\right)\ f_0(0) - z\ f_0(1) = 0. \tag{13}$$

Using the initial conditions (4) in (12), we have

$$U(z) = \left(z^2 - z + 1\right) F(z) - z. \tag{14}$$

Since $z^2 - z + 1 = (z-c)\ (z-d)$, where $c = \dfrac{1+\sqrt{-3}}{2}$ and $d = \dfrac{1-\sqrt{-3}}{2}$. Then (14) becomes,

$$F(z) - \frac{z}{(z-c)\ (z-d)} = \frac{U(z)}{(z-c)\ (z-d)}. \tag{15}$$

Now, we define a function $g(s) = \dfrac{c^s - d^s}{c-d}\ f(1)$, then applying Z-Transforms to $g(s)$, we get

$$Z\left[g(s)\right] = G(z) = \frac{z\ f(1)}{(z-c)\ (z-d)}.$$

Hence

$$(z-c)\ (z-d)\ G(z) - z(z-1)\ f(0) - z\ f(1) = 0. \tag{16}$$

Since, we have $g(0) = f(0) = 0$ and $g(1) = f(1) = 1$. Now,

$$\begin{aligned} Z\left[g(s+2) - g(s+1) + g(s)\right] &= Z\left[g(s+2)\right] - Z\left[g(s+1)\right] + Z\left[g(s)\right] \\ &= \left(z^2 - z + 1\right)\ G(z) - z\left(z-1\right)\ g(0) - z\ g(1) \\ Z\left[g(s+2) - g(s+1) + g(s)\right] &= 0. \qquad [\text{from } (16)] \end{aligned}$$

Since, Z is one-to-one operator, it holds that $g(s+2) - g(s+1) + g(s) = 0$. Therefore, $g(s)$ is a solution of (2). Then we have

$$Z\left[f(s)\right] - Z\left[g(s)\right] = F(z) - G(z) = \frac{U(z)}{(z-c)\ (z-d)} = Z\left[q(s) * u(s)\right],$$

where $q(s) = \dfrac{1}{z}\left\{\dfrac{c^s - d^s}{c-d}\right\}$. Since Z-Transforms is linear and one-to-one, we have $f(s) - g(s) = (q(s) * u(s))$. Now, taking modulus on both sides, we get

$$|f(s) - g(s)| = |q(s) * u(s)| = \left|\sum_{s=-\infty}^{\infty} q\left(s-k\right)\ u(s)\right| \le \sum_{s=-\infty}^{\infty} \left|q\left(s-k\right)\right|\ |u(s)| \le \mathbf{L}(\epsilon)\ \theta(s)$$

for every $s > 0$, where $\mathbf{L} = \sum\limits_{s=-\infty}^{\infty} \left|q\left(s-k\right)\right| = \sum\limits_{s=-\infty}^{\infty} \left|\dfrac{1}{z}\left\{\dfrac{c^{s-k} - d^{s-k}}{c-d}\right\}\right|$ exists for each value of s. Then by the virtue of Definition 2.2, the linear difference equation (2) has the Hyers-Ulam-Rassias stability. □

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