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Holes in L(3, 2, 1)-Labeling

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Abstract: An $L(3,2,1)$-labeling is a simplified model for the channel assignment problem. Given a graph G, an $L(3,2,1)$-labeling of G is a function f from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(u)-f(v)| \geq 1$ if $d(u,v)=3$, $|f(u)-f(v)| \geq 2$ if $d(u,v)=2$ and $|f(u)-f(v)| \geq 3$ if $d(u,v)=1$. The *span* of a labeling f, is the difference between the largest label and the smallest label in an $L(3,2,1)$-labeling. The $L(3,2,1)$-*labeling number* of G, denoted by $\lambda_{3,2,1}(G)$, is the minimum span of all $L(3,2,1)$-labelings of G. A *span labeling* is an $L(3,2,1)$-labeling whose largest label is $\lambda_{3,2,1}(G)$. Let f be an $L(3,2,1)$-labeling that uses labels from 0 to $\lambda_{3,2,1}(G)$. Then $h \in (0, \lambda_{3,2,1}(G))$ is a *hole* if there is no vertex $v \in V(G)$ such that $f(v)=h$. In this paper, we investigate maximum number of holes in $L(3,2,1)$ span labeling of certain classes of graphs.

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1. Introduction

The frequency assignment problem is a problem where the task is to assign frequencies (non-negative integers) to a given group of radio transmitters so that interfering transmitters are assigned frequencies with atleast a minimum allowed separation. The level of interference between any two radio stations relates with the geographic locations of the stations. Closer stations have a stronger interference and thus there must be a greater difference between their assigned channels. The frequency assignment problem was formulated as a vertex labeling problem of graphs by Hale [1]. Two vertices x and y are said to be 'very close' and 'close' if the distance between x and y is 1 and 2 respectively. Griggs and Yeh [2] defined the $L(2,1)$ - labeling of a graph $G=(V,E)$ as a function f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$, where $d(x,y)$ represent the distance between the vertices x and y. By considering those vertices that are at distance three as well, Clipperton [3] introduced the concept of $L(3,2,1)-$ labeling as an extension of $L(2,1)-$ labeling.

Definition 1.1. *The $L(3,2,1)$-labeling of a graph G is a function f from the vertex set $V(G)$ to the set of non-negative integers such that $|f(x)-f(y)| \geq 3$ if $d(x,y)=1$, $|f(x)-f(y)| \geq 2$ if $d(x,y)=2$ and $|f(x)-f(y)| \geq 1$ if $d(x,y)=3$. The span of the labeling f, is the difference between the largest label and the smallest label. The $L(3,2,1)$-labeling number, $\lambda_{3,2,1}(G)$, or simply λ is the smallest non-negative integer k such that G has a $L(3,2,1)$-labeling of span k. An $L(3,2,1)$-labeling f is irreducible if there does not exist an $L(3,2,1)$-labeling g such that $g(u) \leq f(u)$ for all $u \in V(G)$ and $g(v) < f(v)$ for some $v \in V(G)$.*

A span labeling of P_8 is given in the following figure.

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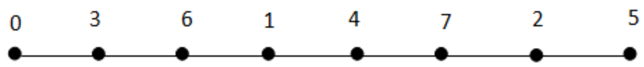

Figure 1. $L(3, 2, 1)$**-labeling of** P_8 **with span 7 where all the 8 labels are used**

It is interesting to note that the span labeling of a graph need not be unique always. The span labeling of P_8 as shown in the above figure uses all the labels. However it is not necessary to use all the labels for a span labeling as evident from the following figure.

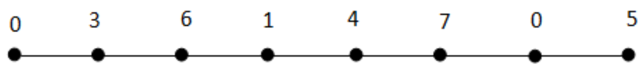

Figure 2. $L(3, 2, 1)$**-labeling of** P_8 **with span 7 with only 7 labels being used**

A natural question that arises regarding the optimal labeling of a graph G is that, "What would be the minimum number of labels required for a span labeling?" In other words, we are interested in knowing the maximum number of lables that have not been used. In fact this question was addressed with respect to $L(2, 1)$-labeling by Laskar [5] by introducing the term *holes* which meant the maximum number of labels that can be spared in an $L(2, 1)$ span labeling. In this paper, we initiate a study on the maximum number of holes in the $L(3, 2, 1)$ span labeling of certain classes of graphs. The problem of finding minimum number of different labels needed to label a graph G is equivalent to the question of finding the maximum number of holes in a span labeling of the graph G. In terms of frequencies, maximum number of holes can be seen as the minimum number of different frequencies required for an interference-free communication in a given network. Throughout this paper, unless mentioned otherwise, we will consider simple, undirected graphs as treated in most of the standard text-books on finite graph theory such as [6].

2. Maximum Number of Holes in Some Families of Graphs

In this section we determine the maximum number of holes in some families of graphs such as complete graphs, paths, complete bipartite graphs and bistars with respect to $L(3, 2, 1)$-labeling. First we state the formal definition of a *hole.*

Definition 2.1. *Let f be an $L(3, 2, 1)$-labeling of a graph G that uses labels from 0 to k. Then, the integer h is said to be a hole, if $h \in (0, k)$ and there is no vertex $v \in V(G)$ such that $f(v) = h$. The maximum number of holes in a span $L(3, 2, 1)$-labeling of a graph G is denoted by $H_\lambda(G)$.*

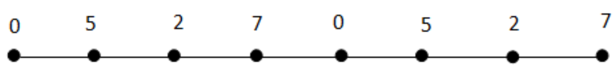

Figure 3. $L(3, 2, 1)$ **span labeling of** P_8 **with holes =** $\{1, 3, 4, 6\}$**.**

We observe that the knowledge of the $L(3, 2, 1)$ labeling number $\lambda_{3,2,1}(G)$ is necessary but not sufficient to determine the maximum number of holes $H_\lambda(G)$, as $L(3, 2, 1)$ span labeling of G need not be unique. This is clear from Figures 1, 2 and 3, as it shows three different $L(3, 2, 1)$ span labelings of P_8. We would be using the following results for further discussions.

Theorem 2.2 ([3])**.** *For any complete graph K_n on n vertices, $\lambda_{3,2,1}(K_n) = 3(n - 1)$.*

Theorem 2.3 ([3])**.** *For any complete bipartite graph $K_{r,s}$, $\lambda_{3,2,1}(K_{r,s}) = 2(r + s) - 1$.*

Theorem 2.4 ([3])**.** *For a path P_n on n vertices,*

$$\lambda_{3,2,1}(P_n) = \begin{cases} 3, & \text{if } n = 2 \\ 5, & \text{if } n = 3,\ 4 \\ 6, & \text{if } n = 5,\ 6,\ 7 \\ 7, & \text{if } n \geq 8 \end{cases}$$

Lemma 2.5 ([4])**.** *For a star $S_n = \{v\} + \bar{K}_n, \lambda_{3,2,1}(S_n) = 2n+1$. Moreover, if f is a (2n+1) - $L(3,2,1)$-labeling of S_n, then $f(v) = 0$ or 2n+1.*

Lemma 2.6 ([4])**.** *If H is a subgraph of G, then $\lambda_{3,2,1}(H) \leq \lambda_{3,2,1}(G)$.*

Using the above results we first determine the value of H_λ for any complete graph.

Theorem 2.7. *For a complete graph K_n on n vertices, $H_\lambda(K_n)$ = 2(n-1).*

Proof. Consider a complete graph K_n on n vertices. By Theorem 2.2, $\lambda_{3,2,1}(K_n) = 3(n-1)$. Hence the set of integers used for the $L(3,2,1)$-labeling will be the set $\{0, 1, 2, \ldots, 3(n-1)\}$ consisting of $3(n-1)+1 = 3n-2$ integers. Since every pair of vertices are adjacent to each other in K_n, n different integers are required for the $L(3,2,1)$-labeling. Therefore the number of labels not used is equal to $3n-2-n = 2(n-1)$. □

Now we determine the maximum number of holes for a path P_n on n vertices.

Theorem 2.8. *For a path P_n on n vertices,*

$$H_\lambda(P_n) = \begin{cases} 2, & \text{if } n = 2,\ 4,\ 5 \\ 3, & \text{if } n = 3 \\ 1, & \text{if } n = 6 \\ 0, & \text{if } n = 7 \\ 4, & \text{if } n \geq 8 \end{cases}$$

Proof. Let $\{v_1, v_2, ..., v_n\}$ be the vertices and (v_i, v_{i+1}) $\forall i = 1, 2, ..., n-1$ be the edges of P_n. Note that we need n different labels to label the vertices of P_n, n $\leq$ 4. We will consider various cases based on the number of vertices.

Case 1: $n = 2$.

Consider P_2, then by Theorem 2.4 $\lambda_{3,2,1}(P_2) = 3$. Therefore the only possible labeling of P_2 is using the labels 0 and 3. Hence $H_\lambda(P_n) = 2$.

Case 2: $n = 3$ or 4

Note that by Theorem 2.4 the $L(3,2,1)$-labeling number of P_3 and P_4 is 5. Therefore the maximum number of labels required for $L(3,2,1)$ span labelings of P_3 and P_4 would be 6. Further we need atleast 3 labels to label P_3, therefore maximum number of labels not used is 6 - 3 = 3; similarly we need atleast 4 labels to label P_4, therefore maximum number of labels not used is 6 - 4 = 2. Hence $H_\lambda(P_3) = 3$ and $H_\lambda(P_4) = 2$.

Case 3: $n = 5, 6, 7$

First we note that $\lambda_{3,2,1}(P_n) = 6$ in this case. Consider the set $\{2, 5, 0, 3, 6, 1, 4\}$ in the same order and obtain the labeling for P_5, P_6 and P_7. We can label P_5 using the first 5 labels i.e $\{2, 5, 0, 3, 6\}$. Hence the number of labels not used in P_5 is

2. Therefore $H_\lambda(P_5) \geq 2$. If $H_\lambda(P_5) > 2$, then atleast one label should be repeated but clearly no labels can be repeated without increasing the $L(3,2,1)$-labeling number ($\lambda_{3,2,1}$). Hence $H_\lambda(P_5) = 2$. Similarly we can label P_6 and P_7 using the first 6 and 7 labels respectively i.e. $\{2,5,0,3,6,1\}$ and $\{2,5,0,3,6,1,4\}$. Hence the maximum number of labels not used in P_6 and P_7 are 1 and 0 respectively. i.e $H_\lambda(P_6) = 1$. and $H_\lambda(P_7) = 0$.

Case 4 : $When n \geq 8$

Note that atleast four different labels to label P_n, $n \geq 4$ satisfying the conditions of $L(3,2,1)$-labeling because same label cannot be used to label four consecutive vertices as the first and the fourth vertex will be at distance 3 and hence label difference should be atleast 1. We also know from Theorem 2.4 that $\lambda_{3,2,1}(P_n) = 7, n \geq 8$ therefore atmost 8 labels are used. Since we need atleast four labels to label P_n therefore atmost four labels are not being used. Therefore

$$H_\lambda \leq 4 \tag{1}$$

Now, consider the labeling f : V(G) $\to \{0,1,...,7\}$ defined by:

$$f(v_k) = \begin{cases} 0, & \text{for k} \equiv 1(mod4) \\ 5, & \text{for k} \equiv 2(mod4) \\ 2, & \text{for k} \equiv 3(mod4) \\ 7, & \text{for k} \equiv 0(mod4) \end{cases}$$

We know that $\lambda(P_n)$=7, n $\geq$ 8; therefore the above defined labeling f is a span $L(3,2,1)$-labeling and it does not use four labels namely 1, 3, 4 and 6. Since H_λ is the maximum number of labels not being used; therefore

$$H_\lambda \geq 4 \tag{2}$$

Therefore, by (1) and (2), $H_\lambda(P_n) = 4$. □

Remark 2.9. *From the above theorem it follows that $H_\lambda(P_8) = 4$ and the labeling of P_8 in Figure 3 is the optimal labeling with maximum number of labels not being used.*

Remark 2.10. *The maximum number of holes for any labeling of P_7 is zero which shows that we need to use all the labels for a span $L(3,2,1)$-labeling of P_7.*

Theorem 2.11. *For a complete bipartite graph $K_{r,s}$, $H_\lambda(K_{r,s}) = r + s$.*

Proof. Let G = $K_{r,s}$ be a complete bipartite graph, and let $U = \{u_1, u_2, ..., u_r\}$ and $V = \{v_1, v_2, ..., v_s\}$ be the two sets of vertices that partition V(G). Note that given any two vertices of U they are at distance 2; similarly any two vertices of V are also at distance 2 and every vertex of U is adjacent every vertex of V. i.e

$$d(u_i, u_j) = 2 \quad \forall \quad u_i, u_j \in U \text{ with } i \neq j; \tag{3}$$

$$d(v_i, v_j) = 2 \quad \forall \quad v_i, v_j \in V \text{ with } i \neq j; \tag{4}$$

$$d(u_i, v_i) = 1 \quad \forall \quad u_i \in U \text{ and } v_i \in V. \tag{5}$$

By Theorem 2.3, $\lambda_{3,2,1}(K_{r,s}) = 2(r+s) - 1$ therefore there exist a $L(3,2,1)$ span labeling say f. Let $f(u_1) = 0$, because of equation (3) we need $|f(u_i) - f(u_j)| \geq 2 \ \forall \quad u_i, u_j \in U$ with $i \neq j$. Let $f(U)$ denote the set of labels assigned to the vertices of U. Thus, $f(U) = \{0, 2, 4, ..., 2(r-1)\}$. Without loss of generality let $f(u_1) \leq f(u_2) \leq ... \leq f(u_r)$, then $f(u_1) = 0, f(u_2) = 2, ..., f(u_n) = 2(r-1)$. Clearly there is exactly one label that have not been used between $f(u_i)$ and $f(u_{i+1})$ $\forall$ i=1,2,...,$r-1$ namely $\{1, 3, 5, ..., 2r-1\}$. Hence there are exactly $r-1$ labels that have not been used in $f(U)$.Since equation (??) holds, $|f(u_i) - f(v_i)| \geq 3$ $\forall$ $u_i \in U$ and $v_i \in V$; hence $f(v_i) \geq f(u_r) + 3 = 2(r-1) + 3 = 2r + 1$ $\forall i = 1, 2, ..., s$. The labeling of vertices in V follows an argument similar to the labeling of vertices in U. Let $f(v_1) = 2r + 1$ and $f(v_i) = 2r + 1 + 2(i-1)$ $\forall i = 1, 2, ..., s$. Let $f(V)$ denote the set of labels assigned to the vertices of V. Then $f(V) = \{2r+1, 2r+3, ..., 2r+1+2(s-1)\}$. Clearly there is exactly one label that have not been used between $f(v_i)$ and $f(v_{i+1})$ $\forall i = 1, 2, ..., s-1$ namely $\{2r+2, 2r+4, ..., 2r+2(s-1)\}$. Hence there are exactly $s-1$ labels that have not been used in $f(V)$. Maximum number of labels that have not been used to label $K_{r,s}$

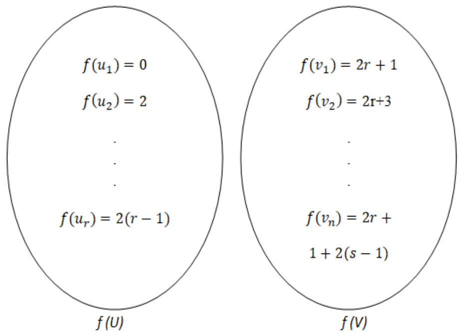

Figure 4. **Assigned labels to the partitioned sets U and V of the vertex set of $K_{r,s}$**

$$H_\lambda(K_{r,s}) = (r-1) + (s-1) + 2 = r + s$$

□

Theorem 2.12. *For a bistar $B_{r,s}$,*

$$H_\lambda(B_{r,s}) = \begin{cases} 2 & if\ r = s \\ 2 + |r-s| & if\ r \neq s \end{cases}$$

Proof. The bistar $B_{r,s}$ is a graph obtained by joining the center (apex) vertices of $K_{1,r}$ and $K_{1,s}$ by an edge. Consider a copy of $K_{1,r}$ and a copy of $K_{1,s}$. Let $v_1, v_2, ..., v_r$ and $u_1, u_2, ..., u_s$ be the corresponding vertices of $K_{1,r}$ and $K_{1,s}$ with apex vertex v and u respectively. Let $e_i = vv_i, e'_i = uu_i$, and $e = uv$ be the edges of the bistar $B_{r,s}$. Note that $|V(B_{r,s})| = r + s + 2 = n$ and $|E(B_{r,s})| = r + s + 1$.

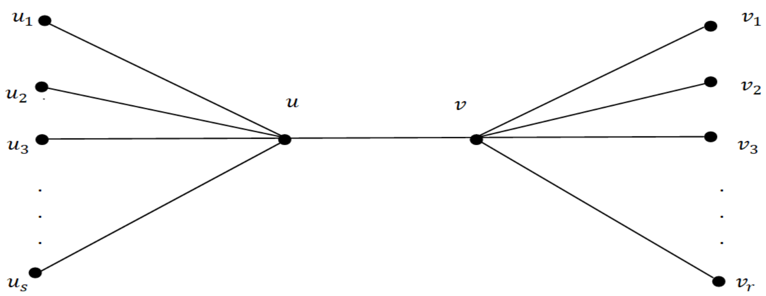

Figure 5. **A Bistar $B_{r,s}$**

Note that the pair of vertices at distance 1 are $(u, u_i), (v, v_i)$ and (u, v). Whereas the vertex pairs $(v_i, v_j), (u_i, u_j), (v_i, u)$ and (u, v_i), $\forall\ i, j$ and $i \neq j$ are distance 2 apart. The pendent vertices (v_i, u_j) $\forall\ i, j$ are at a distance 3. Let f be a $L(3, 2, 1)$-labeling defined on $V(B_{r,s})$ to a set of non-negative integer. Then,

$$|f(v) - f(v_i)| \geq 3,\ |f(u) - f(u_i)| \geq 3 \text{ and } |f(v) - f(u)| \geq 3 \tag{6}$$

$$|f(v_i) - f(v_j)| \geq 2,\ |f(v_i) - f(u)| \geq 2,\ |f(v) - f(u_i)| \geq 2 \text{ and } |f(u_i) - f(u_j)| \geq 2 \tag{7}$$

$$|f(v_i) - f(u_j)| \geq 1 \tag{8}$$

Let $f(u_1) \leq f(u_2) \leq ... \leq f(u_s)$ and $f(v_1) \leq f(v_2) \leq ... \leq f(v_r)$. Consider the star $K_{1,r}$ with central vertex v and $s \leq r$. By Lemma (2.5) vertex v should be given the label either 0 or 2r+1. Without loss of generality let $f(v) = 0$, then by equation (6) $f(v_1) = f(v) + 3 = 0 + 3 = 3$; by equation (7) $f(v_2) = f(v_1) + 2 = 5$;

$$\begin{aligned} f(v_3) &= f(v_2) + 2 = f(v_1) + 2(2); \\ &\vdots \\ f(v_r) &= f(v_1) + (r-1)(2) = 3 + (r-1)(2) \\ \text{and } f(u) &= f(v_1) + r(2) = 3 + r(2) \end{aligned}$$

Let $f(u_i) = f(v_j) - 1\ \ \forall\ \ j = 1, 2, ..., r$ and $i = 1, 2, ..., s$. As per the labeling used above the maximum label used is $3 + 2r$. Therefore $\lambda_{3,2,1}(B_{r,s}) \leq 3 + 2r$. Note that if $r < s$ then according to the above defined labeling there will be atleast one vertex say u_s which has to be assigned a label greater than 3+2r. Hence we follow the above defined labeling. Clearly the diameter of the graph is 3, hence all the vertices of $B_{r,s}$ should be given different labels, hence no label can be repeated or reduced. That is $\lambda_{3,2,1}(B_{r,s}) \not< 3 + 2r$. Therefore $\lambda_{3,2,1}(B_{r,s}) = 3 + 2r$.

Case 1: When $r = s$.

Here $B_{r,s} = B_{r,r}$. As per the labeling f defined as above all the labels from the set $\{0, 1, ..., 3+2r\}$ except labels 1 and 2+2r have been used to label $B_{r,r}$. Hence there are exactly 2 labels that have not be used to label $B_{r,r}$. Therefore maximum number of labels that are not being used will be atleast 2. i.e $H_\lambda(B_{r,r}) \geq 2$. Suppose $H_\lambda(B_{r,r}) > 2$ then atleast one label from the labeling f should be repeated i.e atleast two vertices shoud have the same label which is a contradiction to the fact that all the vertices of $B_{r,r}$ sould be given different labels. Hence $H_\lambda(B_{r,r}) \not> 2$. Therefore $H_\lambda(B_{r,r}) = 2$.

Case 2: When $r \neq s$.

Without loss of generality let $r > s$. As per the labeling f defined as above, $f(u_i) = f(v_j) - 1\ \forall\ \ j = 1, 2, ..., r$ and $i = 1, 2, ..., s$. Since $r > s$ the labels $f(v_j) - 1\ \forall\ \ j = s+1, s+2, ..., r$ will not be used. Therefor all the labels from the set $\{0, 1, ..., 3 + 2r\}$ except labels 1, $f(v_j) - 1\ \forall\ \ j = s+1, s+2, ..., r$ and $2 + 2r$ have been used to label $B_{r,s}$. Hence there are exactly $2 + |r - s|$ labels that have not be used to label $B_{r,s}$. Therefore maximum number of labels that are not being used will be atleast $2 + |r - s|$. i.e $H_\lambda(B_{r,s}) \geq 2 + |r - s|$. Suppose $H_\lambda(B_{r,s}) > 2 + |r - s|$ then atleast one label from the labeling f should be repeated i.e atleast two vertices shoud have the same label which is a contradiction to the fact that all the vertices of $B_{r,s}$ sould be given different labels. Hence $H_\lambda(B_{r,s}) \not> 2 + |r - s|$. Therefore $H_\lambda(B_{r,s}) = 2 + |r - s|$. □

3. Problems For Further Exploration

In this paper we have initiated a study of *holes* in $L(3, 2, 1)$-labeling for simple graphs. We have given the exact value for the maximum number of labels that have not been used for complete graphs, paths, complete bipartite graphs and bistar.

The study could be extended to different labelings and more complex graphs. For certain classes of graphs it is difficult to find the exact value of holes like in interval graphs as it is difficult to even find the $L(3,2,1)$-labeling number as mentioned by Madhumangal in [7]. It is interesting to note that there are many graphs that require all the underlying labels, in other words, graphs that have no-hole labeling. As mentioned earlier the question of finding the the maximum number of holes in a span labeling of the graph G deals with the problem of determining the minimum number of different frequencies required for an interference-free communication in a given network subject to certain conditions. The present study is only a first step in this direction. There are several problems that are open as we consider the concept of holes in $L(3,2,1)$ labeling; some of which are presented below,

Problem 3.1. *To determine the maximum number of holes, H_λ in more classes of graphs like trees and interval graphs.*

Problem 3.2. *Given two graphs G and H, determine the maximum number of holes, H_λ for the various type of product of graphs G and H.*

Problem 3.3. *To find bounds of maximum number of holes, H_λ of any graph G.*

Problem 3.4. *To characterize the graphs which needs to utilize every one of the labels in a $L(3,2,1)$ span labeling.*

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