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A Criterion for the Secondary Unitary Congruence of Conjugate Secondary Normal Matrices

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Abstract: In this paper, conjugate secondary normal (con-s-normal) matrices play the same role in the theory of secondary unitary (s-unitary) congruences as conventional s-normal matrices do in the theory of s-unitary similarities. The aim of this section is to propose a simple criterion for s-unitary congruence for the class of con-s-normal matrices.

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1. Introduction

Let $C_{n\times n}$ be the space of $n \times n$ complex matrices of order n. For $A \in C_{n\times n}$, let A^T, $\overline{A}$, A^*, A^S, $A^\theta \left(= \overline{A}^s\right)$ and A^{-1} denote the transpose, conjugate, conjugate transpose, secondary transpose, conjugate secondary transpose and inverse of matrix A respectively. The conjugate secondary transpose of A satisfies the following properties such as $\left(A^\theta\right)^\theta = A$, $(A+B)^\theta = A^\theta + B^\theta$, $(AB)^\theta = B^\theta A^\theta$ etc.

Definition 1.1. *A matrix $A \in C_{n\times n}$ is said to be normal if $AA^* = A^*A$.*

Definition 1.2. *A Matrix $A \in C_{n\times n}$ is said to be conjugate normal (con-normal) if $AA^* = \overline{A^*A}$.*

Definition 1.3. *A matrix $A \in C_{n\times n}$ is said to be secondary normal (s-normal) if $AA^\theta = A^\theta A$.*

Definition 1.4. *A matrix $A \in C_{n\times n}$ is said to be unitary if $AA^* = A^*A = I$.*

Definition 1.5. *A matrix $A \in C_{n\times n}$ is said to be s-unitary if $AA^\theta = A^\theta A = I$.*

Definition 1.6 ([2])**.** *A matrix $A \in C_{n\times n}$ is said to be a conjugate secondary normal matrix (con-s-normal) if*

$$AA^\theta = \overline{A^\theta A} \quad where \quad A^\theta = \overline{A}^S. \tag{1}$$

Result 1.7 (Specht's Criterion)**.** *Matrices A and B are unitarily similar if and only if*

$$trW(A, A^*) = trW(B, B^*) \tag{2}$$

for every word $W(s,t)$ in the non commuting variables s and t.

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Result 1.8. *Specht's criterion is inefficient because it amounts to the verification of an infinite set of conditions* (2)*. Pearcy [3] found an efficient form for this criterion, showing that, for n-by-n matrices A and B, the verification of* (2) *can be limited to words of a length not exceeding* 2^{n^2}*. However, even this limited verification requires a huge computational effort if n is not very small.*

2. s-unitary Congruence of Matrices

Definition 2.1. *Matrices* $A, B \in M_n(C)$ *are s-unitarily similar if the similarity between A and B can be realized by means of a s-unitary transformation matrix U:*

$$B = U^{\theta} AU \tag{3}$$

To verify that A and B are s-unitarily similar, one can use Specht's classical criterion.

Result 2.2. *Matrices A and B are s-unitarily similar if and only if*

$$trW\left(A, A^{\theta}\right) = trW\left(B, B^{\theta}\right) \tag{4}$$

for every word $W(s,t)$ *in the non commuting variables s and t. The knowledge that A and B belong to a special matrix class sometimes makes it possible to substantially reduce the computational effort required for checking the s-unitary similarity between A and B. In this relation, the class of s-normal matrices is the most striking example. Secondary Normal matrices A and B are s-unitarily similar if and only if they have the same s-eigen values. This latter property can be checked by verifying only n conditions for traces, namely,*

$$tr(A^i) = tr(B^i), \quad i = 1, 2, \ldots, n. \tag{5}$$

Definition 2.3. *Matrices* $A, B \in M_n\left(C\right)$ *are congruent if* $B = S^S AS$ *for a nonsingular matrix S. If the congruence between A and B can be realized by means of a s-unitary transformation matrix U, i.e., if*

$$B = U^S AU, \tag{6}$$

then A and B are said to be s-unitarily congruent.

Remark 2.4. *Unlike in the case of s-unitary similarity, no criterion (even an inefficient one) is currently known for s-unitary congruence. Consider, for instance, the following result from [1].*

Theorem 2.5. *Matrices* $A, B \in M_n\left(C\right)$ *are s-unitarily congruent if and only if there exists a s-unitary matrix V such that* $BB^{\theta} = V^{\theta}\left(AA^{\theta}\right)V$, $B\bar{B} = V^{\theta}\left(A\bar{A}\right)V$, $B^S\bar{B} = V^{\theta}\left(A^S\bar{A}\right)V$.

The s-unitary similarity can be verified for each of the pairs $\left(AA^{\theta}, BB^{\theta}\right)$, $\left(A\bar{A},\, B\bar{B}\right)$, and $\left(A^S\bar{A}, B^S\bar{B}\right)$ by using the Specht-Pearcy criterion. However, there is no method as yet for checking whether all of these three s-unitary similarities can be realized by means of the same s-unitary matrix U. The fact that no criterion is available for the s-unitary congruence between generic matrices A and B does not imply that such criteria cannot exist for special matrix classes. The following result is an analogue of the theorem in [4, 5] on the secondary spectral decomposition of a s-normal matrix.

3. s-unitary Congruence of Con-s-normal Matrices

Theorem 3.1. *Any con-s-normal matrix A is s-unitarily congruent to a block secondary diagonal matrix with the secondary diagonal blocks of orders 1 and 2. The blocks of order 1 are real nonnegative scalars, while each block of order 2 can be given the form of the following 2-by-2 s-hermitian matrix:*

$$\begin{pmatrix} 0 & \mu_j \\ \bar{\mu}_j & 0 \end{pmatrix}, \qquad Im\,\mu_j \neq 0 \tag{7}$$

With each matrix $A \in M_n(C)$, we associate the matrix

$$A_L = \bar{A}A \tag{8}$$

Lemma 3.2. *If A and B is s-unitarily congruent, then A_L and B_L are s-unitarily similar. Indeed, (6) implies that*

$$\begin{aligned} B_L &= \bar{B}B = U^\theta \bar{A}\bar{U}U^S AU \\ &= U^\theta\left(\bar{A}A\right)U = U^\theta A_L U. \end{aligned}$$

Lemma 3.3. *If A is a con-s-normal matrix, then A_L is s-normal in the conventional sense.*

Proof. It follows from Definition 1.6 that $AA^\theta = A^S\bar{A}$ and $\bar{A}A^S = A^\theta A$. Using these equalities, we find that

$$\begin{aligned} A_L A_L^\theta &= \bar{A}AA^\theta A^S = \bar{A}\left(AA^\theta\right)A^S \\ &= \bar{A}\left(A^S\bar{A}\right)A^S = \left(\bar{A}A^S\right)^2 = \left(A^\theta A\right)^2, \\ A_L^\theta A_L &= A^\theta A^S \bar{A}A = A^\theta\left(A^S\bar{A}\right)A \\ &= A^\theta\left(AA^\theta\right)A = \left(A^\theta A\right)^2. \end{aligned}$$

Thus, $A_L A_L^\theta = A_L^\theta A_L$. □

Now, we can formulate the main result of this section.

Theorem 3.4. *Con-s-normal matrices $A, B \in M_n(C)$ are s-unitarily congruent if and only if the corresponding s-normal matrices A_L and B_L are s-unitarily similar.*

Proof. Matrices A and B are s-unitarily congruent if and only if their canonical forms described in Theorem 3.1 are s-unitarily congruent. Let

$$F_A = \lambda_1 \oplus \cdots \oplus \lambda_k \oplus \begin{pmatrix} 0 & \mu_1 \\ \bar{\mu}_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \mu_l \\ \bar{\mu}_l & 0 \end{pmatrix}, \qquad k + 2l = n$$

be the canonical form of A. It is easy to see that $(F_A)_L = \bar{F}_A F_A$ is the diagonal matrix

$$(F_A)_L = diag\left(\lambda_1^2, \lambda_2^2, \ldots, \lambda_k^2, \bar{\mu}_1^2, \mu_1^2, \bar{\mu}_2^2, \mu_2^2, \ldots, \bar{\mu}_l^2, \mu_l^2\right).$$

Thus, the scalars λ_1, λ_2,..., λ_k, μ_1, μ_2,..., μ_l, which define the canonical form of A, are the square roots of the s-eigen values of $(F_A)_L$ or, equivalently, the square roots of the s-eigen values of A_L. A similar conclusion is valid for B. We infer that $(F_A)_L$ and $(F_B)_L$ are s-unitarily congruent (and even coincide, provided that the square roots are consistently chosen) if and only if A_L and B_L have the same s-eigen values. Since A_L and B_L are s-normal matrices, their s-eigen values are identical if and only if A_L and B_L are s-unitarily similar. □

From Theorem 3.4, we immediately obtain the desired criterion.

Result 3.5 (Criterion for s-unitary congruence)**.** *Con-s-normal matrices* $A, B \in M_n(C)$ *are s-unitarily congruent if and only if* $tr\left[\left(\bar{A}A\right)^i\right] = tr\left[\left(\bar{B}B\right)^i\right]$, $i = 1, 2, \ldots, n$.

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