Domino Chromatic Number of Derived Graph of Some Graphs

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Abstract: In a graph G, the distance d(u,v) between a pair of vertices u and v is the length of a shortest path joining them. The Derived graph, G† is the graph whose vertices are same as the vertices of G and two vertices in G† are adjacent if and only if the distance between them in G is two. In this paper we obtain the exact value for χd for Derived Graph of Path, Cycle, Sunlet graph, Bistar graph and Triple Star graph respectively.

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1. Introduction

The field of Graph theory plays an important role in various areas of pure and applied sciences. We begin with simple, connected, finite, undirected graph G = (V(G), E(G)) with p = |V(G)| and q = |E(G)|. For all terminology and notations in graph theory especially defined in this paper, we refer the reader to the standard text books [8] respectively. The minimum and maximum degree of G are denoted by δ = δ(G) and Δ = Δ(G) respectively. The distance dG(vi;vj) between the vertices vi and vj is the length of a shortest path between them. If there is no path between vi and vj then we formally assume that dG(vi;vj) = ∞.

A dominating set, D of a graph G is a subset of the vertices in G such that for each vertex v, NG[v] ∩ D ≠ ∅ The domination number γ(G) of G is the cardinality of a minimum dominating set. The concept of domination in graphs, with its many variations, has been well studied in graph theory [7]. A proper coloring of a graph G is a function from the set of vertices of a graph to a set of colors such that any two adjacent vertices have different colors. A subset of vertices colored with the same color is called a color class. The chromatic number is the minimum number of colors needed in a proper coloring of a graph and is denoted by χ (G). A dominator coloring is a coloring of the vertices of a graph such that every vertex is either alone in its color class or adjacent to all vertices of at least one other color class. The concept of a dominator coloring in a graph was introduced and studied by Gera [4] and studied further by Gera [3, 5] and Chellai and Maffray [1].

Let G be a simple graph. Its derived graph G† is the graph whose vertices are same as the vertices of G and two vertices in G† are adjacent if and only if the distance between them in G is two. Directly from this definition follows that [G1 ∪ G2]† = [G1]† ∪ [G2]†. Double graph [2] of a connected graph G is constructed by taking two copies of G say G′ and G′′, join each vertex u′ in G′ to the neighbour of the corresponding vertices u− in G′′. Double graph of G is denoted by D(G).

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2. Dominator Chromatic Number of Derived Graphs

**Theorem 2.1.** For \( n \geq 3 \), the dominator chromatic number of derived graph of Sunlet graph is

\[
\chi_d(S_n)^\dagger = \begin{cases} 
\lfloor \frac{2n}{3} \rfloor + 1 & \text{when } n \equiv 1 \mod 4 \\
n/2 + 3 & \text{when } n \equiv 2 \mod 4 \\
\lfloor n/2 \rfloor + 2 & \text{when } n \equiv 3 \mod 4 \\
n/2 + 2 & \text{when } n \equiv 0 \mod 4
\end{cases}
\]

**Proof.** Let us define the vertex set \( V \) of \( S_n \) as \( V(S_n) = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \) where \( v_i \) are the vertices of cycles taken in cyclic order and \( u_i \) are the pendent vertices. By the definition of derived graph, \( V(S_n)^\dagger = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \). A procedure to obtain dominator coloring of derived graph of sunlet graph as follows. Define a coloring function \( f \) on \( V(S_n)^\dagger \) such that for all vertices of \( v_i \) and \( u_i \)

**Case 1:** When \( n \not\equiv 1 \mod 4 \)

\[
f(v_i) = \begin{cases} 
i & \text{for } v_{2i-2} \text{ when } i = 2k, 1 \leq k \leq \lfloor n/4 \rfloor \\
i & \text{for } v_{2i-1} \text{ when } i = 2k - 1, 1 \leq k \leq \lfloor n/4 \rfloor
\end{cases}
\]

Remaining vertices are colored in the following sub cases.

**Sub Case 1:** \( n \equiv 2 \mod 4 \)

\[
f(v_i, u_i) = \begin{cases} 
n/2 + 2 & \text{for } u_i : 1 \leq i \leq n \\
n/2 + 3 & \text{otherwise}
\end{cases}
\]

**Sub Case 2:** \( n \equiv 3 \mod 4 \)

\[
f(v_i, u_i) = \begin{cases} 
\lfloor n/2 \rfloor + 1 & \text{for } v_{2i-1} \text{ when } i = 2k, 1 \leq k \leq \lfloor n/4 \rfloor \\
\lfloor n/2 \rfloor + 1 & \text{for } v_{2i} \text{ when } i = 2k, 1 \leq k \leq \lfloor n/4 \rfloor \\
\lfloor n/2 \rfloor + 2 & \text{for } u_i : 1 \leq i \leq n
\end{cases}
\]

**Sub Case 3:** \( n \equiv 0 \mod 4 \)

\[
f(v_i, u_i) = \begin{cases} 
n/2 + 2 & \text{for } v_{2i-1}, v_{2i} \text{ when } i = 2k, 1 \leq k \leq \lfloor n/4 \rfloor \\
n/2 + 1 & \text{for } u_i : 1 \leq i \leq n
\end{cases}
\]

**Case 2:** When \( n \equiv 1 \mod 4 \)

\[
f(v_i, u_i) = \begin{cases} 
i & \text{for } v_{2i-2} \text{ when } i = 2k, 1 \leq k \leq \lfloor n/4 \rfloor \\
i & \text{for } v_{2i-1} \text{ when } i = 2k - 1, 1 \leq k \leq \lfloor n/4 \rfloor \\
\lfloor 2n/3 \rfloor & \text{for } u_i : 1 \leq i \leq n.
\end{cases}
\]

\[
\lfloor 2n/3 \rfloor + 1 \text{ otherwise}
\]

This Completes the proof of the theorem.

**Theorem 2.2.** For any \( n \) the dominator chromatic number of derived graph of double Star graph is,

\[
\chi_d(K_{1,n,n})^\dagger = n + 1
\]
Proof. Let $V(K_{1,n})^1 = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ be the vertices of derived graph of double star graph. The following procedure gives the dominator chromatic number of derived graph of double star graph. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_n, c_{n+1}\}$ Here $v_i : 1 \leq i \leq n$ forms a clique of order $n$, so we have to assign $c_i$ colors to $v_i : 1 \leq i \leq n$. Next assign the color $c_{n+1}$ to $v$ and $c_{n-1}$ color to $u_i : 1 \leq i \leq n$. By the definition of dominator coloring every vertex of $v_i$ dominates any one color class $c_i$ and $v$ dominates itself. Next the vertices $u_i : 1 \leq i \leq n$ dominates the color class $c_{n+1}$. Hence an easy observation shows that $\chi_d(K_{1,n})^1 = n + 1$, where $c_i = \{v_i : 1 \leq n - 1\}$, $c_n = \{v, u_i : 1 \leq i \leq n\}$, $c_{n+1} = v$. This completes the proof of theorem.

Theorem 2.3. For $n \geq 3$, the dominator chromatic number of derived graph of triple star graph is $n + 2$ i.e.,

$$\chi_d(K_{1,n,n})^1 = n + 2$$

Proof. Let $V(K_{1,n,n,n})^1 = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n\}$ be the vertices of derived graph of triple star graph. The following procedure gives the dominator chromatic number of derived graph of triple star graph. In $V(K_{1,n,n,n})^1$ induced subgraph $v_i : 1 \leq i \leq n$ forms a clique of order $n$, so we have to assign color $c_i$ to $v_i : 1 \leq i \leq n$. For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $w_i$, $u_i : 1 \leq i \leq n$ and assign the color $c_{n+2}$ to $v$. By the definition of dominator coloring $w_i$ dominates the color class $c_i : 1 \leq i \leq n$ and $u_i : 1 \leq i \leq n$ dominates the color class $c_{n+2}$ and $v$ dominates itself. Hence $\chi_d(K_{1,n,n,n})^1 = n + 2$, where $c_i = \{v_i : 1 \leq i \leq n\}$, $c_{n+1} = \{u_i,w_i : 1 \leq i \leq n\}$, $c_{n+2} = v$.

Theorem 2.4. Let $m, n \geq 3$, the dominator chromatic number of derived graph of Bi-star graph is,

$$\chi_d(B_{m,n})^1 = \begin{cases} m + 2 & \text{when } m > n \\ n + 2 & \text{when } n \geq m \end{cases}$$

Proof. Consider the Bistar $B_{m,n}$, let $\{u_i : 1 \leq i \leq m\}$ be the $m$ pendant edges attached to the vertex $u$ and $\{v_i : 1 \leq i \leq n\}$ be the another $n$ pendant edges attached to the vertex $v$. By the definition of derived,

$$V(B_{m,n})^1 = \{u, u_i : 1 \leq i \leq m\} \cup \{v, v_i : 1 \leq i \leq n\}.$$ 

In $(B_{m,n})^1$ the vertices $u_i : 1 \leq i \leq m$ along with $v$ forms a clique of order $n + 1$. Also we see that the vertices $v_i : 1 \leq i \leq n$ together with $u$ forms another clique of order $n + 1$. Consider the following cases.

Case 1: When $m > n$. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_m, c_{m+1}, c_{m+2}\}$.

- Assign the color $c_i$ to $u_i : 1 \leq i \leq m$ and assign color $c_{m+1}$ for $u$.

- For $1 \leq i \leq n$ assign color $c_i$ to $v_i$ and assign color $c_{m+2}$ to $v$.

By the definition of dominator coloring every vertex of $u_i$ dominates the color class $c_{m+2}$ and $v_i$ dominates the color class $c_{m+1}$. Vertices $u,v$ dominates itself. Hence $C = \{c_1, c_2, c_3, \ldots, c_m, c_{m+1}, c_{m+2}\}$ is a dominator coloring of $(B_{m,n})^1$, where

$$c_i = \{v_i : 1 \leq i \leq n, u_i : 1 \leq i \leq m\},$$

$$c_{m+1} = \{u\}, c_{m+2} = \{v\}.$$ 

Case 2: When $n \geq m$. Consider the color class $C = \{c_1, c_2, c_3, \ldots, c_m c_{m+1}, c_{m+2}\}$. Since $u_i : 1 \leq i \leq m$ along with $v$ forms a clique of order $n + 1$. Also we see that the vertices $v_i : 1 \leq i \leq n$ together with $u$ forms another clique of order $n + 1$ for the reason that assign color $c_i$ to $u_i : 1 \leq i \leq m$ and $v_i : 1 \leq i \leq n$. Next assign the color $c_{m+1}$ to the vertex $v$ and $c_{m+2}$ to $u$. Thus $\chi_d(B_{m,n})^1 \leq n + 2$. On the other hand if we assign $c_{m+1}$ colors to all the vertices in $(B_{m,n})^1$ then it contradicts the definition of dominator coloring. Hence an easy observation shows that $\chi_d(B_{m,n})^1 = n + 2$. 

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The \(n\)-centipede \(C_n\) is a tree with \(2n\) vertices and \(2n - 1\) edges obtained by joining the bottoms of \(n\) copies of the path graph \(P_2\) laid in a row with edges.

**Theorem 2.5.** For \(n \geq 3\), the dominator chromatic number of derived graph of Centipede graph is

\[
\chi_d(C_n) = \begin{cases} 
\lceil n/2 \rceil + 2 & \text{when } n \equiv 3, 0 \mod 4 \\
n/2 + 3 & \text{when } n \equiv 2 \mod 4 \\
\lceil n/2 \rceil + 3 & \text{when } n \equiv 1 \mod 4.
\end{cases}
\]

**Proof.** Let us define the vertex set \(V\) of \(C_n\) as \((C_n)^\dagger = \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}\). A procedure to obtain dominator coloring of derived graph of centipede graph as follows. Define a coloring function \(f\) on \((C_n)^\dagger\) such that for any vertex \(v_i\) and \(u_i\).

**Case 1:** When \(n \equiv 3, 0 \mod 4\)

\[
f(v_i, u_i) = \begin{cases} 
i & \text{for } v_{2i}, \text{ when } i = 2k - 1, 1 \leq k \leq \lceil n/4 \rceil \\
i & \text{for } v_{2i - 1}, \text{ when } i = 2k, 1 \leq k \leq \lceil n/4 \rceil \\
\lceil n/2 \rceil + 1 & \text{for } u_i : 1 \leq i \leq n.
\end{cases}
\]

**Case 2:** When \(n \equiv 2 \mod 4\)

\[
f(v_i, u_i) = \begin{cases} 
i & \text{for } v_{2i}, \text{ when } i = 2k - 1, 1 \leq k \leq \lceil n/4 \rceil \\
i & \text{for } v_{2i - 1}, \text{ when } i = 2k, 1 \leq k \leq \lceil n/4 \rceil \\
\lceil n/2 \rceil & \text{for } u_n \\
\lceil n/2 \rceil + 1 & \text{for } v_n \\
\lceil n/2 \rceil + 2 & \text{for } u_i : 1 \leq i \leq n.
\end{cases}
\]

**Case 3:** When \(n \equiv 1 \mod 4\)

\[
f(v_i, u_i) = \begin{cases} 
i & \text{for } v_{2i}, \text{ when } i = 2k - 1, 1 \leq k \leq \lceil n/4 \rceil \\
i & \text{for } v_{2i - 1}, \text{ when } i = 2k, 1 \leq k \leq \lceil n/4 \rceil \\
\lceil n/2 \rceil + 1 & \text{for } u_n : \\
\lceil n/2 \rceil + 2 & \text{for } u_i : 1 \leq i \leq n.
\end{cases}
\]

Hence this completes the proof of the theorem.

**Theorem 2.6.** For \(n \geq 9\), the dominator chromatic number of derived graph of cycle is

\[
\chi_d(C_n) = \begin{cases} 
\lceil n/3 \rceil + 2 & \text{when } n \equiv 0, 1 \mod 3 \\
2 \lceil n/6 \rceil + 2 & \text{when } n \equiv 2 \mod 3
\end{cases}
\]
Proof. By the definition of derived graph, \(\{v_1, v_2, v_3, \ldots, v_n\}\) be the vertices of derived graph of cycle graph. The following procedure gives the dominator chromatic number of derived graph of cycle graph. Consider the color classes \(C = \{c_1, c_2, c_3, \ldots, \lceil n/3 \rceil + 2\} \).

Case 1: When \(n \equiv (0, 1) \mod 3\)

For \(v_i\) where \(i = 3k - 2, 1 \leq k \leq \lceil n/3 \rceil\) are colored by color \(c_{\lceil i/3 \rceil}\) and assign the color \(c_{\lceil n/3 \rceil + 2}, c_{\lceil n/3 \rceil + 1}\) to the remaining vertices of \(v_i\). By the definition of dominator coloring \(v_i\) where \(i = 3k - 2, 1 \leq k \leq \lceil n/3 \rceil\) Dominate itself and the remaining vertices dominates at least any one color classes \(c_i, i = 3k - 2, 1 \leq k \leq \lceil n/3 \rceil\).

Case 2: \(n \equiv 2 \mod 3\)

First Assign the color \(c_{\lceil i/3 \rceil}\) to \(v_i, i = 6k - 5\) where \(1 \leq k \leq \lceil n/3 \rceil - 2\) and assign color \(c_{\lceil i/3 \rceil} + 1 \) to \(v_i, i = 6k - 4, 1 \leq k \leq \lceil n/3 \rceil - 2\). Next the color \(c_{\lceil n/3 \rceil + 2}\) and \(c_{\lceil n/3 \rceil + 1}\) are assigned to the remaining vertices of \(v_i\). Hence,

\[
\chi_d(G^1(C_n)) = \begin{cases} 
\lceil n/3 \rceil + 2 & \text{when } n \equiv (0, 1) \mod 3 \\
\lceil n/3 \rceil + 2 & \text{when } n \equiv 2 \mod 3
\end{cases}
\]

References


