Nano $O_{16}$-closed Sets

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Abstract: In this paper, we investigated a new class of sets called nano $O_{16}$-closed sets and nano $O_{16}$-open sets in nano topological spaces and its properties are studied.

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1. Introduction

Lellis Thivagar [5] introduced a nano topological space with respect to a subset $X$ of an universe which is defined in terms of lower approximation and upper approximation and boundary region. The classical nano topological space is based on an equivalence relation on a set, but in some situation, equivalence relations are nor suitable for coping with granularity, instead the classical nano topology is extend to general binary relation based covering nano topological space. Bhuvaneswari [4] introduced and investigated nano $g$-closed sets in nano topological spaces. Recently, Parvathy and Bhuvaneswari the notions of nano $gpr$-closed sets which are implied both that of nano $rg$-closed sets. In this paper, we defined nano $O_{16}$-closed sets and obtained some of its basic properties as results.

2. Preliminaries

Throughout this paper $(U, \tau_{R}(X))$ (or $X$) represent nano topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $H$ of a space $(U, \tau_{R}(X))$, $Ncl(H)$ and $Nint(H)$ denote the nano closure of $H$ and the nano interior of $H$ respectively. We recall the following definitions which are useful in the sequel.

Definition 2.1 ([7]). Let $U$ be a non-empty finite set of objects called the universe and $R$ be an equivalence relation on $U$ named as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernable with one another. The pair $(U, R)$ is said to be the approximation space. Let $X \subseteq U$.

(1) The lower approximation of $X$ with respect to $R$ is the set of all objects, which can be for certain classified as $X$ with respect to $R$ and it is denoted by $L_{R}(X)$. That is, $L_{R}(X) = \bigcup_{x \in U} \{ R(x) : R(x) \subseteq X \}$, where $R(x)$ denotes the equivalence class determined by $x$.

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(2). The upper approximation of $X$ with respect to $R$ is the set of all objects, which can be possibly classified as $X$ with respect to $R$ and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x \in U} \{R(x) : R(x) \cap X \neq \emptyset\}$.

(3). The boundary region of $X$ with respect to $R$ is the set of all objects, which can be classified neither as $X$ nor as not-$X$ with respect to $R$ and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

**Property 2.2** ([5]). If $(U, R)$ is an approximation space and $X, Y \subseteq U$; then

1. $L_R(X) \subseteq X \subseteq U_R(X)$;
2. $L_R(\emptyset) = U_R(\emptyset) = \emptyset$ and $L_R(U) = U_R(U) = U$;
3. $U_R(X \cup Y) = U_R(X) \cup U_R(Y)$;
4. $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$;
5. $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$;
6. $L_R(X \cap Y) \subseteq L_R(X) \cap L_R(Y)$;
7. $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$;
8. $U_R(X^c) = [L_R(X)]^c$ and $L_R(X^c) = [U_R(X)]^c$;
9. $U_R U_R(X) = L_R U_R(X) = U_R(X)$;
10. $L_R L_R(X) = U_R L_R(X) = L_R(X)$.

**Definition 2.3** ([5]). Let $U$ be the universe, $R$ be an equivalence relation on $U$ and $\tau_R(X) = \{U, \emptyset, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then by the Property 2.2, $R(X)$ satisfies the following axioms:

1. $U$ and $\emptyset \in \tau_R(X)$,
2. The union of the elements of any sub collection of $\tau_R(X)$ is in $\tau_R(X)$,
3. The intersection of the elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ is a topology on $U$ called the nano topology on $U$ with respect to $X$. We call $(U, \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the dual nano topology of $[\tau_R(X)]$.

**Remark 2.4** ([5]). If $[\tau_R(X)]$ is the nano topology on $U$ with respect to $X$, then the set $B = \{U, \emptyset, L_R(X), B_R(X)\}$ is the basis for $\tau_R(X)$.

**Definition 2.5** ([5]). If $(U, \tau_R(X))$ is a nano topological space with respect to $X$ and if $H \subseteq U$, then the nano interior of $H$ is defined as the union of all nano open subsets of $H$ and it is denoted by $Nint(H)$. That is, $Nint(H)$ is the largest nano open subset of $H$. The nano closure of $H$ is defined as the intersection of all nano closed sets containing $H$ and it is denoted by $Ncl(H)$. That is, $Ncl(H)$ is the smallest nano closed set containing $H$.

**Definition 2.6.** A subset $H$ of a nano topological space $(U, \tau_R(X))$ is called

1. nano semi-open [5] if $H \subseteq Ncl(Nint(H))$;
2. nano $\alpha$-open [6] if $H \subseteq N(int(Ncl(Nint(H))))$. 


(3) nano $\beta$-open [10] if $H \subseteq Ncl(Ncl(Ncl(H)))$.

(4) nano regular-open [5] if $H = Nint(Ncl(H))$.

(5) nano pre-open [5] if $H \subseteq Nint(Ncl(H))$.

The complements of the above mentioned sets is called their respective closed sets.

**Definition 2.7.** A subset $H$ of a nano topological space $(U, \tau_{R}(X))$ is called:

(1) nano $g$-closed [3] if $Ncl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano open.

(2) nano gp-closed set [4] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano open.

(3) nano gpr-closed set [6] if $Npcl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano regular open.

(4) nano sg-closed set [9] if $Nscl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano semi-open.

(5) nano ag-closed [12] if $Nacl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano open.

(6) nano ga-closed [12] if $Nacl(H) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano $\alpha$-open.

3. On Nano $\mathcal{O}_{16}$-closed Sets

**Definition 3.1.** A subset $H$ of space $(U, \tau_{R}(X))$ is called nano $\mathcal{O}_{16}$-closed if $Ncl(Nint(Ncl(H))) \subseteq G$, whenever $H \subseteq G$ and $G$ is nano open. The complement of nano $\mathcal{O}_{16}$-open if $H^{c} = U - H$ is nano $\mathcal{O}_{16}$-closed.

**Example 3.2.** Let $U = \{a, b, c, d\}$ with $U/R = \{\{a\}, \{b, c\}, \{d\}\}$ and $X = \{b, d\}$. Then the nano topology $\tau_{R}(X) = \{\phi, \{d\}, \{b, c\}, \{b, c, d\}, U\}.$

(1) then $\{a, b\}$ is nano $\mathcal{O}_{16}$-closed.

(2) then $\{c\}$ is nano $\mathcal{O}_{16}$-open.

**Theorem 3.3.** In a space $(U, \tau_{R}(X))$, every nano $ga$-closed set is nano $\mathcal{O}_{16}$-closed.

**Proof.** Let $H$ be a nano $ga$-closed set in $U$ and $H \subseteq G$ where $G$ is nano $\alpha$-open. Now nano $\alpha$-open implies that $G$ is nano open. Also $Ncl(Nint(Ncl(H))) \subseteq Ncl(H) \subseteq Nacl(H) \subseteq G$. Thus $H$ is nano $\mathcal{O}_{16}$-closed set in $U$.

**Remark 3.4.** The converse of the theorem 3.3 need not be true as seen from the following example.

**Example 3.5.** In Example 3.2, then $\{b, c, d\}$ is nano $\mathcal{O}_{16}$-closed but not nano $ga$-closed.

**Theorem 3.6.** In a spaces $(U, \tau_{R}(X))$, the union of two nano $\mathcal{O}_{16}$-closed is nano $\mathcal{O}_{16}$-closed.

**Proof.** Assume that $H$ and $K$ are nano $\mathcal{O}_{16}$-closed set in $U$. Let $G$ be a nano open in $U$ such that $H \cup K \subseteq G$. Then $H \subseteq G$ and $K \subseteq G$. Since $H$ and $K$ are nano $\mathcal{O}_{16}$-closed, $Ncl(Nint(Ncl(H))) \subseteq G$ and $Ncl(Nint(Ncl(K))) \subseteq G$. Thus $Ncl(Nint(Ncl(H) \cup K)) = Ncl(Nint(Ncl(H))) \cup Ncl(Nint(Ncl(K))) \subseteq G$. That is $Ncl(Nint(Ncl(H \cup K))) \subseteq G$. Hence $H \cup K$ is nano $\mathcal{O}_{16}$-closed set in $U$.

**Example 3.7.** In Example 3.2, then $P = \{a, c\}$ and $Q = \{a, d\}$ is nano $\mathcal{O}_{16}$-closed clearly $P \cup Q = \{a, c, d\}$ nano $\mathcal{O}_{16}$-closed.

**Remark 3.8.** In a space $(U, \tau_{R}(X))$, the intersection of two nano $\mathcal{O}_{16}$-closed is but not nano $\mathcal{O}_{16}$-closed.
Example 3.9. In Example 3.2, then \( P = \{a, b, d\} \) and \( Q = \{b, c, d\} \) is nano \( O_{16} \)-closed clearly \( P \cap Q = \{b, d\} \) is but not nano \( O_{16} \)-closed.

Theorem 3.10. In a space \( (U, \tau_R(X)) \), if a subset \( H \) of \( U \) then \( Ncl(Nint(Ncl(H))) - H \) does not contain any non empty nano open set.

Proof. Suppose that \( H \) is nano \( O_{16} \)-closed set in \( U \). Let \( G \) be a nano open set such that \( (Ncl(Nint(Ncl(H))) - H \subseteq G \) and \( Ncl(Nint(Ncl(H))) \supseteq G \). Now \( G \subseteq Ncl(Nint(Ncl(H))) - H \). Hence \( G \subseteq U - G \). Since \( G \) is nano open set, \( U - G \) is also nano open in \( U \). Since \( H \) is nano \( O_{16} \)-closed sets in \( U \), by definition we have \( Ncl(Nint(Ncl(H))) \subseteq U - G \). Therefore \( G \subseteq U - Ncl(Nint(Ncl(H))) \). Also \( G \subseteq Ncl(Nint(Ncl(H))) \). Hence \( G \subseteq Ncl(Nint(Ncl(H))) \cap (U - Ncl(Nint(Ncl(H)))) = \phi \). This shows that \( G = \phi \) which is contradiction. Thus \( Ncl(Nint(Ncl(H))) - H \) does not contains any non empty nano open set in \( U \).

Remark 3.11. The converse of the above theorem need not be true as seen from the following example.

Example 3.12. In Example 3.2, let \( H = \{b, c, d\} \) be a subset of nano \( O_{16} \)-closed. Then \( Ncl(Nint(Ncl(H))) - H = U - \{b, c, d\} = \{a\} \) does not contain any non-empty nano open set but \( H \) is not nano \( O_{16} \)-closed.

Theorem 3.13. In a space \( (U, \tau_R(X)) \), if \( H \) is nano regular closed then nano \( O_{16} \)-closed.

Proof. Assume that \( H \subseteq G \) and \( G \) is nano open in \( U \). Now \( G \subseteq U \) is nano open if and only if \( G \) is the union of a nano semi open set and nano pre open set. Let \( H \) be a nano regular closed subset of \( U \). Hence \( H = Ncl(Nint(Ncl(H))) \). Every nano regular closed set is nano semi open set and every nano semi open set is nano open set. Hence \( Ncl(Nint(Ncl(H)) \subseteq (\{a, b\}, U) \). This implies that \( H \) is not nano regular closed.

Remark 3.14. The converse of the above theorem need not be true as seen from the following example.

Example 3.15. Let \( U = \{a, b, c\} \) with \( U/R = \{\{a, b\}, \{c\}\} \) and \( X = \{b, c\} \). Then the nano topology \( \tau_R(X) = \{\phi, \{a, b\}, U\} \). We have \( H = \{a\} \). Clearly \( H \) nano \( O_{16} \)-closed set but not nano regular closed. Since \( H \neq Nrccl(H) \). This implies that \( H \) is not nano regular closed.

Theorem 3.16. In a space \( (U, \tau_R(X)) \), for an element \( x \in X \), the set \( X - \{x\} \) is nano \( O_{16} \)-closed.

Proof. Suppose \( X - \{x\} \) is not nano open. Then \( X \) is the only nano open set containing \( X - \{x\} \). This implies \( Ncl(Nint(Ncl(X - \{x\})) \subseteq X \). Hence \( X - \{x\} \) is nano \( O_{16} \)-closed.

Theorem 3.17. In a space \( (U, \tau_R(X)) \), if \( H \) is nano regular open and nano \( O_{16} \)-closed then \( H \) is nano regular closed and nano clopen.

Proof. Assume that \( H \) is nano regular open and nano \( O_{16} \)-closed. As every nano regular open set is nano open and \( H \subseteq H \), we have \( Ncl(Nint(Ncl(H)) \subseteq H \). Since \( Ncl(H) \subseteq Ncl(Nint(Ncl(H))) \). We have \( Ncl(H) \subseteq H \). So \( H \subseteq cl(H) \). Therefore \( cl(H) = H \) that means \( H \) is nano closed. Since \( H \) is nano regular open, \( H \) is nano open. Now \( Ncl(Nint(H)) = Ncl(H) = H \). Hence \( H \) is nano regular closed and clopen.

Theorem 3.18. In a space \( (U, \tau_R(X)) \), if \( H \) is nano regular open and nano \( rg \)-closed then \( H \) is nano \( O_{16} \)-closed.

Proof. Let \( H \) be nano regular open and nano \( rg \)-closed in \( U \). Let \( G \) be any nano open set in \( U \) such that \( H \subseteq G \). Since \( H \) is nano regular open and nano \( rg \)-closed, we have \( Ncl(H) \subseteq H \). Then \( Ncl(H) \subseteq H \subseteq G \). Thus \( H \) is nano \( O_{16} \)-closed.

Theorem 3.19. If \( H \) is nano \( O_{16} \)-closed subset in \( U \) such that \( H \subseteq K \subseteq Ncl(H) \), then \( K \) is nano \( O_{16} \)-closed.
Proof. Let $H$ be nano $O_{16}$-closed subset in $U$ such that $H \subseteq K \subseteq Ncl(H)$. Let $G$ be a nano open set of $U$ such that $K \subseteq G$. Then $H \subseteq G$. Since $H$ is nano $O_{16}$-closed. We have $Ncl(H) \subseteq G$. Now $Ncl(K) \subseteq Ncl(Ncl(H)) = Ncl(H) \subseteq G$. Hence $K$ is nano $O_{16}$-closed.

Remark 3.20. The converse of the above theorem need not be true as seen from the following example.

Example 3.21. In Example 3.15, we have $S = \{a\}$ and $T = \{a, b\}$. Then $S$ and $T$ are nano $O_{16}$-closed, but $S \subseteq T$ is not subset in $Ncl(S)$.

Theorem 3.22. In a space $(U, \tau_R(X))$, let $H$ be nano $O_{16}$-closed then $H$ is closed if and only if $Ncl(H) - H$ is nano open.

Proof. Assume that $H$ is nano closed in $U$. Then $Ncl(H) = H$ and so $Ncl(H) - H = \emptyset$, which is nano open in $U$. Conversely, suppose $Ncl(H) - H$ is nano open in $U$. Since $H$ is nano $O_{16}$-closed, by Theorem 3.10, $Ncl(H) - H$ does not contain any non-empty nano open set in $U$. Then $Ncl(H) - H = \emptyset$, thus $H$ is nano closed in $U$.

Theorem 3.23. If $H$ is both nano open and nano $g$-closed in $U$ then nano $O_{16}$-closed

Proof. Let $H$ be nano open and nano $g$-closed in $U$. Let $H \subseteq G$ and let $G$ be open in $U$. Now $H \subseteq H$. By hypothesis $Ncl(H) \subseteq H$. That is $Ncl(H) \subseteq G$. Hence $H$ is nano $O_{16}$-closed.

Remark 3.24. We obtain Definitions, Theorems, Remarks and Examples follows from the implications.

nano regular-closed $\rightarrow$ nano $O_{16}$-closed $\leftarrow$ nano $g\alpha$-closed

None of the above implications are reversible.

References
