Domination Number of A Line Graph Formed from the Cartesian Product of Cycle and Path Graphs

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Abstract: In this paper we discussed the domination of line graph formed from the Cartesian product of cycle and path graphs. Also we have introduced disconnected total irregular dominating set and disconnected inverse irregular dominating set and various parameters of domination are applied on line graph and some relation between them are discussed with some examples.

Keywords: Domination number, Inverse Domination number, Total Domination number, disconnected inverse irregular dominating set, disconnected total irregular dominating set Line graph, Cartesian product, Path graph, Cycle graph.

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1. Introduction

In this paper we have taken the graphs to be undirected, finite and simple graph. One of the important concepts in the field of research is the domination in graphs. In 1958, the concept of domination in graph was defined by Claude Berge and Ore [1]. He also called domination number the Coefficient of External Stability. The first paper Optimal Domination in Graphs was published in 1975 by E. J. Cockayne and S. T Hedetniemi[2] and the notation \( \gamma(G) \) was first introduced by these two mathematicians. Domination has many applications which is used in chessboard, communication network, wireless Ad-hoc network, maps etc. also it has many parameters and in this paper we discussed about total domination number and inverse domination number. The concept of inverse domination Kulli and Sigarkanti [3] was the first to publish the paper on the Inverse Domination in Graphs. Cockayne, Dawes and Hedetniemi formalized total domination in graphs [4]. The term domination for undirected graphs was first published by Ore in 1962. Cartesian product of graphs have been described by Whitehead and Russel in 1912 according to Imrich and Klavzar [5]. The concept of line graph was invented by H. Whitney in 1932. We begin with some basic definitions and notations:

Definition 1.1. The Cartesian product of simple graphs \( G \) and \( H \) is denoted by \( G \times H \) whose vertex set is \( V(G) \times V(H) \) and \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are adjacent if \( u_1 = v_1 \) and \( u_2 \) is adjacent to \( v_2 \) in \( H \) or \( u_1 \) is adjacent to \( v_1 \) in \( G \) and \( u_2 = v_2 \) in \( H \).

Definition 1.2. The line graph of a simple graph \( G \) is obtained by means of associating a vertex with every edge of the graph and connecting two vertices with an edge iff the corresponding edges of \( G \) have a vertex in common. The Line graph of \( G \) is denoted by \( L(G) \).

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Definition 1.3. A subset $D$ of $V(G)$ is said to be a dominating set of graph $G$ if every $v \in V - D$ is adjacent to atleast one vertex in $D$. The minimum cardinality of a dominating set $D$ is called domination number. It is denoted by $\gamma(G)$.

Definition 1.4. If $D$ is a dominating set in $G$ and if a dominating set exists in $V/D$ say $D'$ then $D'$ is called the inverse dominating set. The inverse domination number denoted by $\gamma'(G)$ is the smallest size of the inverse dominating set.

Definition 1.5. If every vertex of $V$ is adjacent to some vertex in $D$ then the dominating set $D$ is said to be total dominating set. The minimum cardinality of total dominating set is called the total domination number and is denoted by $\gamma_t(G)$.

Definition 1.6. If the induced subgraph $(D)$ is disconnected then the total dominating set $D$ is said to be disconnected total dominating set. The disconnected total domination number is the minimum size of the disconnected total dominating set and is denoted by $\gamma_{td}(G)$.

Definition 1.7. A disconnected total dominating set is called disconnected total irregular dominating set if the induced subgraph $(D)$ is not regular graph. The disconnected total irregular domination number is the minimum size of the disconnected total irregular dominating set. It is denoted by $\gamma_{tdcir}(G)$.

Definition 1.8. A disconnected inverse dominating set is called disconnected inverse irregular dominating set if the induced subgraph $(D)$ is not regular graph. The disconnected inverse irregular domination number is the minimum size of the disconnected inverse irregular dominating set. It is denoted by $\gamma'_{tdcir}(G)$.

2. Relation Between $\gamma(G)$, $\gamma_t(G)$, $\gamma_{tdcir}(G)$, $\gamma'(G)$ and $\gamma'_{tdcir}(G)$ of $L(C_m \times C_n)$

Theorem 2.1. Let $C_m$ and $C_n$ be two cycle graphs with $m$ and $n$ vertices respectively and let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\gamma(G)$, $\gamma'(G)$, $\gamma_{tdcir}(G)$ are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected total irregular domination number of $G$ then $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] < p$ where $p$ is the no. of vertices in $L(G) \forall m, n > 2$.

Proof. The vertex set of $C_m$ be $U = \{u_1, u_2, \ldots, u_m\}$ and the vertex set of $C_n$ be $W = \{w_1, w_2, \ldots, w_n\}$. The graph $G$ has $mn$ vertices and $2mn$ edges and the line graph $L(G)$ has $2mn$ vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and inverse disconnected irregular domination number we consider the following cases:

Case 1: When $m$ and $n$ are even and distinct (i.e) $m = 4, n = 6$ the graph $G = C_4 \times C_6$ has 24 vertices and 48 edges in $G$ and $L(G)$ has 48 vertices.

Let $D = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}\}$ be the dominating set of $L(G)$ and so $\gamma[L(G)] = 12$. Let $D = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}, v_{16}, v_{17}, v_{18}\}$ be the inverse dominating set and $\gamma'[L(G)] = 12$. By definition of total dominating set we get $\gamma_t[L(G)] = 12$ and since the induced subgraph of $D$ is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{tdcir}[L(G)] = 12$. And also since the induced subgraph of $D'$ is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number we have $\gamma'_{tdcir}[L(G)] = 12$. Hence $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] = 12$ which is less than $p = 48$.

Case 2: When $m$ is even and $n$ is odd (i.e) $m = 6, n = 3$. The graph $G = C_6 \times C_3$ has 18 vertices and 36 edges and $L(G)$ has 36 vertices.

Let $D = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the dominating set of $L(G)$ which is the minimum dominating set of $(G)$.
and hence $\gamma[L(G)] = 9$. Let $D' = \{v_4, v_5, v_11, v_16, v_23, v_25, v_26, v_31, v_35\}$ be the inverse dominating set and so $\gamma'[L(G)] = 9$.

By definition of total dominating set we get $\gamma_t(G) = 9$. Since the induced subgraph of $D$ and $D'$ is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma_{dirc}[L(G)] = 9$ and disconnected total irregular domination number $\gamma_{tdecr}[L(G)] = 9$. Thus $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{dirc}[L(G)] = \gamma'[L(G)] = \gamma_{dirc}[L(G)] = 9$ which is less than $p = 36$.

**Case 3:** When $m$ and $n$ are odd and equal (i.e.) $m = 5, n = 5$. The graph $G = C_5 \times C_5$ has 25 vertices and 50 edges and $L(G)$ has 50 vertices.

Let $D = \{v_1, v_3, v_11, v_12\}$ be the dominating set of $L(G)$ and so $\gamma[L(G)] = 4$. Let $D' = \{v_4, v_5, v_6, v_{10}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 4$. By definition of total dominating set we get $\gamma_t(G) = 4$ and since the induced subgraph of $D$ is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{dirc}[L(G)] = 4$. Since the induced subgraph of $D'$ is disconnected and it is not regular graph so we have $\gamma_{dirc}[L(G)] = 4$. Therefore $\gamma[L(G)] = \gamma_t(G) = \gamma_{dirc}[G] = \gamma'[L(G)] = \gamma_{dirc}[L(G)] = 4$ which is less than $p = 50$.

**Case 4:** When $m$ and $n$ are even and equal (i.e.) $m = 4, n = 4$. The graph $G = C_4 \times C_4$ has 16 vertices and 32 edges and $L(G)$ has 32 vertices.

Let $D = \{v_6, v_{17}, v_{22}, v_{27}, v_{29}, v_{30}, v_{31}\}$ be the dominating set of $L(G)$ and so $\gamma[L(G)] = 8$. Let $D' = \{v_2, v_3, v_8, v_{11}, v_{12}, v_{18}, v_{23}, v_{24}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 8$. By definition of total dominating set we get $\gamma_t(G) = 8$ and since the induced subgraph of $D$ is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{dirc}[L(G)] = 8$. Since the induced subgraph of $D'$ is disconnected and it is irregular graph so by definition of disconnected inverse irregular domination number we get $\gamma_{dirc}[L(G)] = 8$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{dirc}[L(G)] = \gamma'[L(G)] = \gamma_{dirc}[L(G)] = 8$ which is less than $p = 32$.

**Case 5:** When $m$ and $n$ are odd and distinct (i.e.) $m = 3, n = 5$. The graph $G = C_3 \times C_5$ has 15 vertices and 30 edges and $L(G)$ has 30 vertices.

Since $D = \{v_2, v_4, v_{10}, v_{14}, v_{16}, v_{27}, v_{29}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 7$. Let $D = \{v_5, v_7, v_{12}, v_{16}, v_{19}, v_{28}, v_{30}\}$ be the inverse dominating set and hence $\gamma'[L(G)] = 7$. By definition of total dominating set we get $\gamma_t(L(G)) = 7$. Since the induced subgraph of $D$ and $D'$ is disconnected and it is irregular graph so by definition we have disconnected total irregular domination number $\gamma_{dirc}[L(G)] = 7$ and disconnected inverse irregular domination number $\gamma_{dirc}[L(G)] = 7$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{dirc}[L(G)] = \gamma'[L(G)] = \gamma_{dirc}[L(G)] = 7$ which is less than $p = 30$.

From all these cases we generalized that for all $m, n > 2$ we have $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{dirc}[L(G)] = \gamma'[L(G)] = \gamma_{dirc}[L(G)] < p$ where $p$ is the no. of vertices in $L(G)$.

**Example 2.2.** When $m = 4, n = 4$. The Cartesian product of $G = C_4 \times C_4$ is given in figure (1).

**Figure 1.** $C_4 \times C_4$

The line graph of figure (1) is shown in figure (2).
In Figure 2 we get the dominating set \( D = \{v_6, v_{17}, v_{22}, v_{25}, v_{27}, v_{29}, v_{31}\} \) and so \( \gamma[L(G)] = 8 \). Then the inverse dominating set is \( D' = \{v_2, v_8, v_{11}, v_{12}, v_{14}, v_{23}, v_{24}\} \) and hence \( \gamma'[L(G)] = 8 \) and by definition of disconnected total irregular domination number and disconnected inverse irregular domination number we have \( \gamma_{tdcir}[L(G)] = 8 \) and \( \gamma'_{tdcir}[L(G)] = 8 \).

3. Relation Between \( \gamma(G), \gamma_t(G), \gamma_{tdcir}(G), \gamma'(G) \) and \( \gamma'_{tdcir}(G) \) of \( L(P_m \times P_n) \)

**Theorem 3.1.** Let \( P_m \) and \( P_n \) be two path graphs with \( m \) and \( n \) vertices respectively and let \( G = P_m \times P_n \) be the Cartesian product of two path graphs. Let \( L(G) \) be the line graph and \( \gamma(G), \gamma_t(G), \gamma_{tdcir}(G), \gamma'(G), \gamma'_{tdcir}(G) \) are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected inverse irregular domination number of \( G \) then \( \gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] < p \) where \( p \) is the no. of vertices in \( L(G) \) and \( \gamma \ n \geq 2 \).

**Proof.** Let \( U = \{u_1, u_2, \ldots, u_m\} \) be the vertex set in \( P_m \) and the vertex set of \( P_n \) be \( W = \{w_1, w_2, \ldots, w_n\} \). The graph \( G \) has \( mn \) vertices and \( 2mn - (m + n) \) edges and the line graph \( L(G) \) has \( 2mn(m + n) \) vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and disconnected inverse irregular domination number we consider the following cases:

**Case 1:** When \( m \) and \( n \) are even and distinct (i.e.) \( m = 8, n = 2 \) then the graph \( G = P_8 \times P_2 \) has 16 vertices and 22 edges in \( G \) and \( L(G) \) has 22 vertices.

Let \( D = \{v_2, v_3, v_8, v_{10}, v_{13}, v_{17}, v_{18}, v_{22}\} \) and hence \( \gamma[L(G)] = 4 \). If \( D' = \{v_1, v_4, v_6, v_{10}, v_{15}, v_{20}, v_{21}\} \) then \( \gamma'[L(G)] = 4 \).

By definition of total dominating set we get \( \gamma_t(G) = 4 \). Since the induced subgraph of \( D \) and \( D' \) is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number \( \gamma'_{tdcir}[L(G)] = 4 \) and disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 4 \). Therefore \( \gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] = 4 \) which is less than \( p \).

**Case 2:** When \( m \) is odd and \( n \) is even (i.e.) \( m = 3, n = 6 \) the graph \( G = P_3 \times P_6 \) has 18 vertices and 27 edges and \( L(G) \) has 27 vertices.

If \( D = \{v_4, v_7, v_8, v_{12}, v_{20}, v_{21}, v_{26}, v_{27}\} \) then \( \gamma[L(G)] = 8 \). Let \( D = \{v_2, v_{10}, v_{14}, v_{15}, v_{17}, v_{18}, v_{23} \text{ and } v_{24}\} \) then \( \gamma'[L(G)] = 8 \).

By definition of total dominating set we get \( \gamma_t[L(G)] = 8 \) and since the induced subgraph of \( D \) is disconnected and not regular graph we have disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 8 \). Since the induced subgraph of \( D \) is disconnected and it is not regular graph so we have \( \gamma'_{tdcir}[L(G)] = 8 \). Therefore \( \gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] = 8 \).
\[ \gamma'(L(G)) = \gamma_{dcir}[L(G)] = 8 < 27. \]

**Case 3:** When \( m \) and \( n \) are odd and equal (i.e.) \( m = 3, n = 3 \). The graph \( G = P_3 \times P_3 \) has 9 vertices and 12 edges and \( L(G) \) has 12 vertices.

Let \( D = \{v_1, v_3, v_{11}, v_{12}\} \) and thus \( \gamma[L(G)] = 4 \). Let \( D' = \{v_4, v_5, v_6, v_{10}\} \) and hence \( \gamma'[L(G)] = 4 \). By definition of total dominating set we get \( \gamma_t[L(G)] = 4 \). Since the induced subgraph of \( D \) and \( D' \) is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number \( \gamma_{dcir}[L(G)] = 4 \) and disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 4 \).

**Case 4:** When \( m \) and \( n \) are even and equal (i.e.) \( m = 4, n = 4 \). The graph \( G = P_4 \times P_4 \) has 16 vertices and 24 edges and \( L(G) \) has 24 vertices.

Let \( D = \{v_2, v_3, v_7, v_{11}, v_{13}, v_{18}, v_{23}, v_{24}\} \) be the dominating set and \( \gamma[L(G)] = 8 \). Let \( D' = \{v_5, v_6, v_8, v_9, v_{12}, v_{17}, v_{20}, v_{21}\} \) and thus \( \gamma'[L(G)] = 8 \). By definition of total dominating set we get \( \gamma_t[L(G)] = 8 \) and since the induced subgraph of \( D \) is disconnected and not regular graph we have disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 8 \). Since the induced subgraph of \( D' \) is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number so we have \( \gamma_{dcir}[L(G)] = 8 \). Therefore \( \gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] = 8 < 24. \)

**Case 5:** When \( m \) and \( n \) are odd and distinct (i.e.) \( m = 5, n = 3 \). The graph \( G = P_5 \times P_3 \) has 15 vertices and 22 edges and \( L(G) \) has 22 vertices.

Let \( D = \{v_4, v_5, v_{11}, v_{12}, v_{17}, v_{18}, v_{19}\} \) be the dominating set of \( V[L(G)] \) and hence \( \gamma[L(G)] = 7 \). If \( D' = \{v_2, v_7, v_8, v_9, v_{14}, v_{16}, v_{20}\} \) then \( \gamma'[L(G)] = 7 \). By definition of total dominating set we get \( \gamma_t[L(G)] = 7 \). Since the induced subgraph of \( D \) and \( D' \) is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number \( \gamma_{dcir}[L(G)] = 7 \) and disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 7 \). Therefore \( \gamma[L(G)] = \gamma_t[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma_{dcir}[L(G)] = 7 \) which is less than \( p = 22. \)

From all these cases we conclude that for all \( m, n > 2 \) we have \( \gamma[L(G)] = \gamma_t(G) = \gamma_{tdcir}(G) = \gamma'[L(G)] = \gamma_{dcir}[L(G)] < p \) where \( p \) is the no. of vertices in \( L(G). \)

**Example 3.2.** The Cartesian product of \( G = P_3 \times P_6 \) is given in figure (3).

![Figure 3](image.png)

**Figure 3.** \( G = P_3 \times P_6 \)

The line graph of \( G = P_3 \times P_6 \) is shown in figure (4).
From figure 3. we have the dominating set \( D = \{v_4, v_7, v_8, v_{12}, v_{20}, v_{21}, v_{26}, v_{27}\} \) and so \( \gamma(L(G)) = 8 \). By definition of inverse dominating set, \( D' = \{v_2, v_{10}, v_{14}, v_{15}, v_{17}, v_{23}, v_{24}\} \) and hence \( \gamma'(L(G)) = 8 \) and the total dominating set \( t \gamma(L(G)) = 8 \). By definition of disconnected total irregular domination number and disconnected inverse irregular domination number we have \( \gamma_{tdcir}(L(G)) = 8 \) and \( \gamma'_{tdcir}(L(G)) = 8 \).

4. Relation Between \( \gamma(G), \gamma_{t}(G), \gamma_{tdcir}(G), \gamma'(G) \) and \( \gamma'_{tdcir}(G) \) of \( L(P_m \times C_n) \)

**Theorem 4.1.** Let \( P_m \) and \( C_n \) be two graphs with \( m \) and \( n \) vertices respectively and let \( G = P_m \times C_n \) be the Cartesian product of two cycle graphs. Let \( L(G) \) be the line graph and \( \gamma(G), \gamma_{t}(G), \gamma_{tdcir}(G), \gamma'(G), \gamma'_{tdcir}(G) \) are the domination number, total domination number, disconnected total irregular domination number, inverse domination number, disconnected inverse irregular domination number of \( G \) then \( \gamma(L(G)) = \gamma_{t}(L(G)) = \gamma_{tdcir}(L(G)) = \gamma'(L(G)) = \gamma'_{tdcir}(L(G)) < p \) where \( p \) is the no. of vertices in \( L(G) \).

**Proof.** The vertex set of \( P_m \) be \( U = \{u_1, u_2, \ldots, u_m\} \) and the vertex set of \( C_n \) be \( W = \{w_1, w_2, \ldots, w_n\} \). The graph \( G \) has \( mn \) vertices and \( 2mn - n \) edges and the line graph \( L(G) \) has \( 2mn - n \) vertices. To find the domination number, total domination number, disconnected total irregular domination number, inverse domination number and inverse disconnected irregular domination number we consider the following cases:

**Case 1:** When \( m \) and \( n \) are even and distinct (i. e) \( m = 4, n = 6 \) the graph \( G = P_4 \times C_6 \) has 24 vertices and 42 edges in \( G \) and \( L(G) \) has 42 vertices.

Since \( D = \{v_5, v_6, v_{11}, v_{12}, v_{21}, v_{26}, v_{29}, v_{30}, v_{33}, v_{36}, v_{37}, v_{41}\} \) is the dominating set of \( L(G) \) and hence \( \gamma[L(G)] = 13 \). Let \( D' = \{v_3, v_8, v_9, v_{13}, v_{14}, v_{17}, v_{18}, v_{24}, v_{28}, v_{31}, v_{32}, v_{35}, v_{39}\} \) be the inverse dominating set and \( \gamma'[L(G)] = 13 \). By definition of total dominating set we get \( \gamma[L(G)] = 13 \) and since the induced subgraph of \( D \) is disconnected and not regular graph we have disconnected total irregular domination number \( \gamma_{tdcir}[L(G)] = 13 \). Since the induced subgraph of \( D' \) is disconnected and it is not regular graph so by definition of disconnected inverse irregular domination number we have \( \gamma'_{tdcir}[L(G)] = 13 \).

Hence \( \gamma[L(G)] = \gamma_{t}(L(G)) = \gamma_{tdcir}[L(G)] = \gamma[L(G)] = \gamma_{tdcir}[L(G)] = \gamma'[L(G)] = \gamma'_{tdcir}[L(G)] = 13 < 42 \).

**Case 2:** When \( m \) is even and \( n \) is odd (i. e) \( m = 6, n = 3 \). The graph \( G = P_6 \times C_3 \) has 18 vertices and 33 edges in \( G \) and \( L(G) \) has 33 vertices.

Let \( D = \{v_4, v_6, v_7, v_{10}, v_{12}, v_{21}, v_{22}, v_{24}, v_{30}, v_{31}\} \) be the dominating set of \( V[L(G)] \) which is the minimum dominating set of \( G \) and hence \( \gamma[L(G)] = 10 \). Let \( D' = \{v_2, v_8, v_{15}, v_{17}, v_{23}, v_{27}, v_{28}, v_{29}\} \) be the inverse dominating set and so \( \gamma'[L(G)] = 10 \). By definition of total dominating set we get \( \gamma[L(G)] = 10 \). Since the induced sub-
graph of $D$ and $D'$ is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma_{\text{deir}}[L(G)] = 10$ and disconnected total irregular domination number $\gamma_{\text{deir}}[L(G)] = 10$. Thus $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{\text{deir}}[L(G)] = \gamma'[L(G)] = \gamma_{\text{deir}}[L(G)] = 10$ which is less than $p = 33$.

**Case 3:** When $m$ and $n$ are odd and equal (i.e.) $m = 3, n = 3$. The graph $G = P_3 \times C_3$ has 9 vertices and 15 edges and $L(G)$ has 15 vertices. Since $D = \{v_2, v_6, v_8, v_{10}, v_{12}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 5$. Let $D' = \{v_1, v_6, v_7, v_{11}, v_{13}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 5$. By definition of total dominating set we get $\gamma_t[L(G)] = 5$ and since the induced subgraph of $D$ is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{\text{deir}}[L(G)] = 5$. Since the induced subgraph of $D'$ is disconnected and it is not regular graph so we have $\gamma_{\text{deir}}[L(G)] = 5$.

Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{\text{deir}}[L(G)] = \gamma'[L(G)] = \gamma_{\text{deir}}[L(G)] = 5$ which is less than $p = 50$.

**Case 4:** When $m$ and $n$ are even and equal (i.e.) $m = 4, n = 4$. The graph $G = P_4 \times C_4$ has 16 vertices and 28 edges and $L(G)$ has 28 vertices. Let $D = \{v_1, v_4, v_5, v_{12}, v_{14}, v_{17}, v_{20}, v_{25}, v_{28}\}$ is the dominating set of $L(G)$ and so $\gamma[L(G)] = 9$. Let $D' = \{v_3, v_6, v_7, v_{11}, v_{13}, v_{16}, v_{18}, v_{23}, v_{27}\}$ be the inverse dominating set of $V[L(G)]$ and hence $\gamma'[L(G)] = 9$. By definition of total dominating set we get $\gamma_t[L(G)] = 9$. Since the induced subgraph of $D$ and $D'$ is disconnected and it is not regular graph so by definition we have disconnected inverse irregular domination number $\gamma_{\text{deir}}[L(G)] = 9$ and disconnected total irregular domination number $\gamma_{\text{deir}}[L(G)] = 9$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{\text{deir}}[L(G)] = \gamma'[L(G)] = \gamma_{\text{deir}}[L(G)] = 9$ which is less than $p = 28$.

**Case 5:** When $m$ and $n$ are odd and distinct (i.e.) $m = 5, n = 3$. The graph $G = P_5 \times C_3$ has 15 vertices and 27 edges and $L(G)$ has 27 vertices. Since $D = \{v_1, v_4, v_9, v_{13}, v_{20}, v_{22}, v_{23}\}$ is the dominating set of $L(G)$ and hence $\gamma[L(G)] = 7$. Let $D' = \{v_3, v_8, v_{11}, v_{12}, v_{21}, v_{25}, v_{26}\}$ be the inverse dominating set and hence $\gamma'[L(G)] = 7$. By definition of total dominating set we get $\gamma_t[L(G)] = 7$ and since the induced subgraph of $D$ is disconnected and not regular graph we have disconnected total irregular domination number $\gamma_{\text{deir}}[L(G)] = 7$. Since the induced subgraph of $D'$ is disconnected and it is irregular graph so by definition of disconnected inverse irregular domination number we have $\gamma_{\text{deir}}[L(G)] = 7$. Therefore $\gamma[L(G)] = \gamma_t[L(G)] = \gamma_{\text{deir}}[L(G)] = \gamma'[L(G)] = \gamma_{\text{deir}}[L(G)] = 7 < 27$.

From all these cases we generalized that for all $m, n > 2$ we have $\gamma[L(G)] = \gamma'[L(G)] = \gamma_{\text{deir}}[L(G)] < p$ where $p$ is the no. of vertices in $L(G)$.

**Example 4.2.** When $m = 4, n = 4$. The Cartesian product of $G = P_4 \times C_4$ is given in figure (5).

![Figure 5](image)

**Figure 5.** $G = P_5 \times P_3$

The line graph of $G = P_4 \times C_4$ is shown in figure (6).
Theorem 5.1. Let \( L(G) \) be the line graph and \( \gamma \) be the domination set, then \( \gamma[\text{L}(G)] = 2 \). When \( \gamma \) is the domination number we have \( \gamma_1[L(G)] = 9 \) and \( \gamma'_d[L(G)] = 9 \). By definition of inverse domination number and inverse disconnected irregular domination number we get \( \gamma_1[L(G)] = 9 \). By definition of total domination set we get \( \gamma_t[L(G)] = 9 \). When we consider the following cases:

Case 1: When \( m \) and \( n \) are odd (i.e.) \( m = 5, n = 5 \). The graph \( G_1 = (C_m \times C_n) \) has 25 vertices and 40 edges and \( L(G) \) has 40 vertices. The dominating set of [\( L(G) \)] denoted is \( D = \{v_1, v_4\} \) (i.e.) \( \gamma[L(G)] = 2 \). The dominating set of \( V \)-D denoted by \( D' = \{v_2, v_{12}\} \) (i.e.) \( \gamma'[L(G)] = 2 \). Therefore \( \gamma[L(G)] = \gamma'[L(G)] = 2 < 40 \).

Case 2: When \( m = n = 6 \) and \( m = 4, n = 6 \) and \( m = 5, n = 4 \). The graph \( G_2 = (P_m \times P_n) \) has 24 vertices and 38 edges and \( L(G) \) has 38 vertices. The dominating set of \( L(G) \) is \( D = \{v_1, v_5\} \). (i.e.) \( \gamma[L(G)] = 2 \). The dominating set of \( V \)-D denoted by \( D' = \{v_{15}, v_{34}\} \). (i.e.) \( \gamma'[L(G)] = 2 \). Therefore \( \gamma[L(G)] = \gamma'[L(G)] = 2 < 38 \).

Case 3: When \( m = n \) odd and \( m = 5, n = 3 \). The graph \( G = (P_m \times P_n) \) has 15 vertices and 22 edges and

5. Domination Number of Complement of \( L(P_m \times P_n) \)

In this section we obtain the domination number of complement of line graph formed from the Cartesian product of two path graphs.

Theorem 5.1. Let \( P_m \) and \( P_n \) be two path graphs with \( m \) and \( n \) vertices respectively and let \( G = P_m \times P_n \) be the Cartesian product of two path graphs. Let \( L(G) \) be the line graph and \( L(G) \) be the complement of line graph then the domination number then \( \gamma[L(G)] = \gamma'[L(G)] = 2 \forall m, n > 2 \).

Proof. Let \( U = \{u_1, u_2, \ldots, u_m\} \) be the vertex set in \( P_m \) and the vertex set of \( P_n \) be \( W = \{w_1, w_2, \ldots, w_n\} \). The graph \( G \) has \( mn \) vertices and \( 2mn - (m + n) \) edges and the line graph \( L(G) \) has \( 2mn(m + n) \) vertices. To find the domination number, inverse domination number and inverse disconnected regular domination number we consider the following cases:

Case 1: When \( m = 5, n = 5 \). The graph \( G = P_5 \times P_5 \) has 25 vertices and 40 edges and \( L(G) \) has 40 vertices. The dominating set of \( L(G) \) denoted is \( D = \{v_1, v_4\} \) (i.e.) \( \gamma[L(G)] = 2 \). The dominating set of \( V \)-D denoted by \( D' = \{v_2, v_{12}\} \) (i.e.) \( \gamma'[L(G)] = 2 \). Therefore \( \gamma[L(G)] = \gamma'[L(G)] = 2 < 40 \).

Case 2: When \( m = n \) even and \( m = 4, n = 6 \). The graph \( G = P_4 \times P_6 \) has 24 vertices and 38 edges and \( L(G) \) has 38 vertices. The dominating set of \( L(G) \) is \( D = \{v_1, v_5\} \). (i.e.) \( \gamma[L(G)] = 2 \). The dominating set of \( V \)-D denoted by \( D' = \{v_{15}, v_{34}\} \). (i.e.) \( \gamma'[L(G)] = 2 \). Therefore \( \gamma[L(G)] = \gamma'[L(G)] = 2 < 38 \).

Case 3: When \( m = n \) odd and \( m = 5, n = 3 \). The graph \( G = P_5 \times P_3 \) has 15 vertices and 22 edges and

Figure 6. \( L(G) \)
$L(G)$ has 22 vertices.

The dominating set of $[L(G)]$ denoted by $D$ is $v_1, v_9$. (i. e) $\gamma[L(G)] = 2$. The dominating set of $V-D$ denoted by $D'$ is $v_2$ and $v_8$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[L(G)] = \gamma'[L(G)] = 2$ which is less than $p = 22$.

**Case 4:** When $m$ and $n$ are even and equal (i. e) $m = 6, n = 6$. The graph $G = P_6 \times P_6$ has 36 vertices and 60 edges and $L(G)$ has 60 vertices.

The dominating set of $[L(G)]$ is $D = \{v_2, v_5\}$. (i. e) $\gamma[L(G)] = 2$. The dominating set of $V-D$ is denoted by $D' = \{v_{40}, v_{47}\}$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[L(G)] = \gamma'[L(G)] = 2 < 60$.

**Case 5:** When $m$ is even and $n$ is odd (i. e) $m = 5, n = 4$. The graph $G = P_5 \times P_4$ has 20 vertices and 31 edges and $L(G)$ has 31 vertices.

The dominating set of $[L(G)]$ is denoted by $D = \{v_4, v_9\}$. (i. e) $\gamma[L(G)] = 2$. The dominating set of $V-D$ is denoted by $D' = \{v_3, v_5\}$. (i. e) $\gamma'[L(G)] = 2$. Therefore $\gamma[L(G)] = \gamma'[L(G)] = 2 < p = 31$.

From all these cases we can generalize that $\gamma[L(G)] = \gamma'[L(G)] = 2 \forall m, n > 2$.

**Example 5.2.** When $m = 3, n = 3$. The Cartesian product of $G = P_3 \times P_3$ is given in figure (7).

![Figure 7](image1.png)

Figure 7. $G = P_3 \times P_3$

The complement of $L(G)$ is shown in figure (8).

![Figure 8](image2.png)

Figure 8. $\overline{L(G)}$

**Theorem 5.3.** Let $C_m$ and $C_n$ be two cycle graphs with $m$ and $n$ vertices respectively and let $G = C_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\overline{L(G)}$ be the complement of line graph $\gamma$ be the domination number then $\gamma[L(G)] = 2 \forall m, n > 2$.

**Proof.** The proof of this theorem follows from the Theorem 5.1. □
**Theorem 5.4.** Let $P_m$ and $C_n$ be two graphs with $m$ and $n$ vertices respectively and let $G = P_m \times C_n$ be the Cartesian product of two cycle graphs. Let $L(G)$ be the line graph and $\overline{L(G)}$ be the complement of line graph $\gamma$ be the domination number then $\gamma[L(G)] = 2 \forall m \geq 2$ and $n > 2$.

**Proof.** The proof of this theorem follows from the Theorem 5.1.

6. Conclusion

The concept of domination number of line graphs formed from the Cartesian product of path and cycle graphs are defined and the relation between many domination parameters are characterized and described with few examples. This work can be used for further research to any other simple graph products.

References