Int. J. Math. And Appl., 6(2–B)(2018), 25–32
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Certain Investigations on Digital Plane

S. Pious Missier[1] and K. M. Arifmohammed[1,*]

1 P.G. and Research Department of Mathematics, V.O.Chidambaram College (Affiliation of Manonmaiam Sundaranar University), Abishekapatti, Tirunelveli, Tamilnadu, India.

Abstract: We introduce the concept of $^{\#}g\hat{\alpha}$-closed sets in a topological space and characterize it using $^{*}g\alpha o$-kernel and τ^{α}-closure. Moreover, we investigate the properties of $^{\#}g\hat{\alpha}$-closed sets in digital plane. The family of all $^{\#}g\hat{\alpha}$-open sets of $(\mathbb{Z}^2, \kappa^2)$, forms an alternative topology of $\mathbb{Z}^2$. We prove that this plane $(\mathbb{Z}^2, ^{\#}g\hat{\alpha}O)$ is $T_{1/2}$ and $T_{3/4}$. It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$, even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

MSC: 54A05, 54D01, 54F65, 54G05.

Keywords: Preopen sets, generalized closed sets, α-open sets, $^{*}g\alpha$-closed sets, $^{\#}g\hat{\alpha}$-open sets, $T_{1/2}$-spaces, digital lines and digital planes.

Accepted on: 21.04.2018

1. Introduction

In 1970, N. Levine [8] introduced and investigated the concept of generalized closed sets in a topological space. He studied most fundamental properties and also introduced a separation axiom $T_{1/2}$. The digital line is typical example of a $T_{1/2}$ space [2]. After Levine's works, many authors defined and investigated various notions to Levine's g-closed sets and related topics [4]. In 1970, E. Khalimsky [6] introduced digital line. In 1990, K. Kopperman and R. Meyer [5] developed finite analog of the Jordan curve theorem motivated by a problem in computer graphics (cf. [5, 7]). In this paper, we introduce the concept of $^{\#}g\hat{\alpha}$-closed sets in a topological space and characterize it using $^{*}g\alpha o$-kernel and τ^{α}-closure. Moreover, we investigate the properties of $^{\#}g\hat{\alpha}$-closed sets in digital plane. We prove that this plane $(\mathbb{Z}^2, ^{\#}g\hat{\alpha}O)$ is $T_{1/2}$ and $T_{3/4}$. It is well known that the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$, even if $(\mathbb{Z}, \kappa)$ is $T_{1/2}$.

2. Preliminaries

Throughout this paper, (X, τ) or X denotes the topological spaces. For a subset A of X, the closure, the interior and the complement of A are denoted by $cl(A)$, $int(A)$ and A^c respectively. We recall some basic definitions that are used in the sequel.

Definition 2.1. *A subset A of a topological space (X, τ) is called α-open [10] if $A \subseteq int(cl(int(A)))$. Moreover, A is said to be α-closed if $X \backslash A$ is α-open. The collection of all α-open subsets in (X, τ) is denoted by τ^{α}. The α-closure of a subset A is the smallest α-closed set containing A and this is denoted by τ^{α}-$cl(A)$ in this paper.*

* *E-mail: arifjmc9006@gmail.com (Research Scholar)*

Definition 2.2. *A subset A of a topological space (X, τ) is called $^*g\hat{\alpha}$-closed [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an $^*g\alpha$-open set in (X, τ). Moreover, A is said to be $^*g\hat{\alpha}$-open if $X \backslash A$ is $^*g\hat{\alpha}$-closed.*

Lemma 2.3 ([9])**.** *For a subset A of (X, τ), the following conditions are equivalent:*

(1). A is $^*g\alpha$-closed in (X, τ).

(2). τ^α-$cl(A) \subseteq go$-$Ker(A)$ holds.

Lemma 2.4 ([9])**.** *Let a subset A of $(\mathbb{Z}^2, \kappa^2)$.*

(1). go-$Ker(A) = U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$, where $U(A_{\mathcal{F}^2}) = \bigcup\{U(x) | x \in A_{\mathcal{F}^2}\}$.

(2). For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, a subset $\{x\} \cup (U(x))_{\kappa^2}$ is preopen and hence it is α-open in $(\mathbb{Z}^2, \kappa^2)$.

Definition 2.5 ([2])**.** *A space (X, τ) is $T_{3/4}$ if and only if every singleton $\{x\}$ of X is closed or regular open in (X, τ).*

3. $^\# g\hat{\alpha}$-closed Sets and its Properties

In this section we introduce the concept of $^\# g\hat{\alpha}$-closed sets and study some of their properties and relations with other known classes of subsets.

Definition 3.1. *A subset A of a space (X, τ) is called a $^\# g\hat{\alpha}$-closed set if τ^α-$cl(A) \subseteq U$ whenever $A \subseteq U$ and U is a $^*g\alpha$-open set in (X, τ). The class of $^\# g\hat{\alpha}$-closed subsets of (X, τ) is denoted by $^\# g\hat{\alpha}C(X, \tau)$.*

Theorem 3.2. *Finite union of $^\# g\hat{\alpha}$-closed sets is a $^\# g\hat{\alpha}$-closed set in (X, τ).*

Proof. Let A_i's are $^\# g\hat{\alpha}$-closed sets, where $i = 1, 2, 3, ..., n$ and $n \in \mathbb{N}$. Let $\bigcup_{i=1}^n A_i \subseteq U$, U is a $^*g\alpha$-open set xin (X, τ). Since A_i's are $^\# g\hat{\alpha}$-closed sets, τ^α-$cl(A_i) \subseteq U, \forall A_i \subseteq U$. This implies that τ^α-$cl(\bigcup_{i=1}^n A_i) = \bigcup_{i=1}^n \tau^\alpha$-$cl(A_i) \subseteq U$. Therefore $\bigcup_{i=1}^n A_i$ is $^\# g\hat{\alpha}$-closed. □

Remark 3.3. *Finite intersection of $^\# g\hat{\alpha}$-open sets is a $^\# g\hat{\alpha}$-open set in (X, τ).*

Proof. Proof is obvious, since $X \backslash A$ is $^\# g\hat{\alpha}$-open, whenever A is $^\# g\hat{\alpha}$-closed. □

The following example shows that intersection of two $^\# g\hat{\alpha}$-closed sets need not be $^\# g\hat{\alpha}$-closed in (X, τ).

Example 3.4. *Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then, $\{a, b\}$ and $\{a, c\}$ are $^\# g\hat{\alpha}$-closed but their intersection $\{a\}$ is not $^\# g\hat{\alpha}$-closed in (X, τ).*

Theorem 3.5. *If A be a $^\# g\hat{\alpha}$-closed set in (X, τ), then τ^α-$cl(A) \backslash A$ does not contain any non empty $^*g\alpha$-closed set.*

Proof. Suppose that A is $^\# g\hat{\alpha}$-closed and let F be an non-empty $^*g\alpha$-closed set with $F \subseteq \tau^\alpha$-$cl(A) \backslash A$. Then $A \subseteq X \backslash F$ ans so τ^α-$cl(A) \subseteq X \backslash \tau^\alpha$-$cl(A)$. Henece $F \subseteq X \backslash \tau^\alpha$-$cl(A)$, a contradiction. □

Theorem 3.6. *Let (X, τ) be a space, A and B subsets.*

(1). If A is $^*g\alpha$-open and $^\# g\hat{\alpha}$-closed, then A is α-closed in (X, τ).

(2). If A is $^\# g\hat{\alpha}$-closed set of (X, τ) such that $A \subseteq B \subseteq \tau^\alpha$-$cl(A)$, then B is also $^\# g\hat{\alpha}$-closed in (X, τ).

(3). For each $x \in X$, $\{x\}$ is $^*g\alpha$-closed or $X \backslash \{x\}$ is $^\# g\hat{\alpha}$-closed in (X, τ).

(4). Every subset is $^{\#}g\hat{\alpha}$-closed in (X,τ) if and only if every $^{*}g\alpha$-open set is α-closed.

Proof.

(1). Since $A \subseteq A$ and A is both $^{*}g\alpha$-open and $^{\#}g\hat{\alpha}$-closed, $\tau^{\alpha}\text{-}cl(A) \subseteq A$. Therefore A is α-closed.

(2). Let U be a $^{*}g\alpha$-open set such that $B \subseteq U$. Then we have that $\tau^{\alpha}\text{-}cl(A) \subseteq U$ and $\tau^{\alpha}\text{-}cl(B) \subseteq \tau^{\alpha}\text{-}cl(A) \subseteq U$. Therefore, B is $^{\#}g\hat{\alpha}$ closed in (X,τ).

(3). If $\{x\}$ is not $^{*}g\alpha$-closed, then $X\backslash\{x\}$ is not $^{*}g\alpha$-open. Therefore, $X\backslash\{x\}$ is $^{\#}g\hat{\alpha}$-closed in (X,τ).

(4). **Necessity:** Let U be a $^{*}g\alpha$-open set. Then we have that $\tau^{\alpha}\text{-}cl(U) \subseteq U$ and hence U is α-closed. **Sufficiency:** Let A be a subset and U a $^{*}g\alpha$-open set such that $A \subseteq U$. Then $\tau^{\alpha}\text{-}cl(A) \subseteq \tau^{\alpha}\text{-}cl(U) = U$ and hence A is $^{\#}g\hat{\alpha}$-closed. □

We have a characterization of $^{\#}g\hat{\alpha}$-closed sets. We prepare some notations and a lemma. For a subset E of a space (X,τ), we define the following subsets of E:

$E_{\tau} = \{x \in E|\{x\} \in \tau\}$, $E_{\mathcal{F}} = \{x \in E|\{x\} \in \tau^{c}\}$, $E_{^{*}g\alpha o} = \{x \in E|\{x\}$ is $^{*}g\alpha$-open in $(X,\tau)\}$, $E_{^{*}g\alpha c} = \{x \in E|\{x\}$ is $^{*}g\alpha$-closed in $(X,\tau)\}$, $E_{^{\#}g\hat{\alpha}o} = \{x \in E|\{x\}$ is $^{\#}g\hat{\alpha}$-open in $(X,\tau)\}$, $^{*}G\alpha O(X,\tau) = \{U|U$ is $^{*}g\alpha$-open in $(X,\tau)\}$ and $^{*}G\alpha O\text{-}ker(A) = \bigcap\{U|U \in {}^{*}G\alpha O(X,\tau)$ and $A \subseteq U\}$.

Theorem 3.7. *Any subset A is $^{\#}g\hat{\alpha}$-closed if and only if $\tau^{\alpha}\text{-}cl(A) \subseteq {}^{*}G\alpha O\text{-}ker(A)$ holds.*

Proof. **Necessary:** We know that $A \subseteq {}^{*}G\alpha O\text{-}ker(A)$. Since A is $^{\#}g\hat{\alpha}$-closed, $\tau^{\alpha}\text{-}cl(A) \subseteq {}^{*}G\alpha O\text{-}ker(A)$. **Sufficiency:** Let $A \subseteq U$ and U is $^{*}g\alpha$-open. Given that $\tau^{\alpha}\text{-}cl(A) \subseteq {}^{*}G\alpha O\text{-}ker(A)$. If $\tau^{\alpha}\text{-}cl(A) \nsubseteq U$, then $\tau^{\alpha}\text{-}cl(A) \nsubseteq {}^{*}G\alpha O\text{-}ker(A)$, which is a contradiction. Therefore A is $^{\#}g\hat{\alpha}$-closed. □

Lemma 3.8. *For any space (X,τ), $X = X_{^{*}g\alpha c} \cup X_{^{\#}g\hat{\alpha}o}$ holds.*

Proof. Let $x \in X$. By Theorem 3.6(3), $\{x\} \in X_{^{*}g\alpha c}$ or $\{x\} \in X_{^{\#}g\hat{\alpha}o}$. □

4. $^{\#}g\hat{\alpha}$-closed Sets in the Digital Plane

In the digital plane, we investigate explicite forms of $^{*}g\alpha o$-Kernal and Kernal of a subset. The digital line or the so called Khalimsky line is the set of the integers $\mathbb{Z}$, equipped with the topology κ having $\{\{2n-1, 2n, 2n+1\}|n \in \mathbb{Z}\}$ as a subbase. This is denoted by $(\mathbb{Z},\kappa)$. Thus a subset U is open in $(\mathbb{Z},\kappa)$ if and only if whenever $x \in U$ is an even integer, then $x-1, x+1 \in U$. Let $(\mathbb{Z}^2,\kappa^2)$ be the topological product of two digital lines $(\mathbb{Z},\kappa)$, where $\mathbb{Z}^2 = \mathbb{Z}\times\mathbb{Z}$ and $\kappa^2 = \kappa\times\kappa$. This space is called the digital plane in the present paper (cf. [5], [7]). We note that for each point $x \in \mathbb{Z}^2$ there exists the smallest open set containing x, say $U(x)$. For the case of $x = (2n+1, 2m+1), U(x) = \{2n+1\}\times\{2m+1\}$; for the case $x = (2n, 2m), U(x) = \{2n-1, 2n, 2n+1\}\times\{2m-1, 2m, 2m+1\}$; for the case $x = (2n, 2m+1), U(x) = \{2n-1, 2n, 2n+1\}\times\{2m+1\}$; for the case $x = (2n+1, 2m), U(x) = \{2n+1\}\times\{2m-1, 2m, 2m+1\}$, where $n, m \in \mathbb{Z}$. For a subset E of $(\mathbb{Z}^2,\kappa^2)$, we define the following three subsets as follows: $E_{\mathcal{F}^2} = \{x \in E|\{x\}$ is closed in $(\mathbb{Z}^2,\kappa^2)\}$; $E_{\kappa^2} = \{x \in E|\{x\}$ is open in $(\mathbb{Z}^2,\kappa^2)\}$; $E_{mix} = E\backslash(E_{\mathcal{F}^2} \cup E_{\kappa^2})$.

Lemma 4.1. *Let A and E be subsets of $(\mathbb{Z}^2,\kappa^2)$.*

(1). If E be non-empty $^{*}g\alpha$-closed set, then $E_{\mathcal{F}^2} \neq \emptyset$ [1].

(2). If E is $^{*}g\alpha$-closed and $E \subseteq B_{mix} \cup B_{\kappa^2}$ holds for some subset B of $(\mathbb{Z}^2,\kappa^2)$, then $E = \emptyset$ [1].

(3). The set $U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$ is a $^{\#}g\hat{\alpha}$-open set containing A.

Proof. (3) First we claim that $A_{mix} \cup A_{\kappa^2}$ is $^{\#}g\hat{\alpha}$-open set. Let F be a non-empty $^{*}g\alpha$-closed set such that $F \subseteq A_{mix} \cup A_{\kappa^2}$. Then by (2), $F = \emptyset$. Thus, we have that $F \subseteq \tau^{\alpha}\text{-}int(A_{mix} \cup A_{\kappa^2})$. Therefore $A_{mix} \cup A_{\kappa^2}$ is $^{\#}g\hat{\alpha}$-open. Since every open set is $^{\#}g\hat{\alpha}$-open, $U(A_{\mathcal{F}^2})$ is $^{\#}g\hat{\alpha}$-open. Since union of two $^{\#}g\hat{\alpha}$-open sets is $^{\#}g\hat{\alpha}$-open, $U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$ is a $^{\#}g\hat{\alpha}$-open set containing A. □

Theorem 4.2 ([1]). *Let E be a subset of $(\mathbb{Z}^2, \kappa^2)$.*

(1). If E is a non-empty $^{*}g\alpha$-closed set, then $E_{\mathcal{F}^2} \neq \emptyset$.

(2). If E is a $^{*}g\alpha$-closed set and $E \subseteq B_{mix} \cup B^2_{\kappa}$ holds for some subset B of $(\mathbb{Z}^2, \kappa^2)$, then $E = \emptyset$.

Theorem 4.3 ([1]). *Let E be a subset of $(\mathbb{Z}^2, \kappa^2)$.*

(1). $^{*}G\alpha O\text{-}ker(A) = U(A_{\mathcal{F}^2}) \cup A_{mix} \cup A_{\kappa^2}$, $U(A_{\mathcal{F}^2}) = \bigcup\{U(x) | x \in A_{\mathcal{F}^2}\}$.

(2). $G\alpha O\text{-}ker(A) = U(A_{\mathcal{F}^2})$, $U(A_{\mathcal{F}^2}) = \bigcup\{U(x) | x \in A_{\mathcal{F}^2}\}$.

Theorem 4.4. *Let B be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$. If $B_{\mathcal{F}^2} = \emptyset$, then B is $^{\#}g\hat{\alpha}$-open.*

Proof. Let F be a $^{*}g\alpha$-closed set such that $F \subseteq B$. Since $B_{\mathcal{F}^2} = \emptyset$, we have $B = B_{mix} \cup B^2_{\kappa}$. Then by Theorem 4.2(2), we get $F = \emptyset \Rightarrow F \subseteq \tau^{\alpha}\text{-}int(B)$. Therefore, B is $^{\#}g\hat{\alpha}$-open. □

Theorem 4.5. *Let B be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$, the following are equivalent:*

(1). The subset B is $^{\#}g\hat{\alpha}$-open set of $(\mathbb{Z}^2, \kappa^2)$,

(2). $(U(x))_{\kappa^2} \subseteq B$ holds for each point $x \in B_{\mathcal{F}^2}$.

Proof. (1) $\Rightarrow$ (2) Let $x \in B_{\mathcal{F}^2}$. Since $\{x\}$ is closed, $\{x\}$ is $^{*}g\alpha$-closed set and $\{x\} \subseteq B$. By (1), $\{x\} \subset \tau^{\alpha}\text{-}int(B) = B \cap int(cl(int(B)))$ and so $x \in int(cl(int(B)))$. Namely, x is an interior point of the set $cl(int(B))$. Thus, we have that, for the smallest open set $U(x)$ containing x, $U(x) \subset cl(int(B))$. We can set $x = (2s, 2u)$ for some integers s and u, because $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Since $U((2s, 2u)) = \{2s-1, 2s, 2s+1\} \times \{2u-1, 2u, 2u+1\}$, it is shown that $(U(x))_{\kappa^2} = \{(x_1, x_2) \in U(x) | x_1$ and x_2 are odd$\} = \{p_1, p_2, p_3, p_4\}$, where $p_1 = (2s-1, 2u-1), p_2 = (2s-1, 2u+1), p_3 = (2s+1, 2u+1), p_4 = (2s+1, 2u+1)$. For each point $p_i (1 \leq i \leq 4)$, $p_i \in cl(int(B))$ and so $\{p_i\} \cap int(B) \neq \emptyset$. Therefore, $p_i \in B$ for each i with $1 \leq i \leq 4$ and hence $(U(x))_{\kappa^2} \subset B$.
(2) $\Rightarrow$ (1) It follows from the assumption that, for each point $x \in B_{\mathcal{F}^2}$, $\{x\} \cup (U(x))_{\kappa^2} \subset B$ and so $\bigcup\{\{x\} \cup (U(x))_{\kappa^2} | x \in B_{\mathcal{F}^2}\} \subset B$. Put $V_B = \bigcup\{\{x\} \cup (U(x))_{\kappa^2} | x \in B_{\mathcal{F}^2}\}$ and so $V_B \neq \emptyset$, $V_B \subset B$. By Lemma 2.4(2), V_B is preopen and it is α-open. We have that $B = V_B \cup (B \backslash V_B) = V_B \cup \{(B\backslash V_B)_{\mathcal{F}^2} \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)_{mix}\} = V_B \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)_{mix}$, we note that, for a point $y \in (B\backslash V_B)_{mix}, U(y) \subset B$ or $U(y) \nsubseteq B$. we put $(B\backslash V_B)^1_{mix} = \{y \in (B\backslash V_B)_{mix} | U(y) \subseteq B\}$, $U((B\backslash V_B)^1_{mix}) = \bigcup\{U(y) | y \in (B\backslash V_B)^1_{mix}\}$, $(B\backslash V_B)^2_{mix} = \{y \in (B\backslash V_B)_{mix} | U(y) \nsubseteq B\}$. Then, $(B\backslash V_B)_{mix}$ is decomposed as $(B\backslash V_B)_{mix} = (B\backslash V_B)^1_{mix} \cup (B\backslash V_B)^2_{mix}$. Thus, we have that:
$(*^1)$ $B = V_B \cup (B\backslash V_B)_{\kappa^2} \cup (B\backslash V_B)^1_{mix} \cup (B\backslash V_B)^2_{mix}$. Here, V_B is α-open in $(\mathbb{Z}^2, \kappa^2)$; the set $(B\backslash V_B)_{\kappa^2}$ is open and so α-open in $(\mathbb{Z}^2, \kappa^2)$; $U((B\backslash V_B)^1_{mix})$ is open and so α-open in $(\mathbb{Z}^2, \kappa^2)$. Thus, we have that:
$(*^2)$ the subset $V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$ is α-open in $(\mathbb{Z}^2, \kappa^2)$.
Moreover, we conclude that:
$(*^3)$ $B = V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}$ holds.
Proof of $(*^3)$: Since $(B\backslash V_B)^1_{mix} \subseteq U((B\backslash V_B)^1_{mix})$, it is shown that $B \subseteq V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}$(c.f $*^1$). Conversely we have that $V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix} \subseteq B$, because $U((B\backslash V_B)^1_{mix}) \subseteq B$, $V_B \subseteq B$,

$(B\backslash V_B)_{\kappa^2} \subseteq B$ and $(B\backslash V_B)^2_{mix} \subseteq B$ hold. Thus, we have the required equality $(*^3)$. Let F be a nonempty ${}^*g\alpha$-closed set of $(\mathbb{Z}^2, \kappa^2)$ such that $F \subseteq B$. We claim that:

$(*^4)$ $F \cap ((B\backslash V_B)^2_{mix}) = \emptyset$ holds.

Proof of $(*^4)$: Suppose that there exists a point $y \in F \cap ((B\backslash V_B)^2_{mix})$. Then we have that:

$(**) y \in B_{mix}, y \in F_{mix}$ and $U(y) \nsubseteq B$.

By Theorem 2.4[12] for a ${}^*g\alpha$-closed set F and the point $y \in F_{mix}$, it is obtained that $cl(\{y\})\backslash\{y\} \subseteq F$. Since $y \in (\mathbb{Z}^2)_{mix}$, we may put $y = (2s, 2u+1)$(resp. $y = (2s+1, 2u)$), $y^+ = (2s, 2u+2)$(resp. $y^+ = (2s+2, 2u)$),$y^- = (2s, 2u)$(resp. $y^- = (2s, 2u)$), where $s, u \in \mathbb{Z}$. Then $cl(\{y\}) = \{y^+, y, y^-\}$. Thus, we have that $cl(\{y\})\backslash\{y\} = \{y^+, y^-\} \subseteq F$. Since $F \subseteq B$, we have that $y^+ \subseteq B_{\mathcal{F}^2}$ and $y^- \subseteq B_{\mathcal{F}^2}$. For the point y^+, it follows form the assumption (2) that $\{y^+\} \cup (U(y^+))_{\kappa^2} \subseteq B$ and so $U(y) \subseteq B$ which a contradiction to $(**)$. Thus, we have that $F \cap ((B\backslash V_B)^2_{mix}) = \emptyset$. By using $(*^3)$ and $(*^4)$, it is shown that, for the ${}^*g\alpha$-closed set F such that $F \subseteq B$, $F = B \cap F = [V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix}) \cup (B\backslash V_B)^2_{mix}] \cap F \subseteq V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$. We put $E = V_B \cup (B\backslash V_B)_{\kappa^2} \cup U((B\backslash V_B)^1_{mix})$ and so $F \subseteq E \subseteq B$ and E is α-open. Using $(*^2)$ and $(*^3)$, we have that $F \subseteq E \subseteq \tau^\alpha$-$int(B)$ holds. Namely, B is ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. □

Theorem 4.6.

(1). The union of any collection of ${}^\#g\hat{\alpha}$-open sets of $(\mathbb{Z}^2, \kappa^2)$ is ${}^\#g\hat{\alpha}$-open set in $(\mathbb{Z}^2, \kappa^2)$.

(2). The intersection of any collection of ${}^\#g\hat{\alpha}$-closed sets of $(\mathbb{Z}^2, \kappa^2)$ is ${}^\#g\hat{\alpha}$-closed set in $(\mathbb{Z}^2, \kappa^2)$.

Proof.

(1). Let $\{B_i | i \in J\}$ be a collection of ${}^\#g\hat{\alpha}$-open sets of $(\mathbb{Z}^2, \kappa^2)$, where J is an index set and put $V = \bigcup\{B_i | i \in J\}$. First we assume that $V_{\mathcal{F}^2} \neq \emptyset$, there exists a point $x \in (B_j)_{\mathcal{F}^2}$ for some $j \in J$. By Theorem 4.5, it is obtained that $(U(x))_{\kappa^2} \subset B_j$ and hence $(U(x))_{\kappa^2} \subset V$. Again using Theorem 4.5, we conclude that V is ${}^\#g\hat{\alpha}$-open. Finally we assume that $V_{\mathcal{F}^2} = \emptyset$. Then by Theorem 4.4, V is ${}^\#g\hat{\alpha}$-open.

(2). We recall that a subset E is ${}^\#g\hat{\alpha}$-closed if and only if the complement of E is ${}^\#g\hat{\alpha}$-open. It follows from (1) and definition that the intersection of any collection of ${}^\#g\hat{\alpha}$-closed sets is ${}^\#g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$. □

Proposition 4.7. *Let x be a point of $(\mathbb{Z}^2, \kappa^2)$. The following properties on the singleton $\{x\}$ hold.*

(1). If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $\{x\}$ is ${}^\#g\hat{\alpha}$-open; it is not ${}^\#g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$.

(2). If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $\{x\}$ is ${}^\#g\hat{\alpha}$-closed; it is not ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$.

(3). If $x \in (\mathbb{Z}^2)_{mix}$, then $\{x\}$ is ${}^\#g\hat{\alpha}$-is both ${}^\#g\hat{\alpha}$-closed and ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$.

Proof.

(1). It follows from the assumption that $\{x\}$ is open in $(\mathbb{Z}^2, \kappa^2)$ and so it is ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Since $\{x\}$ is ${}^*g\alpha$-open, then there exists a ${}^*g\alpha$-open set $U = \{x\}$ such that τ^α-$cl(\{x\}) \nsubseteq \{x\}$. By Definition 3.1 $\{x\}$ is not ${}^\#g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$.

(2). It follows from the assumption that $\{x\}$ is closed in $(\mathbb{Z}^2, \kappa^2)$ and so it is ${}^\#g\hat{\alpha}$-closed in $(\mathbb{Z}^2, \kappa^2)$. Since $\{x\}$ is ${}^*g\alpha$-closed, then there exists a ${}^*g\alpha$-closed set $B = \{x\}$ such that $\{x\} \nsubseteq \tau^\alpha$-$int(\{x\})$. Therefore $\{x\}$ is not ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$.

(3). Let $x \in (\mathbb{Z}^2)_{mix}$, *i.e*, $x = (2s+1, 2u)$ such that τ^α-$cl(\{x\}) = \{x\} \nsubseteq \{x\} = U$, U is ${}^*g\alpha$-open set. Therefore, $\{x\}$ is ${}^\#g\hat{\alpha}$-closed. Let $x = (2s+1, 2u)$ such that $F = \emptyset \subseteq (2s+1, 2u)$, where F is ${}^*g\alpha$-closed set$\Rightarrow \emptyset \subseteq int(\{x\}) = \emptyset$. Hence $\{x\}$ is ${}^\#g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Similarly we can prove this statement for $x = (2s, 2u+1)$. □

It is well known that the digital line $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ but the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$. By Theorem 4.6 and Remark 3.3, we have a new topology, say $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ of $\mathbb{Z}^2$.

Corollary 4.8. *Let $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ be the family of all $^{\#}g\hat{\alpha}$-open sets in $(\mathbb{Z}^2, \kappa^2)$. Then, the following properties hold.*

(1). The family $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ is a topology of $\mathbb{Z}^2$.

(2). Let $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ be topological space obtained by changing the topology κ^2 of the digital plane $(\mathbb{Z}^2, \kappa^2)$ by $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$. Then $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is a $T_{1/2}$-topological space.

Proof.

(1). It is obvious form Theorem 4.6 and Remark 3.3 that the family $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ is topology of $\mathbb{Z}^2$.

(2). Let $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ be topological space with new topology $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$. Then, it is claimed that the topological space $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. By Proposition 4.7, a singleton set $\{x\}$ is open or closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ and by Theorem 3.1(ii) [3]. Hence the space $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. □

Sometimes, we abbreviate the topology $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ by $^{\#}g\hat{\alpha}O$. For a subset A of $\mathbb{Z}^2$, we denote the closure of A, interior of A and the kernel of A with respect to $^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$ by $^{\#}g\hat{\alpha}O\text{-}cl(A)$, $^{\#}g\hat{\alpha}O\text{-}int(A)$ and $^{\#}g\hat{\alpha}O\text{-}ker(A)$ respectively. The kernel is defined by $^{\#}g\hat{\alpha}O\text{-}ker(A) = \bigcap\{V | V \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2), A \subset V\}$.

Proposition 4.9. *For the topological space $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, we have the properties on the singletons as follows. Let x be a point of $\mathbb{Z}^2$ and $s, u \in \mathbb{Z}$.*

(1). (a) If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\}$ and $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(b) If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\} \cup (U(x))_{\kappa^2} = \{2s, 2u\} \cup \{(2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u-1), (2s-1, 2u+1)\}$ where $x = (2s, 2u)$ and $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(c) If $x \in (\mathbb{Z}^2)_{mix}$, then $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\}$ and $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(2). (a) If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{(2s+1, 2u+1), (2s, 2u+2), (2s, 2u), (2s+2, 2u+2), (2s+2, 2u)\}$ and hence $\{x\}$ is not closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.

(b) If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$.

(c) If $x \in (\mathbb{Z}^2)_{mix}$, then $^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$.

(3). (a) If $x \in (\mathbb{Z}^2)_{\kappa^2}$, then $^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \{x\}$.

(b) If $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, then $^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \emptyset$.

(c) If $x \in (\mathbb{Z}^2)_{mix}$, then $^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \{x\}$.

Proof. (1)(a) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 4.7(1), $\{x\}$ is $^{\#}g\hat{\alpha}$-open in $(\mathbb{Z}^2, \kappa^2)$. Then, we have that $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\}$ and $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(1)(b) Let B be any $^{\#}g\hat{\alpha}$-open set of $(\mathbb{Z}^2, \kappa^2)$ containing the point $x = (2s, 2u) \in (\mathbb{Z}^2)_{\mathcal{F}^2}$. Then, by Theorem 4.5, $\{x\} \cup (U(x))_{\kappa^2} \subset B$ holds and $\{x\} \cup (U(x))_{\kappa^2} \in {}^{\#}g\hat{\alpha}O$. Thus, we have that $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\} \cup (U(x))_{\kappa^2} = \{2s, 2u\} \cup \{(2s+1, 2u+1), (2s+1, 2u-1), (2s-1, 2u-1), (2s-1, 2u+1)\}$. By Lemma 2.4(2) and the fact that $(\kappa^2)^{\alpha} \subset {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$, the kernel $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(1)(c) Let $x \in (\mathbb{Z}^2)_{mix}$. The sigleton set $\{x\}$ is $^{\#}g\hat{\alpha}$-open, because $(\{x\})_{\mathcal{F}^2} = \emptyset$. Thus, we have that $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) = \{x\}$ and $^{\#}g\hat{\alpha}O\text{-}ker(\{x\}) \in {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2)$.

(2)(a) Let $x \in (\mathbb{Z}^2)$. By (1), it is shown that, for a point $y \in \mathbb{Z}^2$, $y \in {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})$ holds if and only if $x \in {}^{\#}g\hat{\alpha}O\text{-}ker(\{y\})$ holds. For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, we put $x = (2s+1, 2u+1)$, where $s, u \in \mathbb{Z}$. For a point $y \in {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})$ holds (i.e., $(y \in {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{\kappa^2}$) if and only if $x \in {}^{\#}g\hat{\alpha}O\text{-}ker(\{y\})$ holds (cf. (1)(a)). Thus we have that ${}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{\kappa^2} = \{x\}$. For a point $y \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $y \in {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})$ holds (i.e., $y \in ({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{\mathcal{F}^2}$) if and only if $x \in {}^{\#}g\hat{\alpha}O\text{-}ker(\{y\})$ holds (i.e., $x \in \{y\} \cup U(y)_{\kappa^2}$ and $x \neq y$ holds) (cf. (1)(b)). Thus, we have that $({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{\mathcal{F}^2} = \{y \in (\mathbb{Z}^2)_{\mathcal{F}^2} | x \in \{y\} \cup U(y)_{\kappa^2}\} = W_x$, where $W_x = \{(2s, 2u), (2s, 2u+2), (2s+2, 2u), (2s+2, 2u+2)\}$ and $x = (2s+1, 2u+1)$. For a point $y \in (\mathbb{Z}^2)_{mix}$, $y \in {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})$ holds (i.e., $y \in ({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{mix}$) if and only if $x \in {}^{\#}g\hat{\alpha}O\text{-}ker(\{y\}) = \{y\}$ holds (cf, 1(c)). Since $y \neq x$, we have that $({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}))_{mix} = \emptyset$. Therefore we obtain ${}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\} \cup W_x$.

(2)(b) For a point $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, by Proposition 4.7(2), it is obtained that ${}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$.

(2)(c) Let a point $x =\in (\mathbb{Z}^2)_{mix}$, by Proposition 4.7(2), it is obtained that ${}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) = \{x\}$.

(3) For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$ (res. $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$, $x \in (\mathbb{Z}^2)_{mix}$), by Proposition 4.7(1) (res. (2), (3)), it is shown that ${}^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \{x\}$ (res. ${}^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \emptyset$, ${}^{\#}g\hat{\alpha}O\text{-}int(\{x\}) = \{x\}$) holds. □

Theorem 4.10. *If $x \in (\mathbb{Z}^2)_{mix}, i.e., x = (2s, 2u+1)$ or $(2s+1, 2u)$, then $\{x\}$ is both regular open and regular closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.*

Proof. For a point $x \in (\mathbb{Z}^2)_{mix}$, by Proposition 4.9(2(c) and 3(c)), ${}^{\#}g\hat{\alpha}O\text{-}cl({}^{\#}g\hat{\alpha}O\text{-}int(\{x\})) = \{x\}$. Therefore $\{x\}$ is regular closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$. Similarly we have, ${}^{\#}g\hat{\alpha}O\text{-}int({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})) = \{x\}$. Therefore, $\{x\}$ is regular open in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$. □

Theorem 4.11. *If $x \in (\mathbb{Z}^2)_{\kappa^2}, i.e., x = (2s+1, 2u+1)$, then $\{x\}$ is not regular closed, moreover $\{x\}$ is semi open and regular open in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$.*

Proof. Let $x \in (\mathbb{Z}^2)_{\kappa^2}$, by Proposition 4.9(2(a) and 3(a)), ${}^{\#}g\hat{\alpha}O\text{-}cl({}^{\#}g\hat{\alpha}O\text{-}int(\{x\})) = {}^{\#}g\hat{\alpha}O\text{-}cl(\{x\}) \supseteq \{x\}$, where $x = (2s+1, 2u+1)$. Therefore $\{x\}$ is not regular closed and hence it is semi-open. By Proposition 4.9(2(a) and 3(a)), ${}^{\#}g\hat{\alpha}O\text{-}int({}^{\#}g\hat{\alpha}O\text{-}cl(\{x\})) = \{x\}$. Therefore, $\{x\}$ is regular open in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$. □

Theorem 4.12. *The space $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$ is $T_{3/4}$ but not T_1.*

Proof. By Theorem 4.11, a singleton $\{x\}$ is regular open in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, where $x \in (\mathbb{Z}^2)_{\kappa^2}$; by Proposition 4.9(2(b)), a singleton $\{x\}$ is closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, where $x \in (\mathbb{Z}^2)_{\mathcal{F}^2}$; by Theorem 4.10, a singleton $\{x\}$ is closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, where $x \in (\mathbb{Z}^2)_{mix}$. Therefore, every singleton $\{x\}$ is regular open of closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$. Namely, it is a $T_{3/4}$. Moreover, it is not T_1. Indeed, by Proposition 4.9(2(a)), a singleton $\{(2s+1, 2u+1)\}$ is not closed in $(\mathbb{Z}^2, {}^{\#}g\hat{\alpha}O(\mathbb{Z}^2, \kappa^2))$, where $s, u \in \mathbb{Z}$. □

Acknowledgment

The first author is thankful to University Grants Commission, New Delhi, for sponsoring this work under grants of Major Research Project-MRP-MATH-MAJOR-2013-30929. F.No. 43-433/2014(SR) Dt. 11.09.2015.

References

[1] R. Devi and M. Vigneshwaran, **GαO-Kernal in the Digital Plane*, Internatioal Journal of Computer Application, 3(6)(2010), 24-26.

[2] J. Dontchev and M. Ganster, *On δ-generalized closed sets and $T_{3/4}$-spaces*, Mem. Fac. Kochi. Univ. Ser A. Math., 15(1997), 259-266.

[3] W. Dunham, *$T_{1/2}$-space*, Koyunpook Math. J., 17(1977), 161-169.

[4] J. Cao, S. Ganster and I. Reilly, *On generalized closed sets*, Topology Appl., 123(2002), 37-46.

[5] E. D. Khalimsky, R. Kopperman and P. R. Meyer, *Computer graphics and connected topologies in finite ordered sets*, Topology Appl., 36(1990), 1-17.

[6] E. D. Khalimsky, *Application of connected ordered topological spaces in topology*, Conference of Math. Dept. of Povolsia, (1970).

[7] T. Y. Kong, R. Kopperman and P. R. Meyer, *A topological approach to digital topology*, Amer. Math. Monthly, 98(1991), 901-907.

[8] N. Levine. *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo, 19(2)(1970), 89-96.

[9] K. Nôno, R. Devi, M. Devipriya, K. Muthukumaraswamy and H. Maki, *On $^*g\alpha$-closed sets and the digital plane*, Bulletin of Fukuoka University of Education, 53(III)(2004), 15-24.

[10] O. Njastad, *On some classes of nearly open sets*, Pacif. J. Math., 15(1965), 961-970.

[11] S. Pious Missier and K. M. Arifmohammed, *On $^*g\hat{\alpha}$-closed sets in digital plane*, Jordan Journal of Mathematics and statistics(communicated).

[12] M. Vigneshwaran and R. Devi, *More on $^*g\alpha$-closed and $^*g\alpha$-open sets in the digital plane*, Journal of Global Research in Mathematical Arichives, 1(1)(2013), 39-48.