Certain Investigations on Digital Plane

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Abstract: We introduce the concept of $\#g\hat{\alpha}$-closed sets in a topological space and characterize it using $^*g\alpha_o$-kernel and $\tau^\alpha$-closure. Moreover, we investigate the properties of $\#g\hat{\alpha}$-closed sets in digital plane. The family of all $\#g\hat{\alpha}$-open sets of $(Z^2, \kappa^2)$ forms an alternative topology of $Z^2$. We prove that this plane $(Z^2, \#g\hat{\alpha}O)$ is $T_{1/2}$ and $T_{3/4}$. It is well known that the digital plane $(Z^2, \kappa^2)$ is not $T_{1/2}$, even if $(Z, \kappa)$ is $T_{1/2}$.

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1. Introduction

In 1970, N. Levine [8] introduced and investigated the concept of generalized closed sets in a topological space. He studied most fundamental properties and also introduced a separation axiom $T_{1/2}$. The digital line is typical example of a $T_{1/2}$ space [2]. After Levine’s works, many authors defined and investigated various notions to Levine’s $g$-closed sets and related topics [4]. In 1970, E. Khalimsky [6] introduced digital line. In 1990, K. Kopperman and R. Meyer [5] developed finite analog of the Jordan curve theorem motivated by a problem in computer graphics (cf. [5, 7]). In this paper, we introduce the concept of $\#g\hat{\alpha}$-closed sets in a topological space and characterize it using $^*g\alpha_o$-kernel and $\tau^\alpha$-closure. Moreover, we investigate the properties of $\#g\hat{\alpha}$-closed sets in digital plane. We prove that this plane $(Z^2, \#g\hat{\alpha}O)$ is $T_{1/2}$ and $T_{3/4}$. It is well known that the digital plane $(Z^2, \kappa^2)$ is not $T_{1/2}$, even if $(Z, \kappa)$ is $T_{1/2}$.

2. Preliminaries

Throughout this paper, $(X, \tau)$ or $X$ denotes the topological spaces. For a subset $A$ of $X$, the closure, the interior and the complement of $A$ are denoted by $\text{cl}(A)$, $\text{int}(A)$ and $A^c$ respectively. We recall some basic definitions that are used in the sequel.

Definition 2.1. A subset $A$ of a topological space $(X, \tau)$ is called $\alpha$-open [10] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. Moreover, $A$ is said to be $\alpha$-closed if $X \setminus A$ is $\alpha$-open. The collection of all $\alpha$-open subsets in $(X, \tau)$ is denoted by $\tau^\alpha$. The $\alpha$-closure of a subset $A$ is the smallest $\alpha$-closed set containing $A$ and this is denoted by $\tau^\alpha\text{-cl}(A)$ in this paper.

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Definition 2.2. A subset $A$ of a topological space $(X, \tau)$ is called $\ast g\alpha$-closed [11] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is an $\ast g\alpha$-open set in $(X, \tau)$. Moreover, $A$ is said to be $\ast g\alpha$-open if $X \setminus A$ is $\ast g\alpha$-closed.

Lemma 2.3 ([9]). For a subset $A$ of $(X, \tau)$, the following conditions are equivalent:

(1). $A$ is $\ast g\alpha$-closed in $(X, \tau)$.

(2). $\tau^\alpha\text{-cl}(A) \subseteq g\text{-Ker}(A)$ holds.

Lemma 2.4 ([9]). Let a subset $A$ of $(\mathbb{Z}^2, \kappa^2)$.

(1). $g\text{-Ker}(A) = U(A_{\kappa^2}) \cup A_{\text{mix}} \cup A_{\kappa^2}$, where $U(A_{\kappa^2}) = \bigcup \{ U(x) | x \in A_{\kappa^2} \}$.

(2). For a point $x \in (\mathbb{Z}^2)_{\kappa^2}$, a subset $\{ x \} \cup (U(x))_{\kappa^2}$ is preopen and hence it is $\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$.

Definition 2.5 ([2]). A space $(X, \tau)$ is $T_{3\frac{1}{2}}$ if and only if every singleton $\{ x \}$ of $X$ is closed or regular open in $(X, \tau)$.

3. $\#g\alpha$-closed Sets and its Properties

In this section we introduce the concept of $\#g\alpha$-closed sets and study some of their properties and relations with other known classes of subsets.

Definition 3.1. A subset $A$ of a space $(X, \tau)$ is called a $\#g\alpha$-closed set if $\tau^\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a $\ast g\alpha$-open set in $(X, \tau)$. The class of $\#g\alpha$-closed subsets of $(X, \tau)$ is denoted by $\#g\alpha C(X, \tau)$.

Theorem 3.2. Finite union of $\#g\alpha$-closed sets is a $\#g\alpha$-closed set in $(X, \tau)$.

Proof. Let $A_i$’s are $\#g\alpha$-closed sets, where $i = 1, 2, 3, ..., n$ and $n \in \mathbb{N}$. Let $\bigcup_{i=1}^{n} A_i \subseteq U$, $U$ is a $\ast g\alpha$-open set in $(X, \tau)$. Since $A_i$’s are $\#g\alpha$-closed sets, $\tau^\alpha\text{-cl}(A_i) \subseteq U$, $\forall A_i \subseteq U$. This implies that $\tau^\alpha\text{-cl}(\bigcup_{i=1}^{n} A_i) = \bigcup_{i=1}^{n} \tau^\alpha\text{-cl}(A_i) \subseteq U$. Therefore $\bigcup_{i=1}^{n} A_i$ is $\#g\alpha$-closed.

Remark 3.3. Finite intersection of $\#g\alpha$-open sets is a $\#g\alpha$-open set in $(X, \tau)$.

Proof. Proof is obvious, since $X \setminus A$ is $\#g\alpha$-open, whenever $A$ is $\#g\alpha$-closed.

The following example shows that intersection of two $\#g\alpha$-closed sets need not be $\#g\alpha$-closed in $(X, \tau)$.

Example 3.4. Let $X = \{ a, b, c \}$ and $\tau = \{ X, \emptyset, \{ a \} \}$. Then, $\{ a, b \}$ and $\{ a, c \}$ are $\#g\alpha$-closed but their intersection $\{ a \}$ is not $\#g\alpha$-closed in $(X, \tau)$.

Theorem 3.5. If $A$ be a $\#g\alpha$-closed set in $(X, \tau)$, then $\tau^\alpha\text{-cl}(A) \setminus A$ does not contain any non empty $\ast g\alpha$-closed set.

Proof. Suppose that $A$ is $\#g\alpha$-closed and let $F$ be an non-empty $\ast g\alpha$-closed set with $F \subseteq \tau^\alpha\text{-cl}(A) \setminus A$. Then $A \subseteq X \setminus F$ ans so $\tau^\alpha\text{-cl}(A) \subseteq X \setminus \tau^\alpha\text{-cl}(A)$. Hence $F \subseteq X \setminus \tau^\alpha\text{-cl}(A)$, a contradiction.

Theorem 3.6. Let $(X, \tau)$ be a space, $A$ and $B$ subsets.

(1). If $A$ is $\ast g\alpha$-open and $\#g\alpha$-closed, then $A$ is $\alpha$-closed in $(X, \tau)$.

(2). If $A$ is $\#g\alpha$-closed set of $(X, \tau)$ such that $A \subseteq B \subseteq \tau^\alpha\text{-cl}(A)$, then $B$ is also $\#g\alpha$-closed in $(X, \tau)$.

(3). For each $x \in X$, $\{ x \}$ is $\ast g\alpha$-closed or $X \setminus \{ x \}$ is $\#g\alpha$-closed in $(X, \tau)$.
Lemma 4.1. (1) If \( \alpha \)-open and \#gā-closed, \( \tau^\alpha\)-cl(A) \( \subseteq \) A. Therefore A is \( \alpha \)-closed.

Proof. (1). Since \( A \subseteq A \) and \( A \) is both \( *g\alpha\)-open and \#gā-closed, \( \tau^\alpha\)-cl(A) \( \subseteq \) A. Therefore A is \( \alpha \)-closed.

(2). Let \( U \) be a \( *g\alpha\)-open set such that \( B \subseteq U \). Then we have that \( \tau^\alpha\)-cl(A) \( \subseteq \) U and \( \tau^\alpha\)-cl(B) \( \subseteq \) \( \tau^\alpha\)-cl(A) \( \subseteq \) U. Therefore, \( B \) is \#gā closed in \( (X, \tau) \).

(3). If \( \{x\} \) is not \( *g\alpha\)-open, then \( X \{x\} \) is not \( *g\alpha\)-open. Therefore, \( X \{x\} \) is \#gā-closed in \( (X, \tau) \).

(4). Necessity: Let \( U \) be a \( *g\alpha\)-open set. Then we have that \( \tau^\alpha\)-cl(U) \( \subseteq \) U and hence \( U \) is \( \alpha \)-closed. Sufficiency: Let A be a subset and \( U \) a \( *g\alpha\)-open set such that \( A \subseteq U \). Then \( \tau^\alpha\)-cl(A) \( \subseteq \) \( \tau^\alpha\)-cl(U) = U and hence A is \#gā-closed.

We have a characterization of \#gā-closed sets. We prepare some notations and a lemma. For a subset \( E \) of a space \( (X, \tau) \), we define the following subsets of \( K \) Khalimsky line is the set of the integers \( * \)

For any space \( (X, \tau) \), we investigate explicite forms of \( *g\alpha\)-Kernal and Kernal of a subset. The digital line or the so called

\[ \text{Lemma 3.8.} \]

For any space \( (X, \tau) \), \( X = X_{g\alpha} \cup X_{\#g\alpha} \) holds.

Proof. Let \( x \in X \). By Theorem 3.6(3), \( \{x\} \in X_{g\alpha} \) or \( \{x\} \in X_{\#g\alpha} \).}

4. \#gā-closed Sets in the Digital Plane

In the digital plane, we investigate explicite forms of \( *g\alpha\)-Kernal and Kernal of a subset. The digital line or the so called Khalimsky line is the set of the integers \( \mathbb{Z} \), equipped with the topology \( \kappa \) having \( \{\{2n-1, 2n, 2n+1\}| n \in \mathbb{Z} \} \) as a subbase. This is denoted by \( (\mathbb{Z}, \kappa) \). Thus a subset \( U \) open in \( (\mathbb{Z}, \kappa) \) if and only if whenever \( x \in U \) is an even integer, \( x - 1 \), \( x + 1 \in U \).

Let \((\mathbb{Z}^2, \kappa^2)\) be the topological product of two digital lines \( (\mathbb{Z}, \kappa) \), where \( \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} \) and \( \kappa^2 = \kappa \times \kappa \). This space is called the digital plane in the present paper (cf. [5], [7]). We note that for each point \( x \in \mathbb{Z}^2 \) there exists the smallest open set containing \( x \), say \( U(x) \). For the case of \( x = (2n+1, 2m+1) \), \( U(x) = \{2n+1\} \times \{2m+1\} \); for the case \( x = (2n, 2m) \), \( U(x) = \{2n-1, 2n, 2n+1\} \times \{2m-1, 2m, 2m+1\} \); for the case \( x = (2n, 2m+1) \), \( U(x) = \{2n-1, 2n, 2n+1\} \times \{2m\} \); for the case \( x = (2n+1, 2m) \), \( U(x) = \{2n+1\} \times \{2m-1, 2m, 2m+1\} \), where \( n, m \in \mathbb{Z} \). For a subset \( E \) of \( (\mathbb{Z}^2, \kappa^2) \), we define the following three subsets as follows: \( E_{x^2} = \{x \in E|\{x\} \text{ is closed in } (\mathbb{Z}^2, \kappa^2)\} \); \( E_{z^2} = \{x \in E|\{x\} \text{ is open in } (\mathbb{Z}^2, \kappa^2)\} \); \( E_{mix} = E \setminus (E_{x^2} \cup E_{z^2}) \).

Lemma 4.1. Let \( A \) and \( E \) be subsets of \( (\mathbb{Z}^2, \kappa^2) \).

(1) If \( E \) be non-empty \( *g\alpha\)-closed set, then \( E_{x^2} \neq \emptyset \).

(2) If \( E \) is \( *g\alpha\)-closed and \( E \subseteq B_{mix} \cup B_z \) holds for some subset \( B \) of \( (\mathbb{Z}^2, \kappa^2) \), then \( E = \emptyset \).

(3) The set \( U(A_{x^2}) \cup A_{mix} \cup A_z \) is a \#gā-open set containing \( A \).
Proof. (3) First we claim that $A_{mix} \cup A_{x2}$ is $^\# g\alpha$-open set. Let $F$ be a non-empty $^*g\alpha$-closed set such that $F \subseteq A_{mix} \cup A_{x2}$. Then by (2), $F = \emptyset$. Thus, we have that $F \subseteq \tau^\alpha$-int$(A_{mix} \cup A_{x2})$. Therefore $A_{mix} \cup A_{x2}$ is $^\# g\alpha$-open. Since every open set is $^\# g\alpha$-open, $U(A_{x2})$ is $^\# g\alpha$-open. Since union of two $^\# g\alpha$-open sets is $^\# g\alpha$-open, $A_{x2} \cup A_{mix} \cup A_{x2}$ is a $^\# g\alpha$-open set containing $A$. \hfill $\Box$

Theorem 4.2 ([1]). Let $E$ be a subset of $(\mathbb{Z}^2, \kappa^2)$.

(1). If $E$ is a non-empty $^*g\alpha$-closed set, then $E_{x2} \neq \emptyset$.

(2). If $E$ is a $^*g\alpha$-closed set and $E \subseteq B_{mix} \cup B_{x2}$ holds for some subset $B$ of $(\mathbb{Z}^2, \kappa^2)$, then $E = \emptyset$.

Theorem 4.3 ([1]). Let $E$ be a subset of $(\mathbb{Z}^2, \kappa^2)$.

(1). $^*GoO$-$ker(A) = U(A_{x2}) \cup A_{mix} \cup A_{x2}$, $U(A_{x2}) = \bigcup \{U(x) | x \in A_{x2}\}$.

(2). GoO-$ker(A) = U(A_{x2}), U(A_{x2}) = \bigcup \{U(x) | x \in A_{x2}\}$.

Theorem 4.4. Let $B$ be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$. If $B_{x2} = \emptyset$, then $B$ is $^\# g\alpha$-open.

Proof. Let $F$ be a $^*g\alpha$-closed set such that $F \subseteq B$. Since $B_{x2} = \emptyset$, we have $B = B_{mix} \cup B_{x2}$. Then by Theorem 4.2(2), we get $F = \emptyset \Rightarrow F \subseteq \tau^\alpha$-int$(B)$. Therefore, $B$ is $^\# g\alpha$-open. \hfill $\Box$

Theorem 4.5. Let $B$ be a non-empty subset of $(\mathbb{Z}^2, \kappa^2)$, the following are equivalent:

(1). The subset $B$ is $^\# g\alpha$-open set of $(\mathbb{Z}^2, \kappa^2)$,

(2). $(U(x))_{x2} \subseteq B$ holds for each point $x \in B_{x2}$.

Proof. (1) $\Rightarrow$ (2) Let $x \in B_{x2}$. Since $\{x\}$ is closed, $\{x\}$ is $^*g\alpha$-closed set and $\{x\} \subseteq B$. By (1), $\{x\} \subset \tau^\alpha$-int$(B) = B \cap cl(int(cl(B)))$ and so $x \in int(cl(int(B)))$. Namely, $x$ is an interior point of the set $cl(int(B))$. Thus, we have that, for the smallest open set $U(x)$ containing $x$, $U(x) \subset cl(int(B))$. We can set $x = (2s, 2u)$ for some integers $s$ and $u$, because $x \in (\mathbb{Z}^2)_{x2}$. Since $U((2s, 2u)) = (2s - 1, 2s, 2s + 1) \times (2u - 1, 2u, 2u + 1)$, it is shown that $(U(x))_{x2} = \{(x1, x2) \in U(x) | x1$ and $x2$ are odd$\} = \{p1, p2, p3, p4\}$, where $p1 = (2s - 1, 2u - 1), p2 = (2s - 1, 2u + 1), p3 = (2s + 1, 2u - 1), p4 = (2s + 1, 2u + 1)$. For each point $p_i (1 \leq i \leq 4), p_i \in cl(int(B))$ and so $\{p_i\} \cap int(B) \neq \emptyset$. Therefore, $p_i \in B$ for each $i$ with $1 \leq i \leq 4$ and hence $(U(x))_{x2} \subseteq B$.

(2) $\Rightarrow$ (1) It follows from the assumption that, for each point $x \in B_{x2}$, $\{x\} \cup (U(x))_{x2} \subseteq B$ and so $\bigcup \{(x) \cup (U(x))_{x2} | x \in B_{x2}\} \subseteq B$. Put $V_B = \bigcup \{(x) \cup (U(x))_{x2} | x \in B_{x2}\}$ and so $V_B \neq \emptyset, V_B \subseteq B$. By Lemma 2.4(2), $V_B$ is preopen and it is $\alpha$-open. We have that $B = V_B \cup (B \setminus V_B) = V_B \cup \{(B \setminus V_B)_{x2} \cup (B \setminus V_B)_{x1} \cup (B \setminus V_B)_{mix}\} = V_B \cup (B \setminus V_B)_{x2} \cup (B \setminus V_B)_{mix}$, we note that, for a point $y \in (B \setminus V_B)_{mix}, U(y) \subset B$ or $U(y) \not\subset B$. We put $(B \setminus V_B)^{1}_{mix} = \{y \in (B \setminus V_B)_{mix} | U(y) \subset B\}, (B \setminus V_B)^{2}_{mix} = \{y \in (B \setminus V_B)_{mix} | U(y) \not\subset B\}$. Then, $(B \setminus V_B)_{mix}$ is decomposed as $(B \setminus V_B)_{mix} = (B \setminus V_B)^{1}_{mix} \cup (B \setminus V_B)^{2}_{mix}$, we thus have that:

$(*)$ $B = V_B \cup (B \setminus V_B)_{x2} \cup (B \setminus V_B)^{1}_{mix} \cup (B \setminus V_B)^{2}_{mix}$. Here, $V_B$ is $\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$; the set $(B \setminus V_B)_{x2}$ is open and so $\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$; $U((B \setminus V_B)^{1}_{mix})$ is open and so $\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$. Thus, we have that:

$(\dagger)$ the subset $V_B \cup (B \setminus V_B)_{x2} \cup U((B \setminus V_B)^{1}_{mix})$ is $\alpha$-open in $(\mathbb{Z}^2, \kappa^2)$.

Moreover, we conclude that:

$(\dagger \dagger)$ $B \subseteq V_B \cup (B \setminus V_B)_{x2} \cup U((B \setminus V_B)^{1}_{mix}) \cup (B \setminus V_B)^{2}_{mix}$ holds.

Proof of $(\dagger \dagger)$: Since $(B \setminus V_B)^{1}_{mix} \subseteq U((B \setminus V_B)^{1}_{mix})$, it is shown that $B \subseteq V_B \cup (B \setminus V_B)_{x2} \cup U((B \setminus V_B)^{1}_{mix}) \cup (B \setminus V_B)^{2}_{mix}$. Conversely we have that $V_B \cup (B \setminus V_B)_{x2} \cup U((B \setminus V_B)^{1}_{mix}) \cup (B \setminus V_B)^{2}_{mix} \subseteq B$, because $U((B \setminus V_B)^{1}_{mix}) \subseteq B, V_B \subseteq B,$
we may put (1). The union of any collection of $S$. Pious Missier and K. M. Arifmohammed

(2). The intersection of any collection of $y$

(3). Let $x$

Proof of (1). It follows from the assumption that $x$

(2). It follows from the assumption that $x$

(3). Let $x$

Proof. 

By Theorem 2.4[12] for a $E$ of $(2,\kappa^2)$ such that $x$

(2). We recall that a subset $g\alpha$ of $(\kappa^2)$ is $a$-open. It follows from (2) and (1), we have that $F \subseteq B$ and $E$ is $a$-open. Using (1) and (2), we have that $F \subseteq E \subseteq \tau^a$-int$(B)$ holds. Namely, $B$ is $\#g\alpha$-open in $(\kappa^2,\kappa^2)$.

Theorem 4.6.

(1). The union of any collection of $\#g\alpha$-open sets of $(\kappa^2,\kappa^2)$ is $\#g\alpha$-open in $(\kappa^2,\kappa^2)$.

(2). The intersection of any collection of $\#g\alpha$-closed sets of $(\kappa^2,\kappa^2)$ is $\#g\alpha$-closed in $(\kappa^2,\kappa^2)$.

Proof.

(1). Let $\{B_i| i \in J\}$ be a collection of $\#g\alpha$-open sets of $(\kappa^2,\kappa^2)$, where $J$ is an index set and put $V = \cup\{B_i| i \in J\}$. First we assume that $V_{\tau^2} \neq \emptyset$, there exists a point $x \in (B_j)_{\tau^2}$ for some $j \in J$. By Theorem 4.5, it is obtained that $(U(x))_{\tau^2} \subseteq B_j$ and hence $(U(x))_{\tau^2} \subseteq V$. Again using Theorem 4.5, we conclude that $V$ is $\#g\alpha$-open. Finally we assume that $V_{\tau^2} = \emptyset$. Then by Theorem 4.4, $V$ is $\#g\alpha$-open.

(2). We recall that a subset $E$ is $\#g\alpha$-closed if and only if the complement of $E$ is $\#g\alpha$-open. It follows from (1) and definition that the intersection of any collection of $\#g\alpha$-closed sets is $\#g\alpha$-closed in $(\kappa^2,\kappa^2)$.

Proposition 4.7. Let $x$ be a point of $(\kappa^2,\kappa^2)$. The following properties on the singleton $\{x\}$ hold.

(1). If $x \in (\kappa^2)_{\tau^2}$, then $\{x\}$ is $\#g\alpha$-open; it is not $\#g\alpha$-closed in $(\kappa^2,\kappa^2)$.

(2). If $x \in (\kappa^2)_{\tau^2}$, then $\{x\}$ is $\#g\alpha$-closed; it is not $\#g\alpha$-open in $(\kappa^2,\kappa^2)$.

(3). If $x \in (\kappa^2)_{\tau^2}$, then $\{x\}$ is $\#g\alpha$-closed both $\#g\alpha$-closed and $\#g\alpha$-open in $(\kappa^2,\kappa^2)$.

Proof.

(1). It follows from the assumption that $\{x\}$ is open in $(\kappa^2,\kappa^2)$ and so it is $\#g\alpha$-open in $(\kappa^2,\kappa^2)$. Since $\{x\}$ is $\#g\alpha$-open, then there exists a $\#g\alpha$-open set $U = \{x\}$ such that $\tau^a$-cl$(\{x\}) \subseteq \{x\}$. By Definition 3.1 $\{x\}$ is not $\#g\alpha$-closed in $(\kappa^2,\kappa^2)$.

(2). It follows from the assumption that $\{x\}$ is closed in $(\kappa^2,\kappa^2)$ and so it is $\#g\alpha$-closed in $(\kappa^2,\kappa^2)$. Since $\{x\}$ is $\#g\alpha$-closed, then there exists a $\#g\alpha$-open set $B = \{x\}$ such that $\{x\} \subseteq \tau^a$-int$(\{x\})$. Therefore $\{x\}$ is not $\#g\alpha$-open in $(\kappa^2,\kappa^2)$.

(3). Let $x \in (\kappa^2)_{\tau^2}$, then $x = (2s + 1,2u)$ such that $\tau^a$-cl$(\{x\}) \subseteq \{x\} = U$. $U$ is $\#g\alpha$-open set. Therefore, $\{x\}$ is $\#g\alpha$-closed. Let $x = (2s + 1,2u)$ such that $F = \emptyset \subseteq (2s + 1,2u)$, where $F$ is $\#g\alpha$-closed set $\Rightarrow \emptyset \subseteq \text{int}(\{x\}) = \emptyset$. Hence $\{x\}$ is $\#g\alpha$-open in $(\kappa^2,\kappa^2)$. Similarly we can prove this statement for $x = (2s,2u + 1)$. 


It is well known that the digital line $(\mathbb{Z}, \kappa)$ is $T_{1/2}$ but the digital plane $(\mathbb{Z}^2, \kappa^2)$ is not $T_{1/2}$. By Theorem 4.6 and Remark 3.3, we have a new topology, say $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ of $\mathbb{Z}^2$.

**Corollary 4.8.** Let $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ be the family of all $\# g\hat{o}$-open sets in $(\mathbb{Z}^2, \kappa^2)$. Then, the following properties hold.

1. The family $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ is a topology of $\mathbb{Z}^2$.

2. Let $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ be topological space obtained by changing the topology $\kappa^2$ of the digital plane $(\mathbb{Z}^2, \kappa^2)$ by $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$. Then $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ is a $T_{1/2}$-topological space.

**Proof.**

1. It is obvious form Theorem 4.6 and Remark 3.3 that the family $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ is topology of $\mathbb{Z}^2$.

2. Let $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ be topological space with new topology $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$. Then, it is claimed that the topological space $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$. By Proposition 4.7, a singleton set $\{x\}$ is open or closed in $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ and by Theorem 3.1(ii) [3]. Hence the space $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$ is $T_{1/2}$.

Sometimes, we abbreviate the topology $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ by $\# g\hat{o}$. For a subset $A$ of $\mathbb{Z}^2$, we denote the closure of $A$, interior of $A$ and the kernel of $A$ with respect to $\# g\hat{o}(\mathbb{Z}^2, \kappa^2)$ by $\# g\hat{o}\text{-}cl(A)$, $\# g\hat{o}\text{-}int(A)$ and $\# g\hat{o}\text{-}ker(A)$ respectively. The kernel is defined by $\# g\hat{o}\text{-}ker(A) = \bigcap\{V|V \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2), A \subset V\}$.

**Proposition 4.9.** For the topological space $(\mathbb{Z}^2, \# g\hat{o}(\mathbb{Z}^2, \kappa^2))$, we have the properties on the singletons as follows. Let $x$ be a point of $\mathbb{Z}^2$ and $s, u \in \mathbb{Z}$.

1. (a) If $x \in (\mathbb{Z}^2)_{x,s}$, then $\# g\hat{o}\text{-}ker(\{x\}) = \{x\}$ and $\# g\hat{o}\text{-}ker(\{x\}) \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2)$.

2. (b) If $x \in (\mathbb{Z}^2)_{x,2}$, then $\# g\hat{o}\text{-}ker(\{x\}) = \{x\} \cup (U(x))_{x,2} = \{(2s, 2u) \cup \{(2s + 1, 2u + 1), (2s + 1, 2u - 1), (2s - 1, 2u + 1), (2s - 1, 2u - 1)\}$ where $x = (2s, 2u)$ and $\# g\hat{o}\text{-}ker(\{x\}) \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2)$.

3. (c) If $x \in (\mathbb{Z}^2)_{mix}$, then $\# g\hat{o}\text{-}ker(\{x\}) = \{x\}$ and $\# g\hat{o}\text{-}ker(\{x\}) \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2)$.

**Proof.**

1.(a) For a point $x \in (\mathbb{Z}^2)_{x,s}$, by Proposition 4.7(1), $\{x\}$ is $\# g\hat{o}$-open in $(\mathbb{Z}^2, \kappa^2)$. Then, we have that $\# g\hat{o}\text{-}ker(\{x\}) = \{x\}$ and $\# g\hat{o}\text{-}ker(\{x\}) \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2)$.

1.(b) Let $B$ be any $\# g\hat{o}$-open set of $(\mathbb{Z}^2, \kappa^2)$ containing the point $x = (2s, 2u) \in (\mathbb{Z}^2)_{x,2}$. Then, by Theorem 4.5, $\{x\} \cup (U(x))_{x,2} \subset B$ holds and $\{x\} \cup (U(x))_{x,2} \in \# g\hat{o}$. Thus, we have that $\# g\hat{o}\text{-}ker(\{x\}) = \{x\} \cup (U(x))_{x,2} = \{(2s, 2u) \cup \{(2s + 1, 2u + 1), (2s + 1, 2u - 1), (2s - 1, 2u + 1), (2s - 1, 2u - 1)\}$.

1.(c) Let $x \in (\mathbb{Z}^2)_{mix}$. The singleton set $\{x\}$ is $\# g\hat{o}$-open, because $(\{x\})_{x,2} = \emptyset$. Thus, we have that $\# g\hat{o}\text{-}ker(\{x\}) = \{x\}$ and $\# g\hat{o}\text{-}ker(\{x\}) \in \# g\hat{o}(\mathbb{Z}^2, \kappa^2)$. 


(2)(a) Let \( x \in (Z^2) \). By (1), it is shown that, for a point \( y \in Z^2 \), \( y \in \# \text{g}_O\text{-cl} \{x\} \) holds if and only if \( x \in \# \text{g}_O\text{-ker}(\{y\}) \) holds. For a point \( x \in (Z^2)_{s2} \), we put \( x = (2s + 1, 2u + 1) \), where \( s, u \in Z \). For a point \( y \in \# \text{g}_O\text{-cl}(\{x\}) \) holds (i.e., \( y \in \# \text{g}_O\text{-cl}(\{x\})_{s2} \) if and only if \( x \in \# \text{g}_O\text{-ker}(\{y\}) \) holds (cf. (1)(a)). Thus we have that \( \# \text{g}_O\text{-cl}(\{x\})_{s2} = \{x\} \).

For a point \( y \in (Z^2)_{s2} \), \( y \in \# \text{g}_O\text{-cl}(\{x\}) \) holds (i.e., \( y \in \# \text{g}_O\text{-cl}(\{x\})_{s2} \) if and only if \( x \in \# \text{g}_O\text{-ker}(\{y\}) \) holds (i.e., \( x \in \{y\} \cup U(y)_{s2} \) and \( x \neq y \) holds) (cf. (1)(b)). Thus, we have that \( \# \text{g}_O\text{-cl}(\{x\})_{s2} = \{y \in (Z^2)_{s2} | x \in \{y\} \cup U(y)_{s2} \} = W_x \), where \( W_x = \{(2s, 2u), (2s + 1, 2u + 2), (2s + 2, 2u), (2s + 2, 2u + 2)\} \) and \( x = (2s + 1, 2u + 1) \). For a point \( y \in (Z^2)_{mix} \), \( y \in \# \text{g}_O\text{-cl}(\{x\}) \) holds (i.e., \( y \in \# \text{g}_O\text{-cl}(\{x\})_{mix} \) if and only if \( x \in \# \text{g}_O\text{-ker}(\{y\}) = \{y\} \) holds (cf, 1(c)). Since \( y \neq x \), we have that \( \# \text{g}_O\text{-cl}(\{x\})_{mix} = \emptyset \). Therefore we obtain \( \# \text{g}_O\text{-cl}(\{x\}) = \{x\} \cup W_x \).

(2)(b) For a point \( x \in (Z^2)_{s2} \), by Proposition 4.7(2), it is obtained that \( \# \text{g}_O\text{-cl}(\{x\}) = \{x\} \).

(2)(c) Let a point \( x = (Z^2)_{mix} \), by Proposition 4.7(2), it is obtained that \( \# \text{g}_O\text{-cl}(\{x\}) = \{x\} \).

(3) For a point \( x \in (Z^2)_{s2} \) (res. \( x \in (Z^2)_{s2} \), \( x \in (Z^2)_{mix} \)), by Proposition 4.7(1) (res. (2), (3)), it is shown that \( \# \text{g}_O\text{-int}(\{x\}) = \{x\} \) (res. \( \# \text{g}_O\text{-int}(\{x\}) = \emptyset \), \( \# \text{g}_O\text{-int}(\{x\}) = \{x\} \) holds.

\[ \text{Theorem 4.10.} \text{ If } x \in (Z^2)_{mix}, \text{i.e., } x = (2s, 2u + 1) \text{ or } (2s + 1, 2u), \text{ then } \{x\} \text{ is both regular open and regular closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \]

\[ \text{Proof.} \text{ For a point } x \in (Z^2)_{mix}, \text{ by Proposition 4.9(2(c) and 3(c)), } \# \text{g}_O\text{-cl}(\# \text{g}_O\text{-int}(\{x\})) = \{x\}. \text{ Therefore } \{x\} \text{ is regular closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \text{ Similarly we have, } \# \text{g}_O\text{-int}(\# \text{g}_O\text{-cl}(\{x\})) = \{x\}. \text{ Therefore, } \{x\} \text{ is regular open in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \]

\[ \text{Theorem 4.11.} \text{ If } x \in (Z^2)_{s2}, \text{i.e., } x = (2s + 1, 2u + 1), \text{ then } \{x\} \text{ is not regular closed, moreover } \{x\} \text{ is semi open and regular open in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \]

\[ \text{Proof.} \text{ Let } x \in (Z^2)_{s2}, \text{ by Proposition 4.9(2(a) and 3(a)), } \# \text{g}_O\text{-cl}(\# \text{g}_O\text{-int}(\{x\})) = \# \text{g}_O\text{-cl}(\{x\}) \supseteq \{x\}, \text{ where } x = (2s + 1, 2u + 1). \text{ Therefore } \{x\} \text{ is not regular closed and hence it is semi-open. By Proposition 4.9(2(a) and 3(a)), } \# \text{g}_O\text{-int}(\# \text{g}_O\text{-cl}(\{x\})) = \{x\}. \text{ Therefore, } \{x\} \text{ is regular open in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \]

\[ \text{Theorem 4.12.} \text{ The space } (Z^2, \# \text{g}_O(Z^2, \kappa^2)) \text{ is } T_{3.5/4} \text{ but not } T_1. \]

\[ \text{Proof.} \text{ By Theorem 4.11, a singleton } \{x\} \text{ is regular open in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)), \text{ where } x \in (Z^2)_{s2}; \text{ by Proposition 4.9(2(b)), a singleton } \{x\} \text{ is closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)), \text{ where } x \in (Z^2)_{s2}; \text{ by Theorem 4.10, a singleton } \{x\} \text{ is closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)), \text{ where } x \in (Z^2)_{mix}. \text{ Therefore, every singleton } \{x\} \text{ is regular open of closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)). \text{ Namely, it is a } T_{3.5/4}. \text{ Moreover, it is not } T_1. \text{ Indeed, by Proposition 4.9(2(a)), a singleton } \{(2s + 1, 2u + 1)\} \text{ is not closed in } (Z^2, \# \text{g}_O(Z^2, \kappa^2)), \text{ where } s, u \in Z. \]

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\section*{References}


