1. Introduction and Preliminaries

During the last four decades or so, several many interesting and useful extensions of the familiar special functions (such as $I$-function which is extension of $H$-function and $G$-function and so on) have been considered by several authors (see, for example, Satyanarayana [10], Saxena [11], Saxena [12–14], Shantha [15] and Vyas and Rathie [19]. The $I$-function, introduced by Rathie [9], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral:

$$I(z) = \frac{1}{2\pi i} \int_L \frac{\chi(s) z^s ds}{s}$$  \hspace{1cm} (1)

For all $z$ different from zero and

$$\chi(s) = \prod_{j=1}^{m} \frac{\Gamma_b'(a_j - \beta_j s)}{\Gamma_{b'}'(1 - a_j + \alpha_j s)} \prod_{j=m+1}^{n} \frac{\Gamma_{b''}(1 - a_j + \alpha_j s)}{\Gamma_{b''}(1 - b_j + \beta_j s)}.$$  \hspace{1cm} (2)

The integral (1) converges when $|\arg z| < \frac{1}{2} \Delta \pi$, if $\Delta > 0$, where

$$\Delta = \sum_{j=1}^{m} B'_j \beta_j - \sum_{j=m+1}^{q} B'_j \beta_j + \sum_{j=1}^{n} A'_j \alpha_j - \sum_{j=m+1}^{p} A''_j \alpha_j.$$  \hspace{1cm} (3)

Recently, the following formulas are defined Qureshi [8].

$$\int_0^\infty \left[ \left( \frac{x}{\rho} + \frac{\tau}{x} \right)^2 + \gamma \right]^{-u-1} dx = \frac{\sqrt{\pi}}{2\rho (4\rho^2 + \gamma)^{u+1/2}} \frac{\Gamma(u+1/2)}{\Gamma(u+1)}.$$  \hspace{1cm} (4)

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\[(\rho > 0; \tau \geq 0; \gamma + 4\rho \tau > 0; (\Re(u) + 1/2) > 0).\]

\[\int_0^\infty \frac{1}{x^2} \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu-1} dx = \frac{\sqrt{\pi}}{2\tau (4\rho \tau + \gamma)^{1/2 + \nu}} \frac{\Gamma(u + 1/2)}{\Gamma(u + 1)}, \quad (5)\]

\[\int_0^\infty \frac{\tau}{x^2} \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu-1} dx = \frac{\sqrt{\pi}}{(4\rho \tau + \gamma)^{1/2 + \nu}} \frac{\Gamma(u + 1/2)}{\Gamma(u + 1)}, \quad (6)\]

\[(\rho > 0; \tau > 0; \gamma + 4\rho \tau > 0; (\Re(u) + 1/2) > 0).\]

The following formulas [16] will be required in our investigation

\[\sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r x^r. \quad (7)\]

2. **Main Results**

**Theorem 2.1.** If \(\rho > 0; \tau \geq 0; \gamma + 4\rho \tau > 0; (\Re(\lambda) + 1/2) > 0, -\frac{1}{2} < (a - b - c) < \frac{1}{2}; |\arg z| < \frac{1}{2} \Delta, \) where \(\Delta > 0\) and given by equation (3), then the following formula holds true:

\[\int_0^\infty \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu-1} 2F_1 \left( a, b, c + \frac{1}{2}; x \right) 2F_1 \left( c - a, c - b, c + \frac{1}{2}; x \right) = \sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r x^r. \quad (8)\]

**Proof.** By virtue of equation (1), (4) and (7), we have

\[= \int_0^\infty \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu-1} \sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r y^r \left\{ (\rho x + \frac{\tau}{x})^2 + \gamma \right\}^{-\nu r} \frac{1}{(2\pi \omega)} \int_L \chi(s) \left( \frac{1}{(4\rho \tau + \gamma)^{1/2 + \nu}} \right) \left( \left( 1 - \lambda - \mu x + \omega s, 1 \right) \Gamma \left( \lambda \right) \Gamma \left( \mu x + \omega s + 1/2 \right) / \Gamma \left( \mu x + \omega s + 1/2 \right) \right) ds \quad (9)\]

Interchanging the order of integration and summation, under the valid conditions, we get

\[= \sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r y^r \frac{1}{(2\pi \omega)} \int_L \chi(s) \left( \int_0^\infty \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu - \omega s - 1} dx \right) ds. \quad (10)\]

With the help of equation (4), the above integral becomes

\[= \frac{\sqrt{\pi}}{2\rho (4\rho \tau + \gamma)^{1/2 + \nu}} \sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r y^r \frac{1}{(2\pi \omega)} \int_L \chi(s) \left( \frac{z}{(4\rho \tau + \gamma)^{1/2 + \nu}} \right) \Gamma \left( \lambda \right) \Gamma \left( \mu x + \omega s + 1/2 \right) / \Gamma \left( \mu x + \omega s + 1/2 \right) ds. \]

Now using the above equation and equation (1), we arrive at (8). This is the completed proof of Theorem 2.1. \(\square\)

**Theorem 2.2.** If \(\rho \geq 0; \tau > 0; \gamma + 4\rho \tau > 0; (\Re(\lambda) + 1/2) > 0, -\frac{1}{2} < (a - b - c) < \frac{1}{2}; |\arg z| < \frac{1}{2} \Delta, \) where \(\Delta > 0\) and given by equation (3), then the following formula holds true:

\[\int_0^\infty \frac{1}{x^2} \left[ (\rho x + \frac{\tau}{x})^2 + \gamma \right]^{-\nu-1} 2F_1 \left( a, b, c + \frac{1}{2}; x \right) 2F_1 \left( c - a, c - b, c + \frac{1}{2}; x \right) = \sum_{r=0}^\infty \frac{(c)_r}{(c + \frac{1}{2})_r} a_r x^r. \quad (11)\]
With the help of equation (5), the above integral becomes

\[ (3) \]

and given by equation

\[ \text{Theorem 2.3.} \]

By virtue of equation (1), (6) and (7), we have

\[ \text{Proof.} \]

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\[ \text{Proof.} \]

By virtue of equation (1), (6) and (7), we have

On interchanging the order of integration and summation under the valid assumption, we get

With the help of equation (6), the above integral becomes

Now using the above equation in view of (1), we arrive at (11). This is the completed proof of Theorem 2.2. \[ \square \]

**Theorem 2.3.** If \( \rho > 0; \tau > 0; \gamma + 4\rho > 0; \; (R(\lambda + 1/2) > 0, -\frac{1}{\tau} < (a - b - c) < \frac{1}{\tau}; |\arg z| < \frac{1}{2}\Delta, \text{where} \Delta > 0 \) and given by equation (3), then the following formula holds true:

\[ \text{Proof.} \]

By virtue of equation (1), (6) and (7), we have

On interchanging the order of integration and summation under the valid assumption, we get

With the help of equation (6), the above integral becomes

Now using the above equation in view of (1), we arrive at (14). This is the completed proof of Theorem 2.3. \[ \square \]
3. Special Cases

In this section, we derive some new integral formulae by using some known $\mathcal{H}$-function, $H$-function, $G$-function which are given in Corollaries (17) to (19), (20) to (22) and (23) to (25) respectively.

(I). If we use the same method as in getting Theorem 2.1 to 2.3, we obtain the following three corollaries which is well known $\mathcal{H}$- function, due to Inayat-Hussain [5], for giving value $A''_j = B''_j = 1$ in equation (1), we get

$$
\int_{p_q}^{m,n} \left( \begin{array}{c} 1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; A_j' \right)_p \\ 1 \left( b_j, \beta_j; B_j \right)_m, m_1 \left( b_j, \beta_j; B_j' \right)_q \end{array} \right) = H_{p_q}^{m,n} \left( \begin{array}{c} 1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; 1 \right)_p \\ 1 \left( b_j, \beta_j; 1 \right)_m, m_1 \left( b_j, \beta_j; B_j' \right)_q \end{array} \right)
$$

Now apply above identity, on the Theorems 2.1-2.3 reduces respectively as:

**Corollary 3.1.** Let the condition of Theorem 2.1 be satisfied, we have

$$
\int_0^{\infty} \Phi^{-\lambda-1}_2 F_1 \left( a, b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) \Phi^{-\lambda-1}_2 F_1 \left( c - a, c - b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) H_{p_q}^{m,n} \left[ z \{ \Phi \}^{-\omega} \right] dx
$$

$$
= \frac{\sqrt{\pi}}{2\rho (4\rho + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r (4\rho + \gamma)^{\mu r}} \alpha_r \gamma_r
$$

$$
\times H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\rho + \gamma)^{2}} \begin{array}{c} \left( \frac{1}{2} - \lambda - \mu r, \omega; 1 \right), n_1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; 1 \right)_p \\ 1 \left( b_j, \beta_j; 1 \right)_m, m_1 \left( b_j, \beta_j; B_j \right)_q, \left( -\lambda - \mu r, \omega; 1 \right) \end{array} \right],
$$

where $\Phi = \left[ (\rho x + \frac{\gamma}{2})^2 + \gamma \right]$.  

**Corollary 3.2.** Let the condition of Theorem 2.2 be satisfied, we obtain

$$
\int_0^{\infty} \frac{1}{x^2} \Phi^{-\lambda-1}_2 F_1 \left( a, b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) \Phi^{-\lambda-1}_2 F_1 \left( c - a, c - b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) H_{p_q}^{m,n} \left[ z \{ \Phi \}^{-\omega} \right] dx
$$

$$
= \frac{\sqrt{\pi}}{2\tau (4\rho + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r (4\rho + \gamma)^{\mu r}} \alpha_r \gamma_r
$$

$$
\times H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\rho + \gamma)^{2}} \begin{array}{c} \left( \frac{1}{2} - \lambda - \mu r, \omega; 1 \right), n_1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; 1 \right)_p \\ 1 \left( b_j, \beta_j; 1 \right)_m, m_1 \left( b_j, \beta_j; B_j \right)_q, \left( -\lambda - \mu r, \omega; 1 \right) \end{array} \right].
$$

**Corollary 3.3.** Let the condition of Theorem 2.3 be satisfied, we get

$$
\int_0^{\infty} \left( \rho + \frac{\gamma}{x^2} \right) \Phi^{-\lambda-1}_2 F_1 \left( a, b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) \Phi^{-\lambda-1}_2 F_1 \left( c - a, c - b, c + \frac{1}{2}; y \{ \Phi \}^{-\mu} \right) H_{p_q}^{m,n} \left[ z \{ \Phi \}^{-\omega} \right] dx
$$

$$
= \frac{\sqrt{\pi}}{(4\rho + \gamma)^{\lambda+1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r (4\rho + \gamma)^{\mu r}} \alpha_r \gamma_r
$$

$$
\times H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\rho + \gamma)^{2}} \begin{array}{c} \left( \frac{1}{2} - \lambda - \mu r, \omega; 1 \right), n_1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; 1 \right)_p \\ 1 \left( b_j, \beta_j; 1 \right)_m, m_1 \left( b_j, \beta_j; B_j \right)_q, \left( -\lambda - \mu r, \omega; 1 \right) \end{array} \right].
$$

(II). If we use $A''_j = B''_j = A'_j = B'_j = 1$ in Theorem 2.1 to 2.3, then $I$-function reduces in to well known $H$- function defined by Fox [4], Braaksma [2] and Mathai and Saxena [7].

$$
\int_{p_q}^{m,n} \left( \begin{array}{c} 1 \left( a_j, \alpha_j; A_j \right)_n, n_1 \left( a_j, \alpha_j; A_j' \right)_p \\ 1 \left( b_j, \beta_j; B_j \right)_m, m_1 \left( b_j, \beta_j; B_j' \right)_q \end{array} \right) = H_{p_q}^{m,n} \left( \begin{array}{c} 1 \left( a_j, \alpha_j; 1 \right)_p \\ 1 \left( b_j, \beta_j; 1 \right)_q \end{array} \right)
$$
Corollary 3.4. Let the condition of Theorem 2.1 be satisfied, we have

$$
\int_0^{\infty} \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) H_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \rho (4\rho r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(z + \frac{\mu}{2})_r} \frac{a_r y^r}{(4\rho r + \gamma)^{\mu r}} H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\rho r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, \alpha_j; 1)_p \right) \left( 1 (b_j, \beta_j; 1)_q, -\lambda - \mu r, \omega; 1 \right) \right].
\tag{20}
$$

Corollary 3.5. Let the condition of Theorem 2.2 be satisfied, we obtain

$$
\int_0^{\infty} \frac{1}{x^2} \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) H_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \tau (4\tau r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r y^r}{(4\tau r + \gamma)^{\mu r}} H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\tau r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, \alpha_j; 1)_p \right) \left( 1 (b_j, \beta_j; 1)_q, -\lambda - \mu r, \omega; 1 \right). \tag{21}
$$

Corollary 3.6. Let the condition of Theorem 2.3 be satisfied, we get

$$
\int_0^{\infty} \left( \rho + \frac{7}{x^2} \right) \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) H_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \tau (4\tau r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r y^r}{(4\tau r + \gamma)^{\mu r}} H_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\tau r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, \alpha_j; 1)_p \right) \left( 1 (b_j, \beta_j; 1)_q, -\lambda - \mu r, \omega; 1 \right). \tag{22}
$$

(III). If we set $A''_j = B''_j = A'_j = B'_j = 1$ and $\alpha_j = \beta_j = 1$ in Theorem 2.1 to 2.3, then $I$-function reduces in to $G$-function. (see Luke [6]).

$$
\int_{p,q}^{m,n} \left[ z \frac{1 (a_j, \alpha_j; A'_j)_n}{1 (a_j, \alpha_j; A''_j)_p} + 1 (a_j, \alpha_j; A'_j)_n \frac{1 (b_j, \beta_j; B''_j)_q}{1 (b_j, \beta_j; B'_j)_p} \right] = G_{p,q}^{m,n} \left[ z \frac{1 (a_j, 1; 1)_p}{1 (b_j, 1; 1)_q} \right] \]

Corollary 3.7. Let the condition of Theorem 2.1 be satisfied, we have

$$
\int_0^{\infty} \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) G_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \rho (4\rho r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(z + \frac{\mu}{2})_r} \frac{a_r y^r}{(4\rho r + \gamma)^{\mu r}} G_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\rho r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, 1; 1)_p \right) \left( 1 (b_j, 1; 1)_q, -\lambda - \mu r, \omega; 1 \right). \tag{23}
$$

Corollary 3.8. Let the condition of Theorem 2.2 be satisfied, we obtain

$$
\int_0^{\infty} \frac{1}{x^2} \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) G_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \tau (4\tau r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r y^r}{(4\tau r + \gamma)^{\mu r}} G_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\tau r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, 1; 1)_p \right) \left( 1 (b_j, 1; 1)_q, -\lambda - \mu r, \omega; 1 \right). \tag{24}
$$

Corollary 3.9. Let the condition of Theorem 2.3 be satisfied, we get

$$
\int_0^{\infty} \left( \rho + \frac{7}{x^2} \right) \Phi^{-\lambda -1} f_1 \left( a, b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) z f_1 \left( c - a, c - b, c + \frac{1}{2}; y \{\Phi\}^{-\mu} \right) G_{p,q}^{m,n} \left[ z \{\Phi\}^{-\omega} \right] dx
$$

$$= \frac{\sqrt{\pi}}{2 \tau (4\tau r + \gamma)^{\lambda + 1/2}} \sum_{r=0}^{\infty} \frac{(c)_r}{(c + \frac{1}{2})_r} \frac{a_r y^r}{(4\tau r + \gamma)^{\mu r}} G_{p+1,q+1}^{m,n+1} \left[ \frac{z}{(4\tau r + \gamma)^{\omega}} \right] \left( \frac{1}{2} - \lambda - \mu r, \omega; 1, 1 (a_j, 1; 1)_p \right) \left( 1 (b_j, 1; 1)_q, -\lambda - \mu r, \omega; 1 \right). \tag{25}
$$
4. Concluding Remark

We have established three new definite integrals on the product of $\bar{I}$-function with the hypergeometric function. We also derived analogous result in the form of $\bar{H}$-function, $H$-function and $G$-function, which have been depicted in corollaries. Further, the results presented in this article are easily converted in terms of a similar type [1, 3, 17, 18] of new interesting integrals with different arguments after some suitable parametric replacements. On account of being general and unified in nature, the results established here yield a large number of known and new results involving simpler functions on suitable specifications of the parameters involved.

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References

