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δ-fuzzy Ideals in MS-algebras

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Abstract: In this paper, we introduce the concept of δ-fuzzy ideals and α-level principal δ-fuzzy ideals in an MS-algebra and study basic properties of these ideals with respect to homomorphisms. It is observed that the class of all δ-fuzzy ideals forms a complete distributive lattice and the class of all α-level principal δ-fuzzy ideals forms a de Morgan algebra. A characterization of δ-fuzzy ideals in terms of α-level principal δ-fuzzy ideals is given.

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1. Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [5]. T.S. Blyth and J.C. Varlet [6] defined a subclass of Ockham algebras so called MS-algebras which generalizes both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by J. Berman [4]. The class of all MS-algebras forms an equational class. T.S. Blyth and J.C. Varlet characterized the subvarieties of MS-algebras [7]. Recently E.Badawy [3] introduced δ-ideals in MS-algebra.

On the other hand, fuzzy set theory introduced by Zadeh [14] is generalization of classical set theory. Next, Rosenfeld [11] applied it to group theory and developed the theory of fuzzy subgroups. Also, many authors have worked on fuzzy lattice theory. They introduced the concept of fuzzy sublattices, fuzzy ideals, fuzzy prime ideals in a lattice and gave some interesting results (see [2, 8–13]). In this paper, we introduce δ-fuzzy ideals and α-level principal δ-fuzzy ideals of an MS-algebra. It is proved that the class of $FI^{\delta}(L)$ of all δ-fuzzy ideals of an MS-algebra L is a complete distributive lattice and the set of all α-level principal δ-fuzzy ideals of an MS-algebra can be made into a de Morgan algebra. Some properties of δ-fuzzy ideals are observed with respect to homomorphisms. The concept of δ-fuzzy ideals preserving homomorphism from an MS-algebra L into another MS-algebra S is introduced as a homomorphism f satisfying the condition $f(\delta(\beta)) = \delta(f(\beta))$, for any δ-fuzzy ideals $\mu = \delta(\beta)$ of L, where β is fuzzy filter of L. It is proved the images and the inverse images, under this homomorphism of δ-fuzzy ideals are again δ-fuzzy ideals. If an MS-algebra L is homomorphic to an MS-algebra S, then lattice $FM^{\circ}(L)$ of all α-level principal δ-fuzzy ideals of L is homomorphic to $FM^{\circ}(S)$ of all α-level principal δ-fuzzy ideals of S and the lattice $FI^{\delta}(L)$ of all δ-fuzzy ideals of L is homomorphic to the lattice $FI^{\delta}(S)$ of all δ-fuzzy ideals of S.

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2. Preliminaries

In this section, we recall some definitions and results which will be used in this paper. For in details in ordinary crisp theory of δ-ideals of MS-algebras, we refer to [3].

Definition 2.1 ([5]). *An MS-algebra is an algebra $(L, \vee, \wedge, ^\circ, 0, 1)$ of type $(2, 2, 1, 0, 0)$, such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to a^\circ$ is a unary operation satisfies: $a \leq a^{\circ\circ}$, $(a \wedge b)^\circ = a^\circ \vee b^\circ$, $1^\circ = 0$.*

A Stone algebra $S = (S, \vee, \wedge, ^*, 0, 1)$ is also a bounded distributive lattice, endowed with a unary operation $x \to x^*$ satisfying $(x \wedge y)^* = x^* \vee y^*, x \wedge x^* = 0$ and $0^* = 1$. A de Morgan algebra is an algebra $(L, \vee, \wedge, ^-, 0, 1)$ of type $(2, 2, 1, 0, 0)$,such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $a \to \overline{a}$ is a unary operation satisfies: $\overline{\overline{a}} = a$, $\overline{(a \wedge b)} = \overline{a} \vee \overline{b}$, $\overline{1} = 0$

Lemma 2.2 ([6]). *For any two elements a, b of an MS-algebra,we have the following :*

(1). $0^\circ = 1$

(2). $a \leq b \Rightarrow b^\circ \leq a^\circ$

(3). $a^{\circ\circ\circ} = a^\circ$

(4). $(a \vee b)^\circ = a^\circ \wedge b^\circ$

(5). $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$

(6). $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$

An element a of an MS-algebra L is called a dense element if $a^\circ = 0$. Let us denote the set of all dense elements in MS-algebra L is by D.

Definition 2.3 ([3]). *Let L be an MS-algebra. Then for any filter F of L, denote the set $\delta(F)$ as follows : $\delta(F) = \{x \in L : x^\circ \in F\}$.*

Lemma 2.4 ([3]). *Let L be an MS-algebra. Then $\delta(F)$ is an ideal of L.*

Definition 2.5 ([3]). *Let L be an MS-algebra. An ideal I of L is called δ ideal if $I = \delta(F)$ for some filter of F of L.*

Definition 2.6 ([8]). *Let μ be a fuzzy subset of $(L, \wedge, \vee, 0, 1)$. For any $\alpha \in [0, 1]$, we shall denote the level subset $\mu^{-1}[\alpha, 1]$ by simply μ_α i.e., $\mu_\alpha = \{x \in L/\alpha \leq \mu(x)\}$.*

Definition 2.7 ([2]). *A fuzzy subset μ of L is proper if it is a non constant function. A fuzzy subset μ such that $\mu(x) = 0$ for all $x \in L$ is an improper fuzzy subset.*

Theorem 2.8 ([12]). *Let μ be a fuzzy subset of L. Then μ is a fuzzy ideal of L if and only if any one of the following sets of conditions is satisfied:*

(1). $\mu(0) = 1$ *and* $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ *for all* $x, y \in L$,

(2). $\mu(0) = 1$ *and* $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$ *and* $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$ *for all* $x, y \in L$.

Theorem 2.9 ([12]). *Let μ be a fuzzy subset of L. Then μ is a fuzzy filter of L if and only if any one of the following sets of conditions is satisfied:*

(1). $\mu(1) = 1$ *and* $\mu(x \wedge y) = \mu(x) \wedge \mu(y)$ *for all* $x, y \in L$,

(2). $\mu(1) = 1$ and $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$ and $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$ for all $x, y \in L$.

Proposition 2.10 ([12]). *Let μ be a fuzzy subset of L. Then*

(1). if μ is a fuzzy ideal of L, then $x \leq y \Rightarrow \mu(x) \geq (y)$ for all $x, y \in L$,

(2). if μ is a fuzzy filter of L, then $x \leq y \Rightarrow \mu(x) \leq \mu(y)$ for all$x, y \in L$

Theorem 2.11 ([2]). *Let μ be a fuzzy subset of L. Then*

(1). μ is a fuzzy ideal of L if and only if, for any $\alpha \in [0,1]$ such that $\mu_\alpha \neq \emptyset$, μ_α is an ideal of L,

(2). μ is a fuzzy filter of L if and only , for any $\alpha \in [0,1]$ such that $\mu_\alpha \neq \emptyset$, μ_α is a filter of L.

Definition 2.12 ([10]). *Let $(L, \vee, \wedge)$ be a distributive lattice and $\mu : L \to [0,1]$. Then*

(1). a proper fuzzy ideal μ of L is called fuzzy prime ideal, if $\mu(x \wedge y) = \mu(x) \vee \mu(y)$ for all $x, y \in L$,

(2). a proper fuzzy filter μ of L is called fuzzy prime filter, if $\mu(x \vee y) = \mu(x) \vee \mu(y)$ for all $x, y \in L$.

Theorem 2.13 ([13]). *Let $f : L \to L^{'}$ be an onto homomorphism. Then, $f(\mu)$ is a fuzzy ideal (dual ideal) of $L^{'}$ if μ is a fuzzy ideal (dual ideal) of L.*

Definition 2.14 ([9]). *Let $(L, \wedge, \vee)$ be a lattice, $x \in L$ and $\alpha \in [0,1]$. Define $\alpha_x : L \to [0,1]$ by $\alpha_x(y) = \begin{cases} 1 & if\ y \leq x \\ \alpha & if\ \ y \nleq x \end{cases}$ for any $y \in L$*

Theorem 2.15 ([9]). *α_x defined above is fuzzy ideal of L for any $\alpha \in [0,1]$ and $x \in L$. It is called the α-level principal fuzzy ideal corresponding to x. Note that $\alpha_0 = \chi_{\{0\}}$ and, if L has largest element 1, then $\alpha_1 = \chi_L$*

Definition 2.16 ([9]). *Let I be a crisp ideal of a lattice L . For any $\alpha \in [0,1]$, define $\alpha_I : L \to [0,1]$ by*

$$\alpha_I(y) = \begin{cases} 1 & if\ y \in I \\ \alpha & if\ \ y \notin I \end{cases}$$

For any $y \in L$. For any $\alpha \in [0,1]$, and α_I can be considered as the characteristic map χ_I of L into $[\alpha, 1]$ and α_I is called the α-level fuzzy ideal corresponding to I.

Definition 2.17 ([9]). *Let $(L, \wedge, \vee)$ be a lattice, $x \in L$ and $\alpha \in [0,1]$. Define $\alpha^x : L \to [0,1]$ by $\alpha^x(y) = \begin{cases} 1 & if\ x \leq y \\ \alpha & if\ \ x \nleq y \end{cases}$ for any $y \in L$*

Theorem 2.18 ([9]). *For any $x \in L$ and $\alpha \in [0,1]$, α^x is fuzzy filter of L. It is called the α-level principal fuzzy filter of L corresponding to x. Note that $\alpha^0 = \chi_L$ and $0^1 = \chi_{\{1\}}$*

We can replace the principal crisp filter $[x)$ by a general crisp filter in α^x as follows.

Definition 2.19 ([9]). *Let F be a crisp filter of a lattice L . For any $\alpha \in [0,1]$, define $\alpha^F : L \to [0,1]$ by*

$$\alpha^F(y) = \begin{cases} 1 & if\ y \in F \\ \alpha & if\ \ y \notin F \end{cases}$$

for any $y \in L$.

Note for any $\alpha \in [0,1]$, and α^F can be considered as the characteristic map χ_F of L into $[\alpha, 1]$ and α^F is called α-level fuzzy filter L corresponding F. For any $x \in L$, consider the principal filter $[x)$ given by $[x) = \{y \in L : x \leq y\}$. Then for any $\alpha \in [0,1]$, $\alpha^x = \alpha^{[x)}$.

Through out in the next two sections L stands for an MS-algebra and $\alpha \in [0,1]$ unless otherwise mentioned.

3. δ-fuzzy Ideals in MS-algebras

In this section, the concepts of δ-fuzzy ideals and α-level principal δ-fuzzy ideals are introduced in MS-algebras. Some properties of these ideals are investigated in the class of MS-algebras.

Definition 3.1. *Let L be MS-algebra. Then for any fuzzy filter μ of L, denote the fuzzy subset $\delta(\mu)$ as follows: $\delta(\mu)(x) = \mu(x^\circ)$ for all $x \in L$.*

Using the above definition, we develop the following two Lemmas.

Lemma 3.2. *Let μ be fuzzy filter of L.Then $\delta(\mu)$ is fuzzy ideal of L.*

Proof. $\delta(\mu)(0) = \mu(0^\circ) = (\mu)(1) = 1$. For any x, y *in* L. Then $\delta(\mu)(x \vee y) = \mu((x \vee y)^\circ) = \mu(x^\circ \wedge y^\circ) = \mu(x^\circ) \wedge \mu(y^\circ) = \delta(\mu)(x) \wedge \delta(\mu)(y)$. Therefore $\delta(\mu)$ is fuzzy ideal of L. □

Lemma 3.3. *For any two fuzzy filters μ and θ of L, we have the following :*

(1). $(\mu \cap \delta(\mu))(x) \leq \alpha$, $\forall\, \alpha \in Im(\mu)$, $\forall\, x \in L$ whenever L is Stone algebra,

(2). $\delta(\mu)(x) = \delta(\mu)(x^{\circ\circ})$, $\forall x \in L$,

(3). $\delta(\mu)(x^\circ) \geq \mu(x)$, $\forall\, x \in L$,

(4). $\mu(x) = 1, \forall x \in L \Leftrightarrow \delta(\mu)(x) = 1, \forall x \in L$,

(5). $\mu \subseteq \theta \Rightarrow \delta(\mu) \subseteq \delta(\theta)$,

(6). $\delta(\mu)(x) \leq \alpha$, $\forall\, \alpha \in Im(\mu) \Leftrightarrow$ $x \in D$,

(7). If μ is fuzz prime filter of L, then $\delta(\mu)$ is fuzzy prime ideal.

Proof.

(1). $(\mu \cap \delta(\mu))(x) = \mu(x) \wedge \delta(\mu)(x) = \mu(x \wedge x^\circ) = \mu(0) \leq \mu(x)$ $\forall$ $x \in L$. Since L is in Stone algebra. Therefore $(\mu \cap \delta(\mu))(x) \leq \alpha$, $\forall\, \alpha \in Im(\mu)$.

(2). $\delta(\mu)(x^{\circ\circ}) = \mu(x^{\circ\circ\circ}) = \mu(x^\circ) = \delta(\mu)(x)$.

(3). $\delta(\mu)(x^\circ) = \mu(x^{\circ\circ}) \geq \mu(x)$, since μ is isotone and $x \leq x^{\circ\circ}$.

(4). For any x in L, $\mu(x) = 1 \Rightarrow \mu(1^\circ) = 1 \Rightarrow \delta(\mu)(1) = 1$, since $\delta(\mu)(1) \leq \delta(\mu)(x)$. Therefore $\delta(\mu)(x) = 1$, $\forall\, x \in L$. Conversely $\delta(\mu)(x) = 1\, \forall\, x \in L \Rightarrow \sigma(\mu)(1) = 1 \Rightarrow \mu(0) = 1$, since $1 = \mu(0) \leq \mu(x)$. Therefore $\mu(x) = 1$, $\forall\, x \in L$.

(5). Since $\mu \subseteq \theta$. Then $\delta(\mu)(x) = \mu(x^\circ) \subseteq \theta(x^\circ) = \delta(\theta)(x), \forall x \in L$.

(6). $\delta(\mu)(x) = \mu(x^\circ) \leq \alpha$, $\forall\, \alpha \in Im(\mu) \Leftrightarrow x^\circ = 0 \Leftrightarrow x \in D$.

(7). For all x, y in L, $\delta(\mu)(x \wedge y) = \mu((x \wedge y)^\circ) = \mu(x^\circ \vee y^\circ) = \mu(x^\circ) \vee \mu(y^\circ) = \delta(\mu)(x) \vee \delta(\mu)(y)$. □

Definition 3.4. *Let L be an MS-algebra. A fuzzy ideal μ of L is called δ-fuzzy ideal if $\mu = \delta(\beta)$ for some fuzzy filter β of L.*

Example 3.5. *Let $L = \{0, x, y, z, 1 : 0 < x < y < z < 1\}$ be a chine of five elements and $x^\circ = x, y^\circ = z^\circ = 1^\circ = 0, 0^\circ = 1$. Clearly $(L, \vee, \wedge, ^\circ, 0, 1)$ is an MS-algebra. Define fuzzy subsets μ and β as follows: $\mu(0) = 1$, $\mu(x) = 0.8$ and $\mu(y) = \mu(z) = \mu(1) = 0.5$ and $\beta(1) = \beta(y) = \beta(z) = 1$, $\beta(x) = 0.8$, $\beta(0) = 0.5$. Clearly μ is fuzzy ideal of L and β is fuzzy filter of L and $\mu = \delta(\beta)$. Then μ is δ-fuzzy ideal of L.*

Theorem 3.6. *I is δ-ideal of an MS-algebra L if and only if χ_I is δ-fuzzy ideal of L.*

Proof. Since I is δ-ideal of L, then there exists a filter F such that $I = \delta(F)$. Clearly χ_I and χ_F are fuzzy ideal and fuzzy filter of L respectively. To prove χ_I is δ-fuzzy ideal enough to show that $\chi_I(x) = \delta(\chi_F)(x)$. Let $x \in L$. If $x \in I = \delta(F)$, then $x^\circ \in F$. This implies $1 = \chi_I(x) = \chi_F(x^\circ) = \delta(\chi_F)(x)$. If $x \notin I = \delta(F)$, then $x^\circ \notin F$. This implies $0 = \chi_I(x) = \chi_F(x^\circ) = \delta(\chi_F)(x)$. Therefore $\chi_I(x) = \delta(\chi_F)(x)$, $\forall\ x \in L$. Conversely suppose χ_I is δ-fuzzy ideal. Now $x \in I \Leftrightarrow 1 = \chi_I(x) = \delta(\beta)(x) = \beta(x^\circ) \Leftrightarrow x^\circ \in \beta_1 \Leftrightarrow x \in \delta(\beta_1)$. Therefore $I = \delta(\beta_1)$ and so I is δ-ideal of L. □

Theorem 3.7. *Let $\alpha \in Im(\mu)$ and μ be proper δ-fuzzy ideal of an MS-algebra L. Then $\mu(x) \leq \alpha$, $\forall\ x \in D$.*

Proof. Suppose $\mu(x) > \alpha$, $\forall\ x \in D$ and μ is proper . Then $x \in \mu_\alpha$. This implies $x^\circ \in \beta_\alpha$ for some level subset of fuzzy filter of L. This implies $\beta(x^\circ) = \beta(0) \geq \alpha$, $\forall\ \alpha \in Im(\mu)$, which is contradiction. Therefore $\mu(x) \leq \alpha$, $\forall\ x \in D$. □

Theorem 3.8. *Let $\{\mu_i : i \in \Omega\}$ be any subfamily of δ-fuzzy ideals in MS-algebra L. Then $\cap_{i\in\Omega}\mu_i$ is δ-fuzzy ideal.*

Proof. Since μ_i is δ-fuzzy ideal $\forall i \in \Omega$, then there exists a fuzzy filter θ_i, $i \in \Omega$ such that $\mu_i = \delta(\theta_i)$. Now $(\cap_{i\in\Omega}\mu_i)(x) = \inf_{i\in\Omega}(\mu_i)(x) = \inf_{i\in\Omega}(\delta(\theta_i))(x) = \inf_{i\in\Omega}(\theta_i)(x^\circ) = (\cap_{i\in\Omega}\theta_i)(x^\circ) = \delta(\cap_{i\in\Omega}\theta_i)(x)$. Since $\cap_{i\in\Omega}\theta_i$ is fuzzy filter, then $\cap_{i\in\Omega}\mu_i$ is δ-fuzzy ideal. □

Let us denote the set of all δ-fuzzy ideals of L by $FI^\delta(L)$. Then in the following Theorem we prove that $FI^\delta(L)$ forms a complete distributive lattice .

Theorem 3.9. *Let L be an MS-algebra. Then $FI^\delta(L)$ forms a complete distributive lattice with relation $\subseteq$.*

Proof. Clearly $\chi_{\{0\}}$ and χ_L are the smallest and the largest δ-fuzzy ideals of L and $(FI^\delta(L), \subseteq)$ is a partial ordered set .For any two δ-fuzzy ideals μ and β of an MS-algebra L . We prove that $\mu \cap \beta$ and $\mu \vee \beta$ are δ-fuzzy ideals. Since μ and β are δ-fuzzy ideals, then there exists filters θ and ν of L such that $\mu = \delta(\theta)$ and $\beta = \delta(\nu)$. So we have to show the following : $\delta(\theta) \cap \delta(\nu) = \delta(\theta \cap \nu)$ and $\delta(\theta) \vee \delta(\nu) = \delta(\theta \vee \nu)$. Now $(\mu \cap \beta)(x) = (\delta(\theta) \cap \delta(\nu))(x) = \delta(\theta)(x) \wedge \delta(\nu)(x) = \theta(x^\circ) \wedge \nu(x^\circ) = \delta(\theta \cap \nu)(x)$ and we to show $\delta(\theta \vee \nu)$ is the smallest fuzzy ideal containing $\delta(\theta)$ and $\delta(\mu)$. Let ϕ be any fuzzy ideal such that $\delta(\theta) \subseteq \phi$ and $\delta(\mu) \subseteq \phi$. Now $(\delta(\theta) \vee \delta(\nu))(x) = (\theta \vee \nu)(x^\circ) = \sup\{\theta(a) \wedge \nu(b) : a \wedge b = x^\circ\} \leq \sup\{\theta(a^{\circ\circ}) \wedge \nu(b^{\circ\circ}) : a \wedge b = x^\circ\} \leq \sup\{\delta(\theta)(a^\circ) \wedge \delta(\nu)(b^\circ) : a \wedge b = x^\circ\} \leq \sup\{\phi(a^\circ) \wedge \phi(b^\circ) : a \wedge b = x^\circ\}$ (since $a \wedge b = x^\circ$, we get $(a \wedge b)^\circ = a^\circ \vee b^\circ = x^{\circ\circ})\} \leq \sup\{\phi(a^\circ) \wedge \phi(b^\circ) : a^\circ \vee b^\circ = x^{\circ\circ}\} \leq \phi(x_1) \wedge \phi(x_2) : x_1^\circ \vee x_2^\circ = x^{\circ\circ}\} = (\phi \vee \phi)(x^{\circ\circ}) \leq \phi(x)$. This implies $\delta(\mu \vee \theta) \subseteq \phi$. Thus implies $\delta(\mu) \vee \delta(\theta) = \delta(\mu \vee \theta)$. This implies $\mu \vee \beta$ is δ- ideals of L and the supremum of both μ and β in $FI^\delta(L)$. For any $\delta(\mu), \delta(\theta)$ and $\delta(\beta) \in FI^\delta(L)$. Then $(\delta(\mu) \cap \delta(\theta)) \vee \delta(\beta) = \delta((\mu \cap \theta) \vee \beta) = \delta(\mu \vee \beta) \cap \delta(\theta \vee \beta)$. Therefore $(FI^\delta(L), \vee, \wedge, ^\circ, 0, 1)$ is bounded distributive lattice. In Theorem 3.9 every subfamily of $FI^\delta(L)$ is greatest lower bound and $FI^\delta(L)$ has greatest element, then $FI^\delta(L)$ is complete lattice . Therefore $FI^\delta(L)$ forms a complete distributive lattice with relation $\subseteq$. □

Definition 3.10. *A δ-fuzzy ideal μ of an MS-algebra L is called α-level principal δ-fuzzy ideal if there exists $x \in L$ such that $\mu = \delta(\alpha^x)$, where $\alpha \in [0, 1]$.*

Theorem 3.11. *Let L be an MS-algebra. Then for any $x \in L$, α_{x° is α-level principal δ-fuzzy ideal of L.*

Proof. It is enough to prove that $\alpha_{x^\circ} = \delta(\alpha^x)$. Suppose $\alpha \neq 1$ and $\alpha_{x^\circ} \neq \delta(\alpha^x)$, then without loss of generality there exists $y \in L$ such that $\alpha_{x^\circ}(y) = 1$ and $\delta(\alpha^x)(y) = \alpha$. Since $\alpha_{x^\circ}(y) = 1 \Rightarrow y \leq x^\circ \Rightarrow x \leq x^{\circ\circ} \leq y^\circ \Rightarrow \alpha^x(y^\circ) = 1 \Rightarrow \delta(\alpha^x)(y) = 1$ which is contradiction. Thus α_{x° is α-level principal δ-fuzzy ideal of L. □

Some properties of α-level principal δ-fuzzy ideals of L are studied in the following.

Lemma 3.12. *Let L be an MS-algebra. Then we have*

(1). for all $x \in L, \delta(\alpha^x) = \alpha_{x^\circ}$,

(2). for all $x \in L, \delta(\alpha^x) = \delta(\alpha^{x^{\circ\circ}})$,

(3). for all $x \in D, \delta(\alpha^x) = \alpha_0$.

Let us denote that $FM^\circ(L) = \{\delta(\alpha^x) : x \in L\} = \{\alpha_{x^\circ} : x \in L\}$. Then in the following theorem, it is observed that $FM^\circ(L)$ is a de Morgan algebra and sublattice of $FI^\delta(L)$.

Theorem 3.13. *For any MS-algebra L, $FM^\circ(L)$ is a sublattice of $FI^\delta(L)$ and $FM^\circ(L)$ can be made in to a de Morgan algebra.*

Proof. Clearly $\delta(\alpha^1) = \alpha_0$, $\delta(\alpha^0) = \alpha_1 \in FM^\circ(L)$, then $FM^\circ(L) \neq \emptyset$. Let $\delta(\alpha^x), \delta(\alpha^y) \in FM^\circ(L)$ fore some $x, y \in L$. Then we get that $\delta(\alpha^x) \cap \delta(\alpha^y) = \delta(\alpha^{x\vee y})$ and $\delta(\alpha^x) \vee \delta(\alpha^y) = \delta(\alpha^{x\wedge y})$ in $FM^\circ(L)$. Therefore $FM^\circ(L)$ is a bounded sublattice of $FI^\delta(L)$. Next we want to $FM^\circ(L)$ is distributive. Let $\delta(\alpha^x), \delta(\alpha^y), \delta(\alpha^z) \in FM^\circ(L)$ fore some $x, y, z \in L$. Then $\delta(\alpha^x) \vee (\delta(\alpha^y) \cap \delta(\alpha^z)) = \delta(\alpha^x) \vee \delta(\alpha^{y\vee z}) = \delta(\alpha^{x\wedge(y\vee Z)}) = \delta(\alpha^{(x\wedge y)\vee(x\wedge z)}) = \delta(\alpha^{x\wedge y}) \cap \delta(\alpha^{x\wedge z}) = (\delta(\alpha^x) \vee \delta(\alpha^y)) \cap (\delta(\alpha^x) \vee \delta(\alpha^z))$. Now define a unary operation "$^-$" *on* $FM^\circ(L)$ by $\overline{\delta(\alpha^x)} = \delta(\alpha^{x^\circ})$. Then we have $\overline{\overline{\delta(\alpha^x)}} = \delta(\alpha^{x^{\circ\circ}}) = \delta(\alpha^x)$ and $\overline{(\delta(\alpha^x) \vee \delta(\alpha^y))} = \overline{\delta(\alpha^{x\wedge y})} = \delta(\alpha^{(x\wedge y)^\circ}) = \delta(\alpha^{x^\circ\vee y^\circ}) = \delta(\alpha^{x^\circ}) \cap \delta(\alpha^{y^\circ}) = \overline{\delta(\alpha^x)} \cap \overline{\delta(\alpha^x)}$ and $\overline{\delta(\alpha^1)} = \delta(\alpha^0)$. Therefore $FM^\circ(L)$ is a de Morgan Algebra. □

Theorem 3.14. *For any MS-algebra L. A mapping $x \to \alpha_{x^\circ}$ is dual homomorphism of L into $FM^\circ(L)$.*

Proof. A mapping $f : L \to FM^\circ(L)$ define by $f(x) = \alpha_{x^\circ}$ for all $x \in L$. Thus $f(0) = \alpha_1$ and $f(1) = \alpha_0$ and $f(x \vee y) = \alpha_{(x\vee y)^\circ} = \delta(\alpha^{x\vee y}) = \delta(\alpha^x) \cap \delta(\alpha^y) = \alpha_{x^\circ} \cap \alpha_{y^\circ} = f(x) \cap f(y)$. Similarly $f(x \wedge y) = f(x) \vee f(y)$. Therefore a mapping $x \to \alpha_{x^\circ}$ is dual homomorphism of L into $FM^\circ(L)$ □

4. δ-fuzzy Ideals and Homomorphisms of MS-algebras

In this section, some properties of the fuzzy homomorphic images and the inverse images of δ-fuzzy ideals are studied.

Theorem 4.1. *Let $f : L \to M$ be a homomorphism of an MS-algebra L onto an MS-algebra M. Then we have*

(1). for any $\alpha \in [0, 1]$ and for any $x \in L$, $f(\delta(\alpha^x)) = \delta(f(\alpha^x))$,

(2). for any fuzzy filter μ of L, $f(\delta(\mu)) = \delta(f(\mu))$,

(3). for any δ-fuzzy ideal μ of L, $f(\mu)$ is a δ-fuzzy ideal of M.

Proof.

(1). For any $y \in M$, $f(\delta(\alpha^x))(y) = \sup\{\delta(\alpha^x)(z) : f(z) = y, z \in L\} = \sup\{(\alpha^x)(z^\circ) : f(z^\circ) = y^\circ\} = f(\alpha^x)(y^\circ) = \delta(f(\alpha^x)(y)$.

(2). analogous to (1).

(3). Since μ is δ-fuzzy ideal of L, then $\mu = \delta(\theta)$ for some fuzzy filter θ of L. For all $y \in M$, $f(\mu)(y) = \sup\{\delta(\theta)(x) : f(x) = y, x \in L\} = \sup\{(\theta)(x^\circ) : f(x^\circ) = y^\circ\} = f(\theta)(y^\circ) = \delta(f(\theta))(y)$. Since f is homomorhism and θ is fuzzy filter of L, then $f(\theta)$ is fuzzy filter of M. Then $f(\mu)$ is δ-fuzzy ideal of M. □

Theorem 4.2. *Let $f : L \to M$ be a homomorphism of an MS-algebra L into an MS-algebra M. Then for any δ-fuzzy ideal θ of M, $f^{-1}(\theta)$ is a δ-fuzzy ideal of L.*

Proof. Since θ is δ-fuzzy ideal of M, then there exists a fuzzy filter of β of M such that $\theta = \delta(\beta)$. Now for any $x \in L$, $f^{-1}(\theta)(x) = f^{-1}(\delta(\beta))(x) = \delta(\beta)(f(x)) = \delta(\beta(f(x))) = \delta(f^{-1}(\beta))(x)$. Clearly $f^{-1}(\beta)$ is fuzzy filter of L. Therefore $f^{-1}(\theta)$ is a δ-fuzzy ideal of L. □

Theorem 4.3. *Let $f : L \to M$ be homomorphism of an MS-algebra L onto an MS-algebra M. Then the mapping $\mu \to f(\mu)$ define a one-to-one correspondence between the set of all f-invariant δ-fuzzy ideals of L and the set of all δ-fuzzy ideals of M.*

Theorem 4.4. *Let $f : L \to M$ be an on to homomorphism between MS-algebras $(L, \vee, \wedge, 0_L, 1_L)$ and $(M, \vee, \wedge, 0_M, 1_M)$. Then we have*

(1). $FM^\circ(L)$ is homomorphic of $FM^\circ(M)$,

(2). $FI^\delta(L)$ is homomorphic of $FI^\delta(M)$.

Proof.

(1). Define $h : FM^\circ(L) \to FM^\circ(M)$ by $h(\delta(\alpha^x)) = \delta(\alpha^{f(x)})$ for any x in L . Clearly $h(\delta(\alpha^{0_L})) = \delta(\alpha^{0_M})$ and $h(\delta(\alpha^{1_L})) = \delta(\alpha^{1_M})$. For every $\delta(\alpha^x), \delta(\alpha^y) \in FM^\circ(L)$, we have get $h(\delta(\alpha^x) \cap \delta(\alpha^y)) = h(\delta(\alpha^{x\vee y})) = \delta(\alpha^{f(x\vee y)}) = \delta(\alpha^{f(x)\vee f(y)}) = \delta(\alpha^{f(x)}) \cap \delta(\alpha^{f(y)}) = h(\delta(\alpha^x)) \cap h(\delta(\alpha^y))$ and $h(\delta(\alpha^x) \vee \delta(\alpha^y)) = h(\delta(\alpha^{x\wedge y})) = \delta(\alpha^{f(x\wedge y)}) = \delta(\alpha^{f(x)\wedge f(y)}) = \delta(\alpha^{f(x)}) \vee \delta(\alpha^{f(y)}) = h(\delta(\alpha^x)) \vee h(\delta(\alpha^y))$ and also $\overline{h(\delta(\alpha^x)} = h(\delta(\alpha^{x^\circ}) = (\delta(\alpha^{f(x^\circ)}) = \overline{(\delta(\alpha^{f(x)})} = h(\overline{(\delta(\alpha^x)})$. Therefore h is a homomorphism of de Morgan algebra $FM^\circ(L)$ and $FM^\circ(M)$.

(2). Define a map $h : FI^\delta(L) \to FI^\delta(M)$ by $h(\mu) = \delta(f(\beta))$, where $\mu = \delta(\beta)$ for some fuzzy filter β of L. Clearly $h(\alpha_{0_L}) = \alpha_{0_M}$ and $h(\alpha_{1_L}) = \alpha_{1_M}$. Let μ and $\theta \in FI^\delta(L)$. Then $\mu = \delta(\beta)$ and $\theta = \delta(\nu)$, where β and ν are some fuzzy filter of L. Then we have to get $h(\mu \cap \theta) = h(\delta(\beta) \cap \delta(\nu)) = h(\delta(\beta \cap \nu)) = \delta(f(\beta \cap \nu)) = \delta(f(\beta) \cap f(\nu)) = \delta(f(\beta)) \cap \delta(f(\nu)) = h(\mu) \cap h(\nu)$. Similarly $h(\mu \vee \theta) = h(\mu) \vee h(\nu)$. Therefore h is homomorphic from $FI^\delta(L)$ to $FI^\delta(M)$. □

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