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Generalization of Fuzzy Boundary in Fuzzy Bitopological Spaces

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Abstract: Focus of this paper is to introduce the concept of fuzzy (τ_i, τ_j)-boundary and fuzzy $\mathfrak{C}$-(τ_i, τ_j)-boundary in fuzzy bitopological space where $\mathfrak{C} : [0, 1] \to [0, 1]$ is a complement function. Several examples are given to illustrate the concepts introduced in this paper.

Keywords: Fuzzy complement function $\mathfrak{C}$, fuzzy $\mathfrak{C}$-(τ_i, τ_j)-boundary in fuzzy bitopological spaces.

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1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by L. A. Zadeh [8] in the year 1965. The theory of fuzzy topological space was introduced and developed by C. L. Chang [3]. A. Kandil [5] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological space. The concept of complement function $\mathfrak{C} : [0, 1] \to [0, 1]$ was introduced by K. Bageerathi and P. Thangavelu in [2]. PU and Liu [7] defined the notion of fuzzy boundary in fuzzy topological spaces in 1980. In this chapter we extend some results of fuzzy boundary to bitopological spaces and characterize their properties using standard and arbitrary complement function and several examples are given to illustrate the concepts introduced in this paper.

2. Preliminaries

In this section we list some definitions and results that are needed. Any function $\mathfrak{C} : [0, 1] \to [0, 1]$ defined from the interval $[0, 1]$ to itself is called a complement function. Throughout the paper $\mathfrak{C}$ denotes an arbitrary complement function and (X, τ_i, τ_j) is a fuzzy bitopological space in the sense of A. Kandil [5]. Throughout this paper, for fuzzy set λ of a fuzzy bitopological space (X, τ_i, τ_j), τ_i-$int\lambda$ and τ_j-$cl_{\mathfrak{C}}\lambda$ means, respectively, the interior and closure of λ with respect to fuzzy topologies τ_i and τ_j.

Definition 2.1 ([2]). *If λ is a fuzzy subset of X then the complement $\mathfrak{C}\lambda$ of a fuzzy set λ is a fuzzy subset with membership function defined by $\mu_{\mathfrak{C}}\lambda(x) = \mathfrak{C}(\mu\lambda(x))$ for all $x \in X$.*

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A subset λ of a fuzzy topological space is fuzzy closed if its standard complement $\lambda\prime$, where $\lambda\prime(x) = 1 - \lambda(x)$ is fuzzy open. Several fuzzy topologists used this type of complement while extending the concepts in general topological spaces to fuzzy topological spaces. But there are other complements available in the fuzzy literature. The properties of fuzzy complement function $\mathfrak{C}$ and $\mathfrak{C}\lambda$ are given in George Klir [4] and Bageerathi [2]. The following lemma will be useful in sequel. Some of the complement functions are given below.

Example 2.2 ([3]).

(1). The standard complement function: $\mathfrak{C}_1(x) = 1 - x$.

(2). The Threshold type complement function for any $t \in [0, 1)$*:*

$$\mathfrak{C}_t(x) = \begin{cases} 1 & for\ 0 \leq x \leq t \\ 0 & for\ t < x \leq 1 \end{cases}$$

(3). Sugeno class complement function for any $\lambda \in (1, \infty)$*:*

$$\mathfrak{C}_{S\lambda}(x) = \frac{1-x}{1+\lambda x},\ for\ x \in [0, 1].$$

(4). Yagor class of complement function for $\omega \in (0, \infty)$*:*

$$\mathfrak{C}_{Y_\omega}(x) = (1 - x^\omega)^{1/\omega},\ for\ x \in [0, 1].$$

Lemma 2.3 ([3]). *The complement functions* $\mathfrak{C}_1$*,* $\mathfrak{C}_t$*,* $\mathfrak{C}_{S\lambda}$ *and* $\mathfrak{C}_{Y_\omega}$ *satisfy the following conditions.*

(1). Boundary condition: $\mathfrak{C}(0) = 1$ *and* $\mathfrak{C}(1) = 0$*;*

(2). Monotonicity: for all $x, y \in [0, 1]$, $x \leq y \Rightarrow \mathfrak{C}(x) \geq \mathfrak{C}(y)$*;*

(3). $\mathfrak{C}$ *is continuous and*

(4). Involutive: $\mathfrak{C}(\mathfrak{C}(x)) = x$ *for all* $x \in [0, 1]$.

Definition 2.4 ([1]). *For a family* $\{A_\alpha : \alpha \in \Delta\}$ *of fuzzy sub sets of X, the union,* $A = \cup\{A_\alpha : \alpha \in \Delta\}$ *and the intersection,* $B = \cap\{A_\alpha : \alpha \in \Delta\}$*, are defined with membership functions respectively* $\mu_A(x) = \sup\{\mu_{A\alpha}(x) : \alpha \in \Delta\}$ *and* $\mu_B(x) = inf\{\mu_{A\alpha}(x) : \alpha \in \Delta\}$, $x \in X$.

Proposition 2.5 ([6]). *If the complement functions* $\mathfrak{C}$ *satisfies the monotonicity and involutive properties, then for any fuzzy subset* λ *of a fuzzy bitopological space, we have*

(1). $\mathfrak{C}(\tau_i - int\lambda) = \tau_i - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)$ *and*

(2). $\mathfrak{C}(\tau_i - cl_{\mathfrak{C}}\lambda) = \tau_i - int(\mathfrak{C}\lambda)$.

Theorem 2.6 ([6]). *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then for any two fuzzy subsets* λ *and* μ *of a fuzzy bitopological space*

(1). $\lambda \leq \tau_i\text{-}cl_{\mathfrak{C}}\lambda$*;*

(2). λ *is fuzzy* $\mathfrak{C}$*-*τ_i*-closed* $\Leftrightarrow \tau_i\text{-}cl_{\mathfrak{C}}\lambda = \lambda$*;*

(3). $\tau_i\text{-}cl_{\mathfrak{C}}(\tau_i\text{-}cl_{\mathfrak{C}}\lambda) = \tau_i - cl_{\mathfrak{C}}\lambda$;

(4). *If* $\lambda \leq \mu$ *then* $\tau_i\text{-}cl_{\mathfrak{C}}\lambda \leq \tau_i\text{-}cl_{\mathfrak{C}}\mu$;

(5). $\tau_i\text{-}cl_{\mathfrak{C}}(\lambda \vee \mu) = \tau_i - cl_{\mathfrak{C}}\lambda \vee \tau_i - cl_{\mathfrak{C}}\mu$ *and*

(6). $\tau_i - cl_{\mathfrak{C}}(\lambda \wedge \mu) \geq \tau_i - cl_{\mathfrak{C}}\lambda \wedge \tau_i - cl_{\mathfrak{C}}\mu$.

Lemma 2.7 ([2])**.** *Let* $\mathfrak{C} : [0,1] \to [0,1]$ *be a complement function that satisfies involutive and monotonic properties. Then for any family* $\{\lambda_a : a \in \Delta\}$ *of fuzzy subsets of X. we have*

(1). $\mathfrak{C}(\vee\{\lambda_a : a \in \Delta\}) = \wedge\{\mathfrak{C}\lambda_a : a \in \Delta\}$ *and*

(2). $\mathfrak{C}(\wedge\{\lambda_a : a \in \Delta\}) = \vee\{\mathfrak{C}\lambda_a : a \in \Delta\}$.

3. Fuzzy (τ_i, τ_j)-boundary

In this section, we introduce the concept of fuzzy (τ_i, τ_j)-boundary in fuzzy bitopological space and discuss some of its properties.

Definition 3.1. *Let* λ *be a fuzzy subset of a fuzzy bitopological space* (X, τ_1, τ_2)*. Then the fuzzy* (τ_i, τ_j)*-boundary of* λ *is defined as* $(\tau_i, \tau_j) - Bd(\lambda) = \tau_i - cl(\tau_j - cl\lambda) \wedge \tau_i - cl(\tau_j - cl(\lambda^c))$*, where* λ^c *is the standard complement of* λ*,* $i, j = 1, 2$ *and* $i \neq j$.

Proposition 3.2. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Then for any fuzzy subset* λ *of X,* $(\tau_i, \tau_j) - Bd\lambda = (\tau_i, \tau_j) - Bd(\lambda^c)$.

Proof. By Definition 3.1, $(\tau_i, \tau_j) - Bd\lambda = \tau_i - cl(\tau_j - cl\lambda) \wedge \tau_i - cl(\tau_j - cl(\lambda^c))$. This implies that $(\tau_i, \tau_j) - B\lambda = \tau_i - cl(\tau_j - cl((\lambda^c)^c)) \wedge \tau_i - cl(\tau_j - cl(\lambda^c)) = (\tau_i, \tau_j) - Bd(\lambda^c)$. Therefore $(\tau_i, \tau_j) - Bd\lambda = (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda^c)$. □

Proposition 3.3. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. If* λ *is fuzzy* τ_i*-closed,* $i = 1, 2$*. Then* $(\tau_i, \tau_j) - Bd\ \lambda \leq \lambda$.

Proof. Let λ be a fuzzy τ_i-closed, $i = 1, 2$. By Definition 3.1, $(\tau_i, \tau_j) - Bd\lambda = \tau_i - cl(\tau_j - cl\lambda) \wedge \tau_i - cl(\tau_j - cl(\lambda^c))^c)$. Since λ is fuzzy τ_i-closed, $\tau_i - cl\lambda = \lambda$, $i = 1, 2$. This implies that $(\tau_i, \tau_j) - Bd\lambda \leq \tau_i - cl(\tau_j - cl(\lambda)) = \lambda$. □

Proposition 3.4. *Let* (X, τ_i, τ_j) *be a fuzzy bitopological space. If* λ *is fuzzy* τ_i*-open,* $i = 1, 2$*. Then* $(\tau_i, \tau_j) - Bd\lambda \leq \lambda^c$.

Proof. Let λ be a fuzzy τ_i-open, $i = 1, 2$. Therefore λ^c is a fuzzy τ_i-closed. By using Proposition 3.3, $(\tau_i, \tau_j) - Bd\lambda \leq \lambda^c$. □

4. Fuzzy $\mathfrak{C} - (\tau_i, \tau_j)$-boundary

In this section, we introduced the concept of fuzzy $\mathfrak{C} - (\tau_i, \tau_j)$-boundary in fuzzy bitopological space and discuss its properties.

Definition 4.1. *Let* λ *be a fuzzy subset of a fuzzy bitopological space* (X, τ_1, τ_2) *and* $\mathfrak{C}$ *be a complement function. Then the fuzzy* $\mathfrak{C} - (\tau_i, \tau_j)$*-boundary of* λ *is defined as* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda) = \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\lambda) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$*. Since arbitrary intersection of fuzzy* $\mathfrak{C} - \tau_i$*-closed sets is fuzzy* $\mathfrak{C} - \tau_i$*-closed,* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda$ *is fuzzy* $\mathfrak{C} - \tau_i$*-closed. We identify* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda$ *with* $(\tau_i, \tau_j) - Bd\lambda$ *when* $\mathfrak{C}(x) = 1 - x$*, the usual complement.*

Example 4.2. *Let* $X = \{a, b, c\}$, $\tau_1 = \{0, \{c_{.4}\}, \{a_{.7}\}, \{a_{.7}, c_{.4}\}, 1\}$ *and* $\tau_2 = \{0, \{c_{.5}\}, \{a_{.6}\}, \{a_{.6}, c_{.5}\}, 1\}$. *Let* $\mathfrak{C}(x) = \frac{1-x}{1+2x}$ *be a complement function satisfies both monotonicity and involutive properties. Then the family of all fuzzy* $\mathfrak{C}-\tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.333}\}, \{a_{.125}, b_1, c_1\}, \{a_{.125}, b_1, c_{.333}\}, 0\}$ *and* $\mathfrak{C}(\tau_2) = \{1, \{a_1, b_1, c_{.25}\}, \{a_{.18}, b_1, c_1\}, \{a_{.18}, b_1, c_{.25}\}, 0\}$. *Let* $\lambda = \{a_{.1}\}$. *Then it can be evaluated that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \tau_1 - Cl_{\mathfrak{C}}\{a_{.18}, b_1, c_{.25}\} = \{a_1, b_1, c_{.333}\}$. *Now* $\mathfrak{C}\lambda = \{a_{.75}, b_1, c_1\}$. *Then it can be found that* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1$. *We see that*

$$\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \{a_1, b_1, c_{.333}\} \;\; and \;\; \tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1 \tag{1}$$

By Definition 4.1 and Equation (1), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda = \{a_1, b_1, c_{.333}\}$.

Proposition 4.3. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and let* $\mathfrak{C}$ *be a complement function that satisfies the involutive property. Then for any fuzzy subset* λ *of* $X, (\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

Proof. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\lambda) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the involutive property, $\mathfrak{C}(\mathfrak{C}\lambda) = \lambda$. This implies that $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\mathfrak{C}(\mathfrak{C}\lambda)) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$. Therefore $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$. □

The following example shows that the involutive property cannot be dropped from the hypothesis of Proposition 4.3 .

Example 4.4. *Let* $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_{.1}, b_{.2}, c_{.4}\}, \{a_{.2}, b_{.1}, c_{.7}\}, \{a_{.1}, b_{.1}, c_{.4}\}, \{a_{.2}, b_{.2}, c_{.7}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.1}, b_{.1}, c_{.2}\}, 1\}$. *Let* $\mathfrak{C}(x) = \sqrt{x^3}, 0 \leq x \leq 1$, *be a complement function. We note that this complement function does not satisfy monotonicity and involutive properties. The family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{0, \{a_{.2}, b_{.3}, c_{.5}\}, \{a_{.3}, b_{.2}, c_{.8}\}, \{a_{.2}, b_{.2}, c_{.5}\}, \{a_{.3}, b_{.3}, c_{.8}\}, 1\}$ *and* $\mathfrak{C}(\tau_2) = \{0, \{a_{.2}, b_{.2}, c_{.3}\}, 1\}$. *Let* $\lambda = \{a_{.2}, b_{.4}, c_{.5}\}$. *Then it can be calculated that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \tau_1 - cl_{\mathfrak{C}}\{1\} = 1$. *Now* $\mathfrak{C}\lambda = \{a_{.1}, b_{.3}, c_{.4}\}$. *Then it can be found that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1$. *We see that*

$$\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = 1 \;\; and \;\; \tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1 \tag{2}$$

By Definition 4.1 and Equation (2) we get $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda = 1$. *Also* $\mathfrak{C}\lambda = \{a_{.1}, b_{.3}, c_{.4}\}$. *Then it can be calculated that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1$. *Now* $\mathfrak{C}(\mathfrak{C}\lambda) = \{a_{.03}, b_{.2}, c_{.3}\}$. *Then it can be found that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.2}, b_{.2}, c_{.5}\}$. *We see that*

$$\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1 \;\; and \;\; \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\mathfrak{C}(\mathfrak{C}\lambda)) = \{a_{.2}, b_{.2}, c_{.5}\}. \tag{3}$$

By Definition 4.1 and Equation (3), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda) = \{a_{.2}, b_{.2}, c_{.5}\}$. *Therefore* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda \neq (\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda)$.

Proposition 4.5. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. If* λ *is fuzzy* $\mathfrak{C} - \tau_i$*-closed,* $i = 1, 2$ *then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \lambda$.

Proof. Let λ be a fuzzy $\mathfrak{C}-\tau_i$-closed, $i = 1, 2$. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\lambda) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Theorem 2.6 (2), $\tau_i - cl_{\mathfrak{C}}\lambda = \lambda$. Hence $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \tau_i - cl_{\mathfrak{C}}\lambda = \lambda$. □

Proposition 4.6. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. If* λ *is fuzzy* τ_i*-open,* $i = 1, 2$ *then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mathfrak{C}\lambda$.

Proof. Let λ be a fuzzy τ_i-open, $i = 1, 2$. Since $\mathfrak{C}$ satisfies the involutive property, $\mathfrak{C}(\mathfrak{C}\lambda)$ is fuzzy τ_i-open. Therefore $\mathfrak{C}\lambda$ is a fuzzy $\mathfrak{C}-\tau_i$-closed. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Proposition 4.5, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mathfrak{C}\lambda) \leq \mathfrak{C}\lambda$. Also by Proposition 4.3, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mathfrak{C}\lambda$. □

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonicity and involutive properties, then the conclusion of Proposition 4.6 is false .

Example 4.7. *Let* $X = \{a, b\}$, $\tau_1 = \{0, \{b_{.2}\}, \{b_{.15}\}, \{a_{.7}, b_{.15}\}, \{a_{.7}, b_{.2}\}, \{a_{.8}, b_{.8}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.8}, b_{.8}\}, 1\}$. *Let* $\mathfrak{C}(x) = \sqrt{x}$ *be a complement function that does not satisfy monotonicity and involutive properties. Then the family of all fuzzy* $\mathfrak{C}-\tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{0, \{b_{.04}\}, \{b_{.2}\}, \{a_{.49}, b_{.2}\}, \{a_{.49}, b_{.04}\}, \{a_{.64}, b_{.64}\}1\}$ *and* $\mathfrak{C}(\tau_2)\{0, \{a_{.64}, b_{.64}\}, 1\}$. *Let* $\lambda = \{a_{.8}, b_{.8}\}$ *then* $\mathfrak{C}\lambda = \{a_{.89}, b_{.89}\}$. *It can be evaluated that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \tau_1 - cl_{\mathfrak{C}}1 = 1$ *and* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \tau_1 - cl_{\mathfrak{C}}1 = 1$. *We see that*

$$\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = 1 \quad and \quad \tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1. \tag{4}$$

By Definition 4.1 and Equation (4), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda = 1$. *This shows that* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda \not\leq \mathfrak{C}\lambda$.

Proposition 4.8. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. If* $\lambda \leq \mu$ *and* μ *is fuzzy* $\mathfrak{C} - \tau_i$*-closed then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mu$.

Proof. Let $\lambda \leq \mu$ and μ be a fuzzy $\mathfrak{C} - \tau_i$-closed. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Theorem 2.6 (4), $\lambda \leq \mu$ implies that $\tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\lambda)) \leq \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mu))$. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\lambda) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\lambda)) \leq \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mu))$, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\mu) \wedge \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) \leq \tau_i - cl_{\mathfrak{C}}(\tau_j - cl_{\mathfrak{C}}\mu)$. Again by Theorem 2.6 (2), $\tau_i - cl_{\mathfrak{C}}\mu = \mu$. This implies that $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mu$. □

The following example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonicity and involutive properties, then the conclusion of Proposition 4.8 is false.

Example 4.9. *Let* $X = \{a, b\}$, $\tau_1 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.2}\}, \{a_{.6}, b_{.2}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$. *Let* $\mathfrak{C}(x) = \frac{x}{2-x}$, $0 \leq x \leq 1$ *be a complement function. We see that complement function* $\mathfrak{C}$ *does not satisfy the monotonicity and involutive properties. Then the family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are given by* $\mathfrak{C}(\tau_1) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.333}\}, \{a_{.8}, b_{.333}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$ *and* $\mathfrak{C}(\tau_2) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.461}\}, \{a_{.8}, b_{.461}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$. *Let* $\lambda = \{a_{.8}, b_{.76}\} \leq \{a_{.857}, b_{.76}\}$. *Then it can be found that* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \tau_1 - cl_{\mathfrak{C}}\{a_{.8}, b_{.8}\} = \{a_{.8}, b_{.8}\}$. *Now* $\mathfrak{C}\lambda = \{a_{.7}, b_{.612}\}$. *Then* $\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \tau_1 - cl_{\mathfrak{C}}\{a_{.8}, b_{.8}\} = \{a_{.8}, b_{.8}\}$. *We see that*

$$\tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}\lambda) = \{a_{.8}, b_{.8}\} \quad and \quad \tau_1 - cl_{\mathfrak{C}}(\tau_2 - cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}\}. \tag{5}$$

By Definition 4.1 and Equation (5), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda) = \{a_{.8}, b_{.8}\} \not\leq \{a_{.857}, b_{.76}\}$. *Therefore the conclusion of Proposition 4.8 is false.*

Proposition 4.10. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. If* $\lambda \leq \mu$ *and* μ *is fuzzy* τ_i*-open then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mathfrak{C}\mu$.

Proof. Let $\lambda \leq \mu$ and μ be a fuzzy τ_i-open. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, $\mathfrak{C}\mu \leq \mathfrak{C}\lambda$. By Theorem 2.6 and $\mathfrak{C}\mu \leq \mathfrak{C}\lambda$ implies that $\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\mu)) \leq \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Lemma 2.7, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) = \mathfrak{C}((\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda)) \vee \mathfrak{C}(\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\mu)) \leq \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)), \mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) \geq \mathfrak{C}((\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda)) \vee \mathfrak{C}(\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\mu))$. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Proposition 2.6 , $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) \geq (\tau_i - Int(\tau_j - Int(\mathfrak{C}\lambda)) \vee \tau_i - Int(\tau_j - Int(\mu)))$. Since μ is fuzzy τ_i- open, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) \geq \mu$. Since $\mathfrak{C}$ satisfies the monotonic properties, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda \leq \mathfrak{C}\mu$. □

The following example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonicity and involutive properties, then the conclusion of Proposition 4.10 is false .

Example 4.11. *Let* $X = \{a, b\}$, $\tau_1 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.2}\}, \{a_{.6}, b_{.2}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.6}, b_{.6}\}, \{a_{.75}, b_{.3}\}, \{a_{.6}, b_{.3}\}, \{a_{.75}, b_{.6}\}, \{a_{.75}, b_{.61}\}, 1\}$. *Let* $\mathfrak{C}(x) = \frac{x}{2-x}$, $0 \leq x \leq 1$, *be a complement function. We see that complement function* $\mathfrak{C}$ *does not satisfy the monotonicity and involutive properties. Then the family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are given by* $\mathfrak{C}(\tau_1) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.333}\}, \{a_{.8}, b_{.333}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$ *and* $\mathfrak{C}(\tau_2) = \{0, \{a_{.8}, b_{.8}\}, \{a_{.857}, b_{.461}\}, \{a_{.8}, b_{.461}\}, \{a_{.857}, b_{.8}\}, \{a_{.857}, b_{.76}\}, 1\}$. *Let* $\lambda = \{a_{.5}, b_{.5}\}$ *and* $\mu = \{a_{.6}, b_{.6}\}$. *Then it can be found that* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\lambda)) = \{a_{.8}, b_{.8}\}$. *Now* $\mathfrak{C}\lambda = \{a_{.333}, b_{.333}\}$. *Then* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}\}$. *We see that*

$$\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\lambda)) = \{a_{.8}, b_{.8}\} \text{ and } \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.8}, b_{.8}\}. \quad (6)$$

By Definition 4.1 and Equation (6), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda) = \{a_{.8}, b_{.8}\} \not\leq \mathfrak{C}\mu = \{a_{.429}, b_{.429}\}$.

Proposition 4.12. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Let* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then for any fuzzy subset* λ *of* X, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) = \tau_i - Int(\tau_j - Int(\lambda)) \vee \tau_i - Int(\tau_j - Int(\mathfrak{C}\lambda))$.

Proof. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Taking complement on both sides, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) = \mathfrak{C}((\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))))$. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Lemma 2.7, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) = \mathfrak{C}(\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda)) \vee \mathfrak{C}(\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)))$. Also by Proposition 2.5, $\mathfrak{C}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda) = (\tau_i - Int(\tau_j - Int(\mathfrak{C}\lambda)) \vee \tau_i - Int(\tau_j - Int(\lambda))$. □

The following example shows that if the monotonicity and involutive properties of the complement function $\mathfrak{C}$ can be dropped, then the conclusion of Proposition 4.12 is false.

Example 4.13. *Let* $X = \{a, b, c\}$, $\tau_1 = \{0, \{a_{.5}, b_{.7}, c_{.8}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.2}, b_{.3}, c_{.1}\}, \{a_{.1}, b_{.5}, c_{.6}\}, \{a_{.1}, b_{.3}, c_{.1}\}, \{a_{.2}, b_{.5}, c_{.6}\}, 1\}$. *Let* $\mathfrak{C}(x) = x^2, 0 \leq x \leq 1$, *be a complement function. We note that this complement function does not satisfy the monotonicity and involutive properties. Then the family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{0, \{a_{.7}, b_{.8}, c_{.9}\}, 1\}$ *and* $\mathfrak{C}(\tau_2) = \{0, \{a_{.4}, b_{.5}, c_{.3}\}, \{a_{.3}, b_{.7}, c_{.8}\}, \{a_{.3}, b_{.5}, c_{.3}\}, \{a_{.4}, b_{.7}, c_{.8}\}, 1\}$. *Let* $\lambda = \{a_{.4}, b_{.7}, c_{.75}\}$. *Then it can be evaluated that* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\lambda) = \{a_{.7}, b_{.8}, c_{.9}\}.Now\mathfrak{C}\lambda = \{a_{.16}, b_{.49}, c_{.56}\}$ *and* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.7}, b_{.8}, c_{.9}\}$. *We see that*

$$\tau_1 - cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\lambda) = \{a_{.7}, b_{.8}, c_{.9}\} \text{ and } \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.7}, b_{.8}, c_{.9}\}. \quad (7)$$

By Definition 4.1 and Equation (7) shows that, $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda = \{a_{.7}, b_{.8}, c_{.9}\}$. *Also* $\mathfrak{C}((\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda) = \{a_{.49}, b_{.64}, c_{.81}\}$. *Now* $\tau_1 - Int(\tau_2 - Int(\lambda)) = \tau_1 - Int\{a_{.2}, b_{.5}, c_{.6}\} = \{0\}$ *and* $\tau_1 - Int(\tau_2 - Int(\mathfrak{C}\lambda)) = \tau_1 - Int\{a_{.1}, b_{.3}, c_{.1}\} = \{0\}$. *Thus we see that* $\mathfrak{C}((\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda) \neq (\tau_1 - Int(\tau_2 - Int(\lambda)) \vee \tau_1 - Int(\tau_2 - Int(\mathfrak{C}\lambda)))$.

Proposition 4.14. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Let* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then for any fuzzy subset* λ *of* X, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = (\tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_i - Int(\tau_j - Int(\lambda)))$.

Proof. By Definition 4.1, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda))$. Since $\mathfrak{C}$ satisfies the monotonicity and involutive properties, by Proposition 2.5, $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda = \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}\lambda) \wedge \mathfrak{C}(\tau_i - Int(\tau_j - Int(\lambda)))$. □

The next example shows that if the complement function $\mathfrak{C}$ does not satisfies the monotonic and involutive properties, then the conclusion of Proposition 4.14 is false.

Example 4.15. *In Example 4.13, let* $\lambda = \{a_{.4}, b_{.2}, c_{.7}\}$. *Then it can be evaluated that* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\lambda) = \tau_1 - Cl_{\mathfrak{C}}\{a_{.4}, b_{.7}, c_{.8}\} = \{a_{.7}, b_{.8}, c_{.9}\}$. *Now* $\mathfrak{C}\lambda = \{a_{.16}, b_{.04}, c_{.49}\}$. *Then* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \tau_1 - Cl_{\mathfrak{C}}\{a_{.3}, b_{.7}, c_{.8}\} = \{a_{.7}, b_{.8}, c_{.9}\}$. *Then we can see that*

$$\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\lambda) = \{a_{.7}, b_{.8}, c_{.9}\} \text{ and } \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_{.7}, b_{.8}, c_{.9}\} \tag{8}$$

By Definition 4.1 and Equation (8) shows that, $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda = \{a_{.7}, b_{.8}, c_{.9}\}$. *Also* $\tau_1 - Int(\tau_2 - Int(\lambda)) = \tau_1 - Int\{0\} = \{0\}$. *Therefore* $\mathfrak{C}(\tau_1 - Int(\tau_2 - Int(\lambda))) = \mathfrak{C}\{0\} = \{0\}$. *This implies that* $\mathfrak{C}(\tau_1 - Int(\tau_2 - Int(\lambda))) = \{0\}$. *Thus we see that* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}\lambda \neq \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\lambda)) \wedge \mathfrak{C}(\tau_1 - Int(\tau_2 - Int(\lambda))) = \{0\}$.

Proposition 4.16. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Let* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then for any fuzzy subset* λ *of* $X, (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\tau_i - Int(\tau_j - Int(\lambda)) \leq (\tau_i, \tau_j) - Bd_{\mathfrak{C}}\lambda$.

The next example shows that if the complement function $\mathfrak{C}$ does not satisfies the involutive property, then the conclusion of Proposition 4.16 is false.

Example 4.17. *Let* $X = \{a, b\}$, $\tau_1 = \{0, \{a_{.98}, b_{.4}\}, \{a_{.5}, b_{.8}\}, \{a_{.98}, b_{.8}\}, \{a_{.5}, b_{.4}\}, 1\}$ *and* $\tau_2 = \{0, \{a_{.6}, b_{.9}\}, \{a_{.5}, b_{.4}\}, 1\}$. *Let* $\mathfrak{C}(x) = \frac{1-x^2}{1+x^2}$, $0 \leq x \leq 1$ *be a complement function. We see that complement function* $\mathfrak{C}$ *does not satisfy the involutive property. Then the family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{1, \{a_{.1}, b_{.65}\}, \{a_{.58}, b_{.33}\}, \{a_{.58}, b_{.65}\}, \{a_{.1}, b_{.33}\}, 0\}$ *and* $\mathfrak{C}(\tau_2) = \{1, \{a_{.5}, b_{.2}\}, \{a_{.1}, b_{.4}\}, 0\}$. *Let* $\lambda = \{a_{.5}, b_{.4}\}$. *Then the value of* $\tau_1 - Int(\tau_2 - Int(\lambda)) = \tau_1 - Int\{a_{.5}, b_{.4}\} = \{a_{.5}, b_{.4}\}$. *This implies* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\tau_1 - Int(\tau_2 - Int(\lambda)))) = \tau_1 - Cl_{\mathfrak{C}}1 = 1$. *Also* $\mathfrak{C}(\tau_1 - Int(\tau_2 - Int(\lambda))) = \mathfrak{C}\{a_{.5}, b_{.4}\} = \{a_{.6}, b_{.7}\}$. *Therefore* $\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}\mathfrak{C}(\tau_1 - Int(\tau_2 - Int(\lambda)))) = \tau_1 - Cl_{\mathfrak{C}}1 = 1$. *By Definition 4.1,*

$$(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\tau_1 - Int(\tau_2 - Int(\lambda))) = 1. \tag{9}$$

We see that

$$\tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\lambda)) = \{a_{.5}, b_{.4}\} and \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = 1. \tag{10}$$

By Definition 4.1 and Equation (10) shows that,

$$(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda) = \{a_{.5}, b_{.4}\}. \tag{11}$$

Hence from Equations (9), (10) and (11), $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\tau_1 - Int(\tau_2 - Int(\lambda))) \not\leq (\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda)$.

Proposition 4.18. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Let* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\lambda)) \leq (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda)$.

Proposition 4.19. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Let* $\mathfrak{C}$ *be a complement function that satisfies the monotonicity and involutive properties. Then* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda \vee \mu) \leq (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda) \vee (\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mu)$.

The following example shows that if the complement function $\mathfrak{C}$ does not satisfy the monotonicity and involutive properties, then the conclusion of Proposition 4.19 is false.

Example 4.20. *Let* $X = \{a, b, c\}$, $\tau_1 = \{0, \{c_{.8}\}, \{a_{.98}\}, \{a_{.98}, c_{.8}\}, 1\}$ *and* $\tau_2 = \{0, \{c_{.95}\}, \{a_{.97}\}, \{a_{.97}, c_{.95}\}, 1\}$. *Let* $\mathfrak{C}(x) = \frac{1-x^2}{1+x^2}$, $0 \leq x \leq 1$, *be a complement function. We see that it satisfies the monotonicity property but does not satisfies the involutive property. Then the family of all fuzzy* $\mathfrak{C} - \tau_i$*-closed sets are* $\mathfrak{C}(\tau_1) = \{1, \{a_1, b_1, c_{.3}\}, \{a_{.1}, b_1, c_1\}, \{a_{.1}, b_1, c_{.3}\}, 0\}$ *and* $\mathfrak{C}(\tau_2) = \{1, \{a_1, b_1, c_{.2}\}, \{a_{.12}, b_1, c_1\}, \{a_{.12}, b_{.1}, c_{.2}\}, 0\}$. *Let* $\lambda = \{a_{.13}\}$ *and* $\mu = \{c_{.2}\}$. *Then* $\mathfrak{C}\lambda = \{a_{.97}, b_1, c_1\}$ *and*

$\mathfrak{C}\mu = \{a_1, b_1, c_{.92}\}$. *Then it can be evaluated that* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda) = \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\lambda)) \wedge \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\mathfrak{C}\lambda)) = \{a_1, b_1, c_{.3}\} \wedge 1 = \{a_1, b_1, c_{.3}\}$. *Also* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\mu) = \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\{c_{.2}\})) \wedge \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\{a_1, b_1, c_{.9}\})) = \{a_1, b_1, c_{.3}\} \wedge 1 = \{a_1, b_1, c_{.3}\}$. *Now* $\lambda \vee \mu = \{a_{.13}, b_0, c_{.2}\}$. *Then* $\mathfrak{C}(\lambda \vee \mu) = \{a_{.97}, b_1, c_{.92}\}$. *Then it can be evaluated that* $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda \vee \mu) = \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\{a_{.13}, b_0, c_{.95}\}) \wedge \tau_1 - Cl_{\mathfrak{C}}(\tau_2 - Cl_{\mathfrak{C}}(\{a_{.97}, b_1, c_{.92}\})) = 1 \wedge 1 = 1$. *We see that*

$$(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda) = \{a_1, b_1, c_{.3}\} \quad and \quad (\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\mu) = \{a_1, b_1, c_{.3}\} \tag{12}$$

$$(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda \vee \mu) = 1 \tag{13}$$

Equations (12) and (13) shows that, $(\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda \vee \mu) \not\leq ((\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\lambda)) \vee ((\tau_1, \tau_2) - Bd_{\mathfrak{C}}(\mu))$.

Theorem 4.21. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space and the complement function* $\mathfrak{C}$ *satisfies the monotonicity and involutive properties. Then for any fuzzy subset* λ *and* μ *of X,* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda \wedge \mu) \leq ((\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\mu))) \vee ((\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\mu) \wedge \tau_i - Cl_{\mathfrak{C}}(\tau_j - Cl_{\mathfrak{C}}(\lambda)))$.

Theorem 4.22. *Let* (X, τ_1, τ_2) *be a fuzzy bitopological space. Suppose the complement function* $\mathfrak{C}$ *satisfies the monotonicity and involutive properties. Then for any fuzzy subset* λ *of X,* $(\tau_i, \tau_j) - Bd_{\mathfrak{C}}((\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda)) \leq ((\tau_i, \tau_j) - Bd_{\mathfrak{C}}(\lambda)$.

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