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BSM European Put Option Pricing Formula for ML-Payoff Function with Mellin Transform

S. J. Ghevariya*

Department of Mathematics, Sardar Patel University, Vallabh Vidya Nagar, Anand, Gujarat, India.

Abstract: This paper contributes to the valuation of BSM European put option pricing formula for Modified-Log (ML) payoff function using Mellin transform. Panini and Srivastav introduced valuation of option through Mellin transform. Mellin transform is related with extended form of well known Laplace and Fourier transforms.

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1. Introduction

In the last five decades, option pricing has become one of the major areas in financial study. The Black-Scholes-Merton (BSM) option pricing formula for plain vanilla payoff function has been used worldwide in financial markets. After that many BSM option pricing formulas have been derived for various payoff functions (see [2, 6]). This indicates the demand of accurate solutions of BSM option pricing formulas. There are various methods for valuing options; namely, analytic method, finite difference method, the fast Fourier transform, probabilistic approach, Mellin transform, etc. The BSM European option pricing formulas for the ML-payoff functions have been derived through analytic approach (see [1]). This is a realistic modification of Paul Wilmott's log payoff function (see [8]). In Section-2, the definition of Mellin transform and its useful properties have been discussed. In Section-3, the valuation of BSM European put option pricing formula for ML-payoff function on an asset paying no dividend has been derived using Mellin transform.

2. Mellin Transform

The Mellin transform is developed by Robert Hjalmar Mellin (1854-1933) for the study of the gamma function, hypergeometric functions, Dirichlet series, the Riemann zeta function and for the solution of partial differential equations. This section presents some fundamental properties of Mellin transform which will be highly used later.

Definition 2.1. *Let $\mathcal{L}(\mathbb{R}^+)$ be the collection of Lebesgue integrable functions on $\mathbb{R}^+$. Then Mellin transform is a mapping from $\mathcal{L}(\mathbb{R}^+)$ to $\mathbb{C}$, defined as*

$$M(f,\omega) = \tilde{f}(\omega) = \int_0^\infty f(x) x^{\omega-1} dx. \tag{1}$$

* *E-mail: sjghevaria@mail.com*

Thus Mellin transform is a complex valued function defined on a vertical strip in the ω-plane whose boundaries are determined by the asymptotic behavior of f whenever $x \to 0^+$ and $x \to \infty$. The conditions $f(x) = \mathcal{O}(x^{\alpha})$ as $x \to 0^+$ and $f(x) = \mathcal{O}(x^{\beta})$ as $x \to \infty$ where $\alpha > \beta$, guarantee the existence of $M(f,\omega)$ in the strip $(-\alpha, -\beta)$. The largest strip (a,b) in which the integral converges is called the fundamental strip. In particular, if $F(f,\omega)$ and $L(f,\omega)$ denote the two-sided Fourier and Laplace transform, respectively, then the relation between these transforms is $M(f(x),\omega) = F(f(e^x), -i\omega) = L(f(e^{-x}),\omega)$.

Definition 2.2. *Let f be integrable with fundamental strip (a,b). Let $c \in (a,b)$ and $f(c+it)$ is integrable. Then*

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega)x^{-\omega}d\omega = M^{-1}(\tilde{f}(\omega)) \tag{2}$$

holds almost everywhere. Further, if f is continuous, then the above equality holds everywhere on $\mathbb{R}^+$.

The following two properties for Mellin transform will be used in the next section.

(i) Scaling Property:

$$M(f(vx),\omega) = \int_0^{\infty} f(vx)x^{\omega-1}dx = v^{-\omega}f(\omega),\ v > 0. \tag{3}$$

(ii) Convolution Property: Let

$$(f * g)(x) = \int_0^{\infty} f\Big(\frac{x}{y}\Big)g(y)\frac{1}{y}dy. \tag{4}$$

Then

$$M((f*g)(x),\omega) = M(f(x),\omega)M(g(x),\omega). \tag{5}$$

3. BSM Formulas for ML-Payoff Functions

Paul Wilmott derived BSM formula for log payoff function (see [8]), while H. V. Dedania and S. J. Ghevariya derived it for modified log payoff function [1]. Both formulas were derived by analytic method. In this section, we derive it for ML-payoff function using Mellin transform.

Theorem 3.1. *The BSM European put option pricing formula for the ML-payoff function* $\max\{S\ln(\frac{K}{S}),0\}$ *is*

$$P(S,t) = S\left[\eta(d)\sigma\sqrt{T-t} - \left(\ln\Big(\frac{S}{K}\Big) + \big(r + \frac{1}{2}\sigma^2\big)(T-t)\right)N(-d)\right],$$

where

$$d = \frac{\ln(\frac{S}{K}) + (r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},\ \eta(d) = \frac{1}{\sqrt{2\pi}}\,e^{-\frac{d^2}{2}} \text{ and } N(x) = \int_{-\infty}^{x}\eta(x)dx.$$

Proof. The BSM partial differential equation along with the boundary conditions for a European put option $P(S,t)$ is as follow:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0 \tag{6}$$

with $P(0,t) = Ke^{-r(T-t)}$, $P(S,t) \to 0$ when $S \to \infty$ and $P(S,T) = \max\{S\ln(\frac{K}{S}),0\}$, where K is the striking price. Now by taking Mellin transform of the equation (6), we get

$$\frac{\partial \tilde{P}(\omega,t)}{\partial t} + \frac{\sigma^2}{2}(\omega^2 + \alpha_1\omega - (1-\alpha_1))\tilde{P}(\omega,t) = 0, \tag{7}$$

where $\alpha_1 = 1 - \frac{2r}{\sigma^2}$. Also at the time of expiration, consider

$$\tilde{P}(\omega,T) = \tilde{g}(\omega).$$

The solution of equation (7) is given by

$$\tilde{P}(\omega,t)=\tilde{g}(\omega)e^{\frac{1}{2}(\omega^2+\alpha_1\omega-(1-\alpha_1))\tau} \tag{8}$$

where $\tau=T-t$. The inverse Mellin transform of equation (8) is given by

$$\begin{aligned} P(S,t) &= g(S)M^{-1}\left(e^{\alpha(\omega+\beta)^2-\alpha\left(\frac{k_1+1}{2}\right)^2}\right) \\ &= e^{-\alpha\left(\frac{k_1+1}{2}\right)^2\tau}g(S)M^{-1}\left(e^{\alpha(\omega+\beta)^2}\right) \end{aligned} \tag{9}$$

where $k_1=\frac{2r}{\sigma^2}$, $\beta=\frac{1-k_1}{2}$ and $\alpha=\frac{\sigma^2}{2}\tau$. Applying convolution property of Mellin transform to equation (9), we get

$$P(S,t)=\frac{e^{-\alpha\left(\frac{k_1+1}{2}\right)^2}}{\sigma\sqrt{2\pi\tau}}\int_0^K\left(\frac{S}{u}\right)^\beta e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{u}\right)}{\sigma\sqrt{\tau}}\right)^2}g(u)\frac{1}{u}du. \tag{10}$$

Note that

$$g(S)=M^{-1}(\tilde{g}(\omega))=P(S,T)=\max\{S\ln(\frac{K}{S}),0\}. \tag{11}$$

Combining equations (10) and (11), we get

$$\begin{aligned} P(S,t) &= \frac{e^{-\alpha\left(\frac{k_1+1}{2}\right)^2}}{\sigma\sqrt{2\pi\tau}}\int_0^K\left(\frac{S}{u}\right)^\beta\ln\left(\frac{K}{u}\right)e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{u}\right)}{\sigma\sqrt{\tau}}\right)^2}du \\ &= \frac{S^\beta e^{-\alpha\left(\frac{k_1+1}{2}\right)^2}}{\sigma\sqrt{\tau}}\left[\ln(K)I_1-I_2\right], \end{aligned} \tag{12}$$

where

$$I_1=\frac{1}{\sqrt{2\pi}}\int_0^K\frac{1}{u^\beta}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{u}\right)}{\sigma\sqrt{\tau}}\right)^2}du$$

and

$$I_2=\frac{1}{\sqrt{2\pi}}\int_0^K\frac{\ln u}{u^\beta}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S}{u}\right)}{\sigma\sqrt{\tau}}\right)^2}du.$$

Now by taking $y=\frac{\ln\left(\frac{S}{u}\right)}{\sigma\sqrt{\tau}}$ in I_2, we have

$$I_2=\sigma\sqrt{\tau}\,S^{-\beta+1}e^{\alpha(\beta-1)^2}\left[\sigma\sqrt{\tau}J_1-\ln SJ_2\right], \tag{13}$$

where

$$J_1=\frac{1}{\sqrt{2\pi}}\int_\infty^{\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}}}ye^{-\frac{1}{2}(y-\sigma\sqrt{\tau}(\beta-1))^2}dy$$

and

$$J_2=\frac{1}{\sqrt{2\pi}}\int_\infty^{\frac{\ln\left(\frac{S}{K}\right)}{\sigma\sqrt{\tau}}}e^{-\frac{1}{2}(y-\sigma\sqrt{\tau}(\beta-1))^2}dy.$$

Taking $t=y-\sigma\sqrt{\tau}(\beta-1)$ and $d=\frac{\ln\left(\frac{S}{K}\right)-\sigma^2\tau(\beta-1)}{\sigma\sqrt{\tau}}=\frac{\ln(\frac{S}{K})+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ in J_1 and J_2, we have

$$J_1=-[\eta(d)+\sigma\sqrt{\tau}(\beta-1)N(-d)] \text{ and } J_2=-N(-d),$$

where

$$\eta(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \text{ and } N(x)=\int_{-\infty}^x\eta(x)dx.$$

Substituting values of J_1 and J_2 in equation (13), we get

$$I_2 = -\sigma\sqrt{\tau}S^{-\beta+1}e^{\alpha(\beta-1)^2}\left[\left(\sigma^2\tau(\beta-1)\right)N(-d) + \sigma\sqrt{\tau}\eta(d) - \ln(S)N(-d)\right].$$

Similarly,

$$I_1 = S^{-\beta+1}e^{\alpha(\beta-1)^2}\sigma\sqrt{\tau}N(-d).$$

Finally, substituting values of I_1 and I_2 in equation (12), we get

$$P(S,t) = S\left[\eta(d)\sigma\sqrt{T-t} - \left(\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right)N(-d)\right].$$

This completes the proof. □

4. Remarks

The plain vanilla option is the most used one in financial market. The Modified log option is very close to plain vanilla option (see [3]). The BSM European put option pricing formula for plain vanilla payoff function has been derived through Mellin transform (see [5]). The BSM European call option pricing formula can not be derived through Mellin transform because in this case $C(S,t) = \mathcal{O}(1)$ as $S \to 0^+$ and $C(S,t) = \mathcal{O}(S)$ as $S \to \infty$. Hence we can not find the strip in which integral converges.

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