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About Isolate Inclusive Sets in Graphs

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Abstract: In this paper we further study isolate domination in graphs. In particular we study the effect of removing an edge from the graph on the isolate domination number of the graph. We prove a necessary and sufficient condition under which the isolate domination number increases when an edge is removed from the graph. Further we also prove a necessary and sufficient condition under which the isoinc number increases when an edge is removed from the graph. We also consider the graphs for which isolate dominating number is equal to 1 or 2.

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Keywords: Isolate dominating set, minimal isolate dominating set, minimum isolate dominating set, isolate domination number, isolate inclusive set, private neighbourhood.

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1. Introduction

The concept of isolate dominating set was studied in [4]. We defined the concept of isolate inclusive set in [2]. We have considered Isoinc sets from different viewpoints. In particular we have considered the operation of removing a vertex from the graph and its effect on Isoinc number of the graph. In this paper, we consider the operation of removing an edge from the graph and observe the effect on Isoinc number of the graph. First we consider the operation of removing an edge from the graph and prove necessary and sufficient conditions under which the Isoinc number increases when this operation is perform. Similarly, we consider the operation of removing an edge from the graph and prove necessary and sufficient conditions under which the isolate domination number increases when this operation is perform. we also consider those graphs for which the isolate domination number is 1 or 2.

2. Preliminaries and Notations

If G is a graph then $V(G)$ denotes the vertex set of the graph G and $E(G)$ denotes the edge set of the graph G. If e is an edge of the graph G then $G-e$ is the subgraph of G obtain by removing from the graph G. We will consider only simple undirected graphs with finite vertex set.

3. Definitions and Examples

Definition 3.1 (Isolate Inclusive set [2])**.** *Let G be a graph and S be a nonempty subset of $V(G)$ then the S is said to be an isolate inclusive set if the $\langle S\rangle$ has an isolated vertex. An isolate inclusive set will be also called isoinc set. An isoinc with*

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maximum cardinality is called a maximum isoinc set and its cardinality is denoted as $\beta_{is}(G)$.

Definition 3.2 (Isolate Dominating Set [2])**.** *Let* G *be a graph and* $S \subset V(G)$ *then* S *is said to be an isolate dominating set if*

(1). S *is a dominating set and*

(2). $\langle S \rangle$ *contains an isolated vertex.*

An isolate dominating set with minimum cardinality is called a minimum isolate dominating set.

The cardinality of a minimum isolate dominating set is called the isolate domination number of the graph G and it is denoted as $\gamma_0(G)$. Obviously for any graph G, $\gamma(G) \leq \gamma_0(G)$, where $\gamma(G)$ denotes the domination number of the graph G.

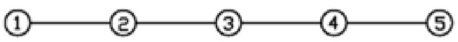
4. Main Result

Now we consider the operation of removing an edge of a graph and its effect on the Isoinc number of the graph. We begin with the following proposition.

Proposition 4.1. *Let* G *be a graph and* e *be an edge of* G *then* $\beta_{is}(G-e) \geq \beta_{is}(G)$.

Proof. Let S be a maximum isoinc sets of G. Then S is also isoinc set of $G-e$. Therefore $\beta_{is}(G-e) \geq |S| = \beta_{is}(G)$. Thus $\beta_{is}(G-e) \geq \beta_{is}(G)$. □

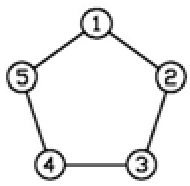
Example 4.2. *Consider the path graph* P_5 *with 5 vertices* $\{1,2,3,4,5\}$.



Here $\beta_{is}(G) = 4$. *Now consider the subgraph* $G-e$ *where* $e = \{45\}$.

Here $\beta_{is}(G-e) = 4$. *Therefore for this graph* $\beta_{is}(G-e) = \beta_{is}(G)$.

Example 4.3. *Consider the cycle graph* C_5 *with 5 vertices* $\{1,2,3,4,5\}$



Here $\beta_{is}(G) = 3$. *Now consider the subgraph* $G-e$ *where* $e = \{15\}$.

Here $\beta_{is}(G-e) = 4$. *Therefore for this graph* $\beta_{is}(G-e) > \beta_{is}(G)$.

Now we state and prove a necessary and sufficient condition under which Iosinc number of a graph increases when an edge is remove from the graph.

Theorem 4.4. *Let G be a graph and $e = \{uv\}$ be an edge of G then $\beta_{is}(G-e) > \beta_{is}(G)$ if and only if there is a subset S of $V(G)$ such that S has no isolated vertices, $|S| > \beta_{is}(G)$, $u,\ v \in S$ and atleast one of u and v is a pendent vertex in the $\langle S \rangle$.*

Proof. First suppose that $\beta_{is}(G-e) > \beta_{is}(G)$. Let S be any maximum Isoinc set of $G-e$.
Claim: $u \in S$ and $v \in S$. If $u \notin S$ or $v \notin S$ then S is a Isoinc set of G then $\beta_{is}(G) \geq |S| = \beta_{is}(G-e)$. Which contradict our assumption that $\beta_{is}(G-e) > \beta_{is}(G)$. Thus $u \in S$ and $v \in S$. Now consider the set S in the graph G. Since $|S| > \beta_{is}(G)$ is cannot be Isoinc set of G but S is an Isoinc set in $G-e$. Therefore u *or* v must be an isolated vertex in S when S is regard as a vertex set of $G-e$. Therefore u or v must be a pendent vertex in the $\langle S \rangle$ when S is regard as a set of vertices of G. Since S is not Isoinc set of G. The $\langle S \rangle$ does not have any isolated vertex.
Conversely, suppose condition is satisfied. Let S be a set of vertices of G such that $|S| > \beta_{is}(G)$, $\langle S \rangle$ has no isolated vertex and $u, v \in S$ and atleast one of u and v is a pendent vertex in the $\langle S \rangle$. Suppose u is a pendent vertex in the $\langle S \rangle$. Now consider S is in the graph $G-e$. Then obviously S is an Isoinc set in $G-e$. Then $\beta_{is}(G-e) \geq |S| > \beta_{is}(G)$. Thus $\beta_{is}(G-e) > \beta_{is}(G)$. □

Theorem 4.5. *Let G be a graph and $v \in V(G) \ni d(v) = d(G) \geq 1$. Then for any edge e whose end vertex is v, $\beta_{is}(G-e) > \beta_{is}(G)$.*

Proof. Let $M = V(G) - N(v)$ then M is a maximum Isoinc set of G. Now consider the graph $G^* = G - e$. Let N^* we denote the set of vertices adjacent to v in G^*. Let $M^* = V(G^*) - N^*(v)$ then M^* is a maximum Isoinc set of G^*. Obviously $|M^*| > |M|$ and Therefore $\beta_{is}(G^*) > \beta_{is}(G)$. □

Remark 4.6.

(1). Let G be a graph with an isolated vertex then $\beta_{is}(G) = |V(G)|$. If we remove any edge from this graph there will be isolated vertex in the new graph and therefore its Iosinc number will also be equal to $|V(G)|$. Thus removing any edge such a graph does not increase the Isoinc number of the graph.

(2). Suppose G is a graph without isolated vertices. Let v be a vertex of $G \ni d(v) = \delta(G)$. As proved above removal of any edge whose one end vertex is v increases the Isoinc number of the graph. Thus in a graph with $\delta(G) \geq 1$ there is always an edge $e \ni \beta_{is}(G-e) > \beta_{is}(G)$.

Thus we have following theorem.

Theorem 4.7. *Let G be a regular graph and e be any edge of G then $\beta_{is}(G-e) > \beta_{is}(G)$.*

Proof. Suppose G is a k_regular graph, $k \geq 1$. Let $e = \{uv\}$ is any edge of G. Now $d(v) = k = \delta(G)$. Therefore by the above remark $\beta_{is}(G-e) > \beta_{is}(G)$. □

Now we consider the operation of removing an edge of a graph on the isolate domination number of a graph.

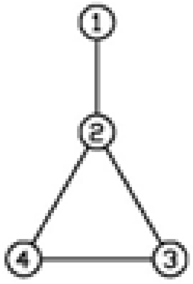
Remark 4.8. *Let G be a graph and $e = \{uv\}$ be any edge of G then any of the following three possibilities exists*

(1). $\gamma_0(G-e) = \gamma_0(G)$.

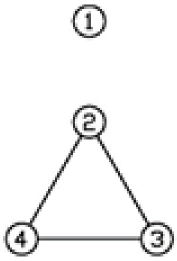
(2). $\gamma_0(G-e) < \gamma_0(G)$.

(3). $\gamma_0(G-e) > \gamma_0(G)$.

Example 4.9. *Let G be a graph with 4 vertices $\{1,2,3,4\}$.*

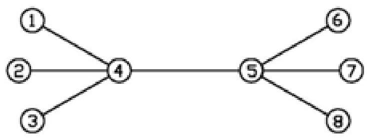


Here $\gamma_0(G) = 1$. Now consider the subgraph $G-e$ where $e = \{12\}$.

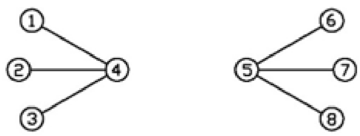


Here $\gamma_0(G-e) = 4$. Therefore for this graph $\gamma_0(G-e) > \gamma_0(G)$. In this graph consider the edge $e = \{34\}$ then $\gamma_0(G-e) = 1 = \gamma_0(G)$.

Example 4.10. *Let G be a graph with 8 vertices $\{1,2,3,4,5,6,7,8\}$.*



Here $\gamma_0(G) = 4$. Now consider the subgraph $G-e$ where $e = \{45\}$.



Here $\gamma_0(G-e) = 2$. Therefore for this graph $\gamma_0(G-e) < \gamma_0(G)$.

Now we state and prove a necessary and sufficient condition under which the isolate domination number of a graph increases when an edge is remove from the graph.

Theorem 4.11. *Let G be a graph with $\gamma_0(G) \geq 2$ and $e = \{uv\}$ be an edge of G then the following statement are equivalent.*

(1). $\gamma_0(G-e) > \gamma_0(G)$.

(2). There is a minimum isolate dominating set S of $G-e \ni u,v \in S$, v is an isolate in $S, P_{extn}[v,S]$ is empty and as has an isolate different from v.

(3). For every minimum isolate dominating set T of G, $u \in T, v \notin T$ and $v \in P_{extn}[u,T]$.

Proof. (1) $\Rightarrow$ (3) Let T be any minimum isolate dominating set of G. If $u,v \in T$ or $u,v \notin T$ then obviously T is an isolate dominating set in $G-e$. Therefore $\gamma_0(G-e) \leq |T| = \gamma_0(G)$. Which is a contradiction. Therefore $u \in T$ and $v \notin T$ or $v \in T$ and $u \notin T$. We may assume that $u \in T$ and $v \notin T$. Now $|T| = \gamma_0(G) < \gamma_0(G-e)$. Therefore T cannot be an isolate dominating set in $G-e$. Now e has isolated vertices then regarded as a set of vertices of $G-e$. Therefore T cannot be dominating set of $G-e$. Therefore v is not adjacent to any vertex of T in $G-v$ but v is adjacent to some vertex of T in G. Therefore $v \in P_n[u,T]$. Thus (1) $\Rightarrow$ (3) is proved.

(3) $\Rightarrow$ (2) Let T be any minimum isolate dominating set of G then $u \in T$, $v \notin T$ and $v \in P_n[u,\ T]$ in G. Obviously T cannot be an isolate dominating set in $G-e$. Let $S = T \cup \{v\}$. Then $v \in S$ and $u \in S$ and v is an isolate in S. Obviously, S is an isolate dominating set of $G-e$. If there is no vertex w outside of S which is adjacent to v then $P_{extn}[v,\ S] = \emptyset$. Suppose there is a vertex $w \ni w$ not in S, w is adjacent to v in $G-e$. Now $w \notin T$ and T is a dominating set of G. Therefore w is adjacent to some vertex z of T. Thus w is adjacent to two distinct vertices of S in $G-e$. Thus $P_{extn}[v,\ S] = \emptyset$. Note that an isolate of T in G is also an isolate of S in $G-e$ and it is different from v. Thus (3) $\Rightarrow$ (2) is proved.

(1) $\Rightarrow$ (2) Let S be minimum isolate dominating set of $G-e$ such that $u \in S$ and $v \in S$ and v is an isolate of S, $P_{extn}[v,S] = \emptyset$ and suppose S has an isolate different from v. Let $T = S - \{v\}$. Let z be any vertex of G which is not in T. If $z = v$ then z is adjacent to u in G. If $z \neq v$ then $z \notin S$. Suppose z is adjacent to v in $G-e$ then z must be adjacent to some other vertex v' of S because $z \notin P_{extn}[v,\ S]$. Then $v' \in T$ and z is adjacent to v' in G. Therefore T is a dominating set in G. Note that T has an isolate because S has an isolate different from v. Thus T is an isolate dominating set of G. Therefore $\gamma_0(G) \leq |T| < |S| = \gamma_0(G-e)$. Thus $\gamma_0(G-e) > \gamma_0(G)$. □

Theorem 4.12. *Let G be a graph with $\gamma_0(G) = 1$. Let $e = \{uv\}$ be any edge of G. Then $\gamma_0(G-e) > \gamma_0(G)$ if and only if the following condition is satisfy*

C: *If $\{z\}$ is an isolate dominating set of G then $z \in \{u,\ v\}$.*

Proof. Suppose condition is satisfy. If $\gamma_0(G-e) = \gamma_0(G)$ then $\gamma_0(G-e) = 1$. Suppose $\{z\}$ is a minimum isolate dominating set of $G-e$. Then $z \neq u$ because v is not adjacent to u in $G-e$. Similarly, if $z \neq v$. This contradicts condition C. Therefore $\gamma_0(G-e) > \gamma_0(G)$ ($\gamma_0(G-e) < \gamma_0(G)$ is not possible because $\gamma_0(G) = 1$).

Conversely, suppose $\gamma_0(G-e) > \gamma_0(G)$. Let z be a vertex of $G \ni z \notin \{u,v\}$. If $\{z\}$ is a minimum isolate dominating set of G then $\{z\}$ is also an isolate dominating set of $G-e$. This would implize that $\gamma_0(G-e) = \gamma_0(G)$. Which is a contradiction. Therefore $z \in \{u,\ v\}$. □

Theorem 4.13. *Let G be a graph with $\gamma_0(G) = 1$ and $e = \{uv\}$ be any edge of G. Suppose $\gamma_0(G-e) > \gamma_0(G)$ then*

(1). $\{u,\ v\}$ is a minimum isolate dominating set of $G-e$.

(2). $\gamma_0(G-e) = 2$.

Proof. By the above Theorem 4.6. Suppose $\{u\}$ is a minimum isolate dominating set of G. Now $\{u\}$ dominates all the vertices of $G-e$ except v. Let $S = \{u,\ v\}$ then obviously S is a minimum isolate dominating set of $G-e$ and $|S| = 2$. Thus (1) and (2) are proved. □

Corollary 4.14. *Let G be a graph with $\gamma_0(G) = 1$ and $e = uv$ be any edge of G then $\gamma_0(G-e) = \gamma_0(G)$ if and only if there is a vertex $z \ni z \notin \{u,\ v\}$ and $\{z\}$ is a minimum isolate dominating set of G.*

Remark 4.15. *If G is a graph with $\gamma_0(G) = 2$ then every minimum isolate dominating set is either an independent dominating set or a total dominating set and also $\gamma(G) = \gamma_0(G) = \gamma_t(G)$. Also note that if $\gamma(G) = 2$ because $\gamma_0(G) = 2$. Also note that if $\gamma_0(G) = 2$ then every minimum dominating set is either an independent dominating set or a total dominating set.*

Now we state and prove a necessary and sufficient condition under which isolate domination number of a graph G (with $\gamma_0(G) = 2$) increases when an edge is remove from the graph.

Theorem 4.16. *Let G be a graph with $\gamma_0(G) = 2$ and $e = uv$ be an edge of G. Then $\gamma_0(G-e) > \gamma_0(G)$ if and only if for every minimum dominating set S of G. The following condition holds:*

(1). If S is an independent dominating set then $u \in S$ and $v \notin S$ and $v \in P_{extn}[u,S]$ or $v \in S$ and $u \notin S$ and $u \in P_{extn}[v,S]$.

(2). If S is a total dominating set of G then $u \notin S$ or $v \notin S$.

Proof. Suppose $\gamma_0(G-e) > \gamma_0(G)$. Suppose S is a minimum dominating set.

(1). Suppose S is an independent set. If $u \in S$ and $v \in S$ then we have an obvious contradiction because S is an independent set. If $u \notin S$ and $v \notin S$ then S is an isolate dominating set in $G-e$. Then $\gamma_0(G-e) \leq |S| = \gamma_0(G)$. Which contradicts the hypothesis. Therefore $u \in S$ and $v \notin S$ or $v \in S$ and $u \notin S$. Suppose $u \in S$ and $v \notin S$. Now suppose $v \notin P_{extn}[u,S]$ then S is an isolate dominating set in $G-e$, which implies that $\gamma_0(G-e) \leq \gamma_0(G)$. Which is a contradiction. Therefore $v \in P_{extn}[u,S]$. Similarly, if $v \in S$ and $u \notin S$ then $u \in P_{extn}[v,S]$.

(2). Suppose S is a total dominating set. If $u \in S$ and $v \in S$ then S is an isolate dominating set in $G-e$. Therefore $\gamma_0(G-e) \leq |S| \leq \gamma_0(G)$. Which is a contradiction. Therefore $u \notin S$ or $v \notin S$.

Conversely, suppose $\gamma_0(G-e) > \gamma_0(G)$. Let $T \subset V(G)$ be such that $T \neq \emptyset$ and $|T| \leq \gamma_0(G)$. Suppose $|T| = 1$. Let $T = \{z\}$. Suppose T is an isolate dominating set in $G-e$ then $z \neq u$ and $z \neq v$. Then $\{z\}$ is an isolate dominating set in G which implies that $\gamma_0(G) = 1$. Which is not true. Therefore any set T with $|T| = 1$ can not be an isolate dominating set of $G-e$. Suppose $T \subset V(G)$ be such that $|T| = 2$ and T is an isolate dominating set of $G-e$. Let $u \in T$ and $v \in T$. Then T is a minimum dominating set of G. Which is a total dominating set and $u, v \in T$. This contradict (2).

Suppose $u \in T$ and $v \notin T$. Now v is an adjacent to some vertex x of T and of course $x \neq u$ because u and v are not adjacent in $G-e$. Now v is an adjacent two distinct vertex u and x of T in the graph G. Therefore $v \notin P_{extn}[u,\ S]$ in G. Similarly, If $v \in T$ and $u \notin T$ then $u \notin P_{extn}[v,\ T]$ in G. Thus, If $u \in T$ and $v \notin T$ or $v \in T$ and $u \notin T$ gives rise to a contradiction. Therefore $u \notin T$ and $v \notin T$. Therefore T is a minimum isolate dominating set of G such that $u \notin T$ and $v \notin T$. Which is contradict (1). Therefore any set T with $|T| = 2$ can not be an isolate dominating set of $G-e$. Thus we conclude that any set T of vertices of $G-e$ with $|T| \leq 2$ can not be an isolate dominating set of $G-e$. Therefore $\gamma_0(G-e) > 2 = \gamma_0(G)$. □

Corollary 4.17. *Let G be a graph with $\gamma_0(G) = 2$ and $e = \{uv\}$ be an edge of G. If $\gamma_0(G-e) > \gamma_0(G)$ then*

(1). $\gamma_0(G-e) = \gamma_0(G) + 1$.

(2). There is a minimum isolate dominating set T of $G - e \ni u,\ v \in T$.

Proof. Let$S$ be a minimum isolate dominating set of G. Since $\gamma_0\left(G-e\right) > \gamma_0\left(G\right)$, $u \in S$ and $v \notin S$ and $v \in P_{extn}\left[u,\ S\right]$ or $v \in S$ and $u \notin S$ and $u \in P_{extn}\left[v,\ S\right]$. Let $T = S \cup \{v\}$ if $v \notin S$ and $T = S \cup \{u\}$ if $u \notin S$. Then T is a minimum isolate dominating set of $G - e$ and $|T| = |S| + 1 = \gamma_0\left(G\right) + 1$. □

Proposition 4.18. *Let G be a graph with $\gamma_0\left(G\right) = 2$ and $e = \{uv\}$ be any edge of G. Then $\gamma_0\left(G-e\right) \geq \gamma_0\left(G\right)$.*

Proof. Suppose $\gamma_0\left(G-e\right) < \gamma_0\left(G\right)$. Then $\gamma_0\left(G-e\right) = 1$. Let$T = \{z\}$ be a minimum isolate dominating set of $G - e$ then $\{z\}$ is also an isolate dominating set of G. This implies that $\gamma_0\left(G\right) = 1$. Which is a contradiction. Therefore $\gamma_0\left(G-e\right) \geq \gamma_0\left(G\right)$. □

Theorem 4.19. *Let G be a graph with $\gamma_0\left(G\right) = 2$ and $e = \{uv\}$ be any edge of G. Then $\gamma_0\left(G-e\right) = \gamma_0\left(G\right)$ if there is a minimum dominating set T of $G \ni u,\ v \in T$.*

Proof. Suppose the condition is satisfy. Let T be a minimum isolate dominating set of $G \ni u,\ v \in T$. Then T is an isolate dominating set of $G - e$. Therefore $\gamma_0\left(G-e\right) \leq |T| = \gamma_0\left(G\right) \leq \gamma_0\left(G-e\right)$. Therefore $\gamma_0\left(G-e\right) = \gamma_0\left(G\right)$. □

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