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A Mixed Quadrature Rule of Precision 5

R. Chiranjeevi[1,*], CH. Mahesh[2], Uma Dixit[1] and S. Madhulatha[2]

1 Department of Mathematics, University Post Graduate College, Secundrabad, Osmania University, Hederabad, India.

2 Department of Mathematics, Osmania University, Hederabad, India.

Abstract: A mixed quadrature rule of higher precision for approximate evaluation of real definite integrals has been constructed using an anti Lobatto rule. The analytical convergence of the rule has been studied. The relative efficiencies of the mixed quadrature rule has been shown with the help of suitable test integrals. The error bound has been determined asymptotically.

MSC: 65D30, 65D32

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1. Introduction

The concept of mixed quadrature rule was first coined by R.N. Das and G. Pradhan [5]. The method of mixed quadrature rule is based on forming a mixed quadrature rule of higher precision by taking a linear / convex combination of two quadrature rule of lower precision. Though in literature we find precision enhancement through Richardson Extrapolation [3] and Kronord extension [3, 10, 11] taking respectively trapezoidal rule and Gaussian quadrature as base rules, these methods are quite cumbersome. On the other hand, the precision enhancement through mixed quadrature method is very simple and easy to handle. Many authors [5, 12–16] have developed mixed quadrature rules for numerical evaluation of real definite integrals. Authors [4, 6–9] have also developed mixed quadrature rules for approximate evaluation of the integrals of analytic functions following F. Lether [2]. So far this is one of the few papers in which an anti-Lobatto quadrature has been used to construct a mixed quadrature rule.

Dirk P. Laurie [1] is first to coin the idea of anti-Gaussian quadrature formula. An anti-Gaussian quadrature formula is an $(n+1)$ point formula of degree $(2n-1)$ which integrates all polynominals of degree upto $(2n+1)$ with an error equal in magnitude but opposite in sign to that of n-point Gaussian formula. If $H^{n+1} = \sum_{i=1}^{n+1} a_i f(x_i)$ be $(n+1)$ point anti-Gaussian formula and $G_n(p)$ be n point Gaussian formula, then by hypothesis $I(p) - H_{n+1}(p) = G_n(p) - I(p), p \in P_{2n+1}$ where p is a polynomial of degree less then or equal to $2n+1$.

In this paper, we incorporate the idea of anti-Gaussian quadrature rule to design an anti-Lobatto quadrature rule following LAURIE. We mix this anti-Lobatto rule with Fejers three point second rule to form a mixed quadrature rule. The relative efficiencies of the mixed rule has been shown by numerically evaluating some test integrals.

* *E-mail: rchiru2004@gmail.com*

2. Construction of Anti-Lobatto Four Point Rule from Lobatto Three Point Rule

We choose the Lobatto three point rule,

$$L_3(f) = \int_{-1}^{1} f(x)dx = \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right) \tag{1}$$

and develop a four point anti-Lobatto rule $L_4(f)$ from three point Lobatto rule $L_3(f)$. Using the principle $I(p) - H_{n+1}(p) = G_n(p) - I(p), p \in P_{2n+1}$ as adopted in Dirk P. Laurie [1], we obtain

$$L_4(f) = 2\int_{-1}^{1} f(x)dx - L_3(f) \tag{2}$$

$$2\int_{-1}^{1} f(x)dx = a_1 f(-1) + a_2 f(x_1) + a_3 f(x_2) + a_4 f(1) + L_3(f) \tag{3}$$

where $L_4(f) = a_1 f(-1) + a_2 f(x_1) + a_3 f(x_2) + a_4 f(1)$. In order to obtain the unknown weights and nodes, we assume that

(1). the rule is exact for all polynomials of degree 3.

(2). the rule integrates all polynomials of degree up to five with an error equal in magnitude and opposite in sign to that of Lobatto rule.

Thus we obtain following system of six equations having six unknowns namely a_i, and x_i for $f(x) = x^i, i = 0, ..., 5$.

$$a_1 + a_2 + a_3 + a_4 = 2 \tag{4}$$

$$-a_1 + a_2 x_1 + a_3 x_2 + a_4 = 0 \tag{5}$$

$$a_1 + a_2 x_1^2 + a_3 x_2^2 + a_4 = 0 \tag{6}$$

$$-a_1 + a_2 x_1^3 + a_3 x_2^3 + a_4 = \frac{2}{3} \tag{7}$$

$$a_1 + a_2 x_1^4 + a_3 x_2^4 + a_4 = 0 \tag{8}$$

$$and\ \ -a_1 + a_2 x_1^5 + a_3 x_2^5 + a_4 = \frac{2}{5} \tag{9}$$

Solving the above system of equation we get, $a_1 = a_4 = -1/9$, $a_2 = a_3 = 10/9$, $x_1 = \sqrt{2/5}$ and $x_2 = -\sqrt{2/5}$. Hence, the anti-Lobatto four point rule becomes,

$$L_4(f) = \frac{10}{9}\left(f\left(\sqrt{\frac{2}{5}}\right) + f\left(-\sqrt{\frac{2}{5}}\right)\right) - \frac{1}{9}\left(f\left(1\right) + f\left(-1\right)\right). \tag{10}$$

The error associated with the rule is computed as

$$EL_4(f) = \int_{-1}^{1} f(x) - L_4(f) = \frac{4f^{(4)}(0)}{5!.3} + \frac{8f^{(6)}(0)}{7!.3} + ... \tag{11}$$

3. Construction of Mixed Quadrature by Using Anti-Lobatto Four Point Rule with Fejer Three Point Second Rule

We have the anti-Lobatto four point rule,

$$L_4(f) = \frac{10}{9}\left(f\left(\sqrt{\frac{2}{5}}\right) + f\left(-\sqrt{\frac{2}{5}}\right)\right) - \frac{1}{9}\left(f\left(1\right) + f\left(-1\right)\right). \tag{12}$$

and Fejer three point second rule taken from [17]:

$$F_3(f) = \frac{2}{3}\left(f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{-1}{\sqrt{2}}\right) + f(0)\right). \tag{13}$$

Each of the rules $L_4(f)$ and $F_3(f)$ is of precision three. Let $EL_4(f)$ and $Ef_4(f)$ denote the errors in approximating the integrals $I(f)$ by the rules $L_4(f)$ and $F_3(f)$ respectively. Now

$$I(f) = L_4(f) + EL_4(f) \tag{14}$$

$$I(f) = F_3(f) + EF_3(f) \tag{15}$$

Using Maclaurines expansion of function in equation (12) and (13). we have,

$$EL_4(f) = \int_{-1}^{1} f(x) - L_4(f) = \frac{4f^{(4)}(0)}{5!.3} + \frac{8f^{(6)}(0)}{7!.3} + ... \tag{16}$$

$$Ef_3(f) = \int_{-1}^{1} f(x) - F_3(f) = \frac{4f(0)}{360} + \frac{8f^{(6)}(0)}{6048} + ... \tag{17}$$

Eliminating $f^{(4)}(0)$ from equation (16) and (17) we have

$$\begin{aligned} I_m(f) &= \frac{1}{3}\left(4F_3(f) - L_4(f) + 4Ef_3(f) - EL_4(f)\right) \\ &= \frac{8}{9}\left(f\left(\frac{1}{\sqrt{2}}\right) + f\left(\frac{-1}{\sqrt{2}}\right) + f(0)\right) - \frac{10}{9}\left(f\left(\sqrt{\frac{2}{5}}\right) + f\left(-\sqrt{\frac{2}{5}}\right)\right) + \frac{1}{9}\left(f\left(1\right) + f\left(-1\right)\right). \end{aligned} \tag{18}$$

This is the desired mixed quadrature rule of precision five. The truncation error generated in this approximation is given by

$$E(f) = \frac{1}{3}\left(4Ef_3(f) - EL_4(f)\right) = \frac{1}{22680} f^{(6)}(0) + ... \tag{19}$$

or

$$|E(f)| = \frac{1}{22680} f^{(6)}(\xi), \quad \xi \in (-1, 1). \tag{20}$$

4. Error Analysis

An asymptotic error estimate and an error bound of the rule (20) are as under.

Theorem 4.1. *Let $f(x)$ be sufficiently differentiable function in the closed interval $[-1, 1]$. Then the error $E(f)$ associated with the rule I_m is given by*

$$|E(f)| = \frac{1}{22680} f^{(6)}(\xi), \quad \xi \in (-1, 1). \tag{21}$$

Proof. The theorem follows from (20) and (21) we have

$$I_m(f) = \frac{1}{3}\left(4F_3(f) - L_4(f)\right)$$

And the truncation error generated in this approximation is given by,

$$E(f) = \frac{1}{3}\left(4Ef_3(f) - EL_4(f)\right)$$

Hence we have,

$$|E(f)| = \frac{1}{22680} f^{(6)}(\xi), \quad \xi \in (-1, 1).$$

□

Theorem 4.2. *The bound of the truncation error*

$$L_4(f) = 2\int_{-1}^{1} f(x)dx - L_3(f)$$

is given by

$$|E(f)| = \frac{M}{270}\left|\theta_1 - \theta_2\right|, \quad \theta_1, \theta_2 \in (-1, 1). \tag{22}$$

where $M = \max_{-1\leq x\leq 1}\left|f^{(5)}(x)\right|$.

Proof. We have

$$EL_3(f) = \frac{1}{90}f^{(4)}(\theta_1)$$

and

$$Ef_4(f) = \frac{1}{360}f^{(4)}(\theta_2)$$

Therefore

$$\begin{aligned} E(f) &= \frac{1}{3}\left(4Ef_3(f) - EL_4(f)\right) \\ &= \frac{1}{3}\left(\frac{4}{360}f^{(4)}(\theta_2 - \frac{1}{90}f^{(4)}(\theta_1)\right) \\ |E(f)| &\leq \frac{1}{270}\left|f^{(4)}(\theta_1) - f^{(4)}(\theta_2)\right| \\ &= \frac{1}{270}\int_{\theta_2}^{\theta_1} f^{(4)}(x)dx \\ &= \frac{M}{270}\left|\theta_1 - \theta_2\right| \end{aligned}$$

where $M = \max_{-1\leq x\leq 1}\left|f^{(5)}(x)\right|$. Which gives a theoretical error bound as θ_1, θ_2 are unknown points in $[-1, 1]$. From this theorem it is clear that the error in approximation will be less if points θ_1, θ_2 are closer to each other. □

5. Numerical Verification Table and Graphs

We applied these methods (i.e $L_3(f), L_4(f)$, $I_m(f)$ and $F_3(f)$) to the following problems then we get table 1

$$\begin{aligned} I_1 &= \int_{-1}^{1} e^x dx \\ I_2 &= \int_{0}^{1} e^{-x^2} dx \\ I_3 &= \int_{0}^{1} e^{x^2} dx \\ I_4 &= \int_{1}^{3} \frac{sin^2 x}{x} dx \\ I_1 &= \int_{0}^{1} \sqrt{x} dx \\ I_1 &= \int_{0}^{1} \sqrt{x} sin(x) dx \\ I_1 &= \int_{0}^{1} sin\sqrt{\pi x} dx \end{aligned}$$

etc. respectively.

Integral	Exact value	$L_3(f)$	$L_4(f)$	$F_3(f)$	$I_m(f)$
I_1	2.35040238	2.3620537	2.35811374	2.3474557	2.35034983
I_2	0.746825	0.74718	0.747054	0.7467297	0.7468162
I_3	1.4627	1.4757305	1.471156	1.459261	1.4624597
I_4	0.794825	0.789451	0.7911007	0.7960751	0.7946843
I_5	2/3	0.6380711	0.6598341	0.6712232	0.6667902
I_6	0.364221	0.3662485	0.36523635	0.3637008	0.364116
I_7	0.849726	0.6798206	0.674672	0.67728655	0.67635138

6. Conclusion

After observation one can smartly draw a conclusion over the efficiency of the rule formed in this paper as follows.

(1). Mixed rule $I_m(f)$ is more efficient than its constituent rules $L_3(f)$, $L_4(f)$ and $F_3(f)$.

(2). In I_1, Rule $L_3(f)$ gives exact value up to one decimal places , $L_4(f)$ and $F_3(f)$ are gives exact value up to two decimal places. But in our result Mixed rule $I_m(f)$ gives exact value up to four decimal places. Which means Mixed rule $I_m(f)$ is more efficient than its constituent rules $L_3(f), L_4(f)$ and $F_3(f)$.

(3). In I_2, The constituent rules $L_3(f)$ and $L_4(f)$ gives the more than (negative values) to exact values and In I_7,the constituent rules $L_3(f), L_4(f)$ and $F_3(f)$ gives less than (positive values) to exact values.

(4). In the same manner In I_2, I_3, I_4, I_5, I_6 and I_7, The Mixed rule $I_m(f)$ gives the best exact value (more efficient) than its constituent rules $L_3(f), L_4(f)$ and $F_3(f)$.

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