Edge-Odd Graceful for Cartesian Product of a Wheel With n Vertices and a Path with Two Vertices

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Abstract: Abhyankar [1] investigated direct methods of gracefully labeling graphs. Bahl [2] got gracefulness labeling for few families of spiders in few terms of merging such graphs. Barrientos [3] obtained graceful labelings for chain graphs. Edwards and Howard [4] analyzed a survey of some classes of graceful graphs. Kaneria [6] obtained graceful labeling for graphs related to cycle. Kaneria [7] made new graceful graphs by merging stars. Kaneria [8] received graceful labeling by attaching cycle to cycles and cycle with a complete bipartite graph. Mishra and Panigrahi [10] investigated new classes of graceful lobsters obtained from diameter four trees. Ramachandran and Sekar [11] got graceful labelling of super subdivision of ladder. A(p, q) connected graph is edge-odd graceful graph if there exists an injective map \( f : E(G) \rightarrow \{1, 3, \ldots , 2q - 1\} \) so that the induced map \( f_+ : V(G) \rightarrow \{0, 1, 2, \ldots , (2k - 1)\} \) defined by \( f_+(x) \equiv \sum f(xy) \pmod{2k} \), where the vertex \( x \) is incident with other vertex \( y \) and \( k = \max\{p, q\} \) makes all the edges distinct and odd. In this article, the edge-odd gracefulness of cartesian product of \( P_2 \) and \( W_n \) is obtained.

Keywords: Graceful graphs, edge-odd graceful labeling, edge-odd graceful graph.

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1. Introduction


2. Edge-Odd Graceful Labeling of Cartesian Product of \( P_2 \) and \( W_n \)

The following definitions are first given.

Definition 2.1 (Graceful graph). A function \( f \) of a graph \( G \) is called a graceful labeling with \( m \) edges, if \( f \) is an injection from the vertex set of \( G \) to the set \( \{0, 1, 2, \ldots , m\} \) such that when each edge uv is assigned the label \( |f(u) - f(v)| \) and the resulting edge labels are distinct. Then the graph \( G \) is graceful.

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Definition 2.2. Edge-odd graceful graph: A graph \( G(p, q) \) connected graph is edge-odd graceful graph if there exists an injective map \( f : E(G) \to \{1, 3, \ldots, 2q-1\} \) so that induced map \( f^+ : V(G) \to \{0, 1, 2, \ldots, (2k-1)\} \) defined by \( f_+(x) \equiv \sum f(x, y) \mod 2k \), where the vertex \( x \) is incident with other vertex \( y \) and \( k = \max\{p, q\} \) makes all the edges distinct and odd. Hence the graph \( G \) is edge-odd graceful.

Theorem 2.3. The cartesian product of \( P_2 \) and \( W_n \) is edge-odd graceful for any positive even integer \( n \).

Proof. The cartesian product of a path \( P_2 \) with 2 vertices and a wheel \( W_n \) with \( n \) vertices is given as follows. The arbitrary labelings for vertices and edges for \( P_2 \times W_n \) are mentioned below.

Let \( n \) be an even positive integer. To find edge-odd graceful, define \( f : E(P_2 \times W_n) \to \{1, 3, \ldots, 2q-1\} \) by

Case (1): For \( n \equiv 0 \mod 8 \)

\[
\begin{align*}
  f(e_1) &= 7, \quad f(e_4) = 1, \\
  f(e_i) &= 2i - 1, \quad i = 2, 3, \ldots, 5, 6, (n - 1), (n + 2), (n + 3), \ldots, (5n - 4) \\
  f(e_n) &= 2n + 1, \quad f(e_{n+1}) = 2n - 1, \\
\end{align*}
\]

Rule (1)

Case (2): For \( n \equiv 2 \mod 8 \)

\[
\begin{align*}
  f(e_1) &= 2i - 1, \quad i = 1, 2, 3, \ldots, (2n - 2), (2n), (2n + 1), \ldots, (3n - 2) \\
  f(e_{3n-1}) &= f(e_{3n-2}) + 4(n - 1) \\
  f(e_{3n}) &= f(e_{3n-2}) + 2(n - 1) \\
  f(e_{3n-1+i}) &= f(e_{3n-1}) - i, \quad i = 2, 4, 6, \ldots, (2n - 6) \\
  f(e_{3n+i}) &= f(e_{3n}) - i, \quad i = 2, 4, 6, \ldots, (2n - 4) \\
  f(e_{3n-4}) &= f(e_{2n-2}) + 2; \quad f(e_{2n-1}) = f(e_{5n-6}) - 2. \\
\end{align*}
\]

Rule (2)

Case (3): For \( n \equiv 4 \mod 8 \)

\[
\begin{align*}
  f(e_1) &= 5, \quad f(e_2) = 3, \quad f(e_3) = 1 \\
  f(e_i) &= 2i - 1, \quad i = 4, 5, \ldots, n, (n + 2), \ldots, (2n - 3), (2n - 1), (2n + 1), \ldots, (4n - 4), (4n - 2), \ldots, (5n - 3), (5n - 5) \\
  f(e_{n+1}) &= 4n - 5, \quad f(e_{5n-4}) = 8n - 7 \\
  f(e_{2n-2}) &= 2n + 1, \quad f(e_{4n-3}) = 10n - 11. \\
\end{align*}
\]

Rule (3)

For \( n \equiv 6 \mod 8 \), the arbitrary labeling for vertices and edges for \( P_2 \times W_n \) are mentioned below.
Case (5): For \( n \equiv 6 \pmod{8} \)

\[
\begin{align*}
  f(e_i) &= 2i - 1, \ i = 1, 3, 4, 5, 6, \ldots, (2n - 3), (2n - 1), (2n), \ldots, (5n - 4) \\
  f(e_2) &= 4n - 5, \ f(e_{2n-2}) = 3
\end{align*}
\]

Rule (4)

Define \( f^+ : V(G) \to \{0,1,2,\ldots,(2k-1)\} \) by \( f^+(v) = \sum f(uv) \pmod{2q} \), where this sum run over all edges through \( v \) \([\text{Rule (9)}]\). Hence the induced map \( f^+ \) provides the distinct labels for vertices and also the edge labeling is distinct.

Hence the cartesian product graph \( P_2 \times W_n \) is edge-odd graceful.

\[\square\]

**Theorem 2.4.** The cartesian product of \( P_2 \) and \( W_n \) is edge-odd graceful for any positive odd integer \( n \).

**Proof.** The cartesian product of a path \( P_2 \) with 2 vertices and a wheel \( W_n \) with \( n \) vertices is given in figure 1 in which the arbitrary labeling for vertices and edges for \( P_2 \times W_n \) are mentioned. Let \( n \) be odd positive integer.

Case (6): For \( n \equiv 3 \pmod{8} \)

\[
\begin{align*}
  f(e_1) &= 3, \ f(e_2) = 1, \\
  f(e_i) &= 2i - 1, \ i = 34, \ldots, (n-1), (2n-1), (2n), \ldots, (5n-4) \\
  f(e_{2n-2-i}) &= f(e_{n-i}) + 2i + 2, \ i = 1, 2, 3, \ldots, (n-4) \\
  f(e_n) &= f(e_{n+2}) + 2, \ f(e_{n+3}) = f(e_{n-1}) + 2, \ f(e_{2n-2}) = f(e_n) + 2
\end{align*}
\]

Rule (6)

Case (7): For \( n \equiv 5 \pmod{8} \)

\[
\begin{align*}
  f(e_1) &= 7, \ f(e_4) = 1 \\
  f(e_i) &= 2i - 1, \ i = 2, 3, 5, 6, \ldots, (5n-4)
\end{align*}
\]

Rule (7)

Case (8): For \( n \equiv 7 \pmod{8} \)

\[
\begin{align*}
  f(e_i) &= 2i - 1, \ i = 1, 2, 34, \ldots, (n-1), (2n), (2n+1), \ldots, (5n-5) \\
  f(e_{2n-1-i}) &= f(e_{n-i}) + 2i, \ i = 1, 2, 3, \ldots, (n-3) \\
  f(e_{2n-1}) &= 10n - 9, \ f(e_{5n-4}) = 10n - 9, \ f(e_{n+1}) = f(e_{n+2}) + 4 \\
  f(e_n) &= f(e_{n+2}) + 2, \ f(e_{n+4}) = 4n - 3
\end{align*}
\]

Rule (8)

Define \( f^+ : V(G) \to \{0,1,2,\ldots,(2q-1)\} \) by \( f^+(v) = \sum f(uv) \pmod{2q} \), where this sum run over all edges through \( v \) \([\text{Rule (9)}]\). Hence the induced map \( f^+ \) provides the distinct labels for vertices and also the edge labeling is distinct.

Hence the cartesian product graph \( P_2 \times W_n \) is edge-odd graceful.

\[\square\]
Example 2.5. The cartesian product graph $P_2 \times W_8$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_8$ is a connected graph with 16 vertices and 36 edges, where $n \equiv 0 \pmod{8}$. Due to the rules (1) & (9) in Theorem 2.1, edge-odd graceful labeling of the required graph is obtained as follows.

![Figure 3. Edge-odd graceful graph $P_2 \times W_8$](image)

Example 2.6. The cartesian product graph $P_2 \times W_{10}$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_{10}$ is a connected graph with 20 vertices and 46 edges, where $n \equiv 2 \pmod{8}$. Due to the rules (2) & (9) in Theorem 2.1, edge-odd graceful labelings of the required graph is obtained as follows.

![Figure 4. Edge-odd graceful graph $P_2 \times W_{10}$](image)

Example 2.7. The cartesian product graph $P_2 \times W_{12}$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_{12}$ is a connected graph with 24 vertices and 56 edges, where $n \equiv 4 \pmod{8}$. Due to the rules (3) & (9) in Theorem 2.1, edge-odd graceful labelings of the required graph is obtained as follows.

![Figure 5. Edge-odd graceful graph $P_2 \times W_{12}$](image)

Example 2.8. The cartesian product graph $P_2 \times W_9$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_9$ is a connected graph with 18 vertices and 41 edges, where $n \equiv 1 \pmod{8}$. Due to the rules (5) & (9) in Theorem 2.2, edge-odd graceful labelings of the required graph is obtained as follows.
Example 2.9. The cartesian product graph $P_2 \times W_{11}$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_{11}$ is a connected graph with 22 vertices and 51 edges, where $n \equiv 3 \pmod{8}$. Due to the rules (6) & (9) in Theorem 2.2, edge-odd graceful labeling of the required graph is obtained as follows.

Example 2.10. The cartesian product graph $P_2 \times W_{5}$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_{5}$ is a connected graph with 10 vertices and 21 edges, where $n \equiv 5 \pmod{8}$. Due to the rules (7) & (9) in Theorem 2.2, edge-odd graceful labeling of the required graph is obtained as follows.

Example 2.11. The cartesian product graph $P_2 \times W_{7}$ is edge-odd graceful.

Proof. The cartesian product graph $P_2 \times W_{7}$ is a connected graph with 14 vertices and 31 edges, where $n \equiv 5 \pmod{8}$. Due to the rules (7) & (9) in Theorem 2.2, edge-odd graceful labeling of the required graph is obtained as follows.
Example 2.12. The cartesian product graph $P_2 \times W_6$ is edge-odd graceful.

**Proof.** The cartesian product graph $P_2 \times W_6$ is a connected graph with 12 vertices and 26 edges, where $n \equiv 6 \pmod{8}$. Due to the rules (4) & (9) in Theorem 2.2, edge-odd graceful labeling of the required graph is obtained as follows.

References


