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On Geometry of Pseudo-Slant Submanifolds of Conformal Sasakian Manifolds

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Abstract: In this paper, we study the geometry of pseudo-slant submanifolds of conformal Sasakian manifolds. We give some results on submanifolds of conformal Sasakian manifolds with parallel canonical structures. Finally we discuss integrability conditions of anti-invariant and slant distributions of pseudo-slant submanifolds of conformal Sasakian manifolds.

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1. Introduction

The study on locally conformal Kaehler manifold was initiated by Libermann [14]. Later Visman [17] proved the necessary and sufficient condition for a locally conformal Kaehler manifold to be a Kaehlar. In search of relationship between Kaehler and Sasakian manifolds, Abedi [1] introduced conformal Sasakian manifolds.

In this connection, we study pseudo-slant submanifolds of conformal Sasakian manifolds. Later Abedi studied different submanifolds of conformal manifolds [2, 3]. The study of slant submanifolds has played an important role in the study of spaces. The study was initiated by Chen ([8, 9]) on complex manifolds. As slant submanifolds are the generalization of invariant and anti-invariant submanifolds, many geometers has shown interest on this study. Lotta [15] introduced the concept of slant immersions in to an almost contact metric manifold. Carriazo introduced another new class of submanifolds called hemi-slant submanifolds (it is also called as anti-slant or pseudo-slant submanifold) [7]. Later many geometers (See [10–13]) studied pseudo-slant submanifolds on various manifolds.

The present paper is organized as follows: Section 2 deals with the basic definitions and results on conformal Sasakian manifolds and submanifolds. We give definition of pseudo-slant submanifolds and some results on parallelism of the canonical structures of the submanifolds of conformal Sasakian manifolds in Section 3. And the last section is devoted to integrability of the anti-invariant and slant distributions of the pseudo-slant submanifolds of the conformal Sasakian manifolds.

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2. Preliminaries

Let $(2m+1)$-dimensional differentiable manifold $\widetilde{M}$ is said to be contact manifold if a global 1-form η satisfies $\eta \wedge (d\eta)^m \neq 0$ everywhere on $\widetilde{M}$. If $\widetilde{M}$ with an almost contact metric structure (ϕ, ξ, η, g) satisfies ([4, 5]):

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{1}$$

$$\phi\xi = 0, \ \eta \circ \phi = 0, \tag{2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X), \tag{4}$$

for any vector fields on $\widetilde{M}$. Then $\widetilde{M}$ is called almost contact metric manifold. Where ϕ is a $(1,1)$-tensor field, ξ is a characteristic vector field and g is a Riemannian metric. Let Φ be the fundamental 2-form on $\widetilde{M}$ defined by $\Phi(X,Y) = g(X, \phi Y) = -\Phi(Y, X)$. Now if $\Phi = d\eta$ then almost contact metric structure becomes contact metric structure. Further an almost contact metric manifold $\widetilde{M}$ is said to be Sasakian if

$$(\widetilde{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \ \ \widetilde{\nabla}_X \xi = -\phi X.$$

Let $\widetilde{M}$ be a smooth manifold of dimension $(2m+1)$, $(\widetilde{M}, \phi, \xi, \eta, g)$ is called a conformal Sasakian manifold if [1]

$$\widetilde{g} = \exp(f)g, \ \widetilde{\phi} = \phi, \ \widetilde{\eta} = (\exp(f))^{1/2}\eta, \ \widetilde{\xi} = (\exp(-f))^{1/2}\xi.$$

Let $\widetilde{\nabla}$ and $\overline{\nabla}$ are the connections of $\widetilde{M}$ with respect to Riemannian metric $\widetilde{g}$ and g respectively, and are related by

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \frac{1}{2}\left\{\omega(X)Y + \omega(Y)X - g(X,Y)\zeta\right\}, \tag{5}$$

where ω is global 1-form defined by $\omega(X) = X(f)$ and ζ is Lee vector field metrically equivalent to ω i.e., $g(\zeta, X) = \omega(X)$. Further for a conformal Sasakian manifold we have

$$(\overline{\nabla}_X \phi)Y = (\exp(f))^{1/2}\left\{g(X,Y)\xi - \eta(Y)X\right\} - \frac{1}{2}\left\{\omega(\phi Y)X - \omega(Y)\phi X + g(X,Y)\phi\zeta - g(X,\phi Y)\zeta\right\}, \tag{6}$$

$$\overline{\nabla}_X \xi = (\exp(f))^{1/2}\phi X + \frac{1}{2}\left\{\eta(X)\zeta - \omega(\xi)X\right\}. \tag{7}$$

Submanifold: Let M be a submanifold of a Riemannian manifold $\widetilde{M}$ with Riemannian metric g. Then for all $X, Y \in TM$ and $V \in T^{\perp}M$ the Gauss and Weingarten formulas with respect to $\overline{\nabla}$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{8}$$

$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{9}$$

where ∇(respectively $\nabla^{\perp}$) is the induced Riemannian connection(respectively normal connection) in TM (respectively $T^{\perp}M$) with respect to $\overline{\nabla}$, A and h are the shape operator and second fundamental form related by

$$g(h(X,Y), V) = g(A_V X, Y). \tag{10}$$

A submanifold M is said to be *totally umbilical* if $h(X,Y) = g(X,Y)H$, where H is the mean curvature of M in $\widetilde{M}$. If $h = 0$ (equivalently $A_V = 0$) then M is called *totally geodesic*. Let for any $X \in TM$ and $V \in T^{\perp}M$ we can write

$$\phi X = TX + NX, \tag{11}$$

$$\phi V = tV + nV, \tag{12}$$

where TX and NX (respectively tV and nV) are the tangential and normal component of ϕX (respectively ϕV). Using (1) in the above equations one can get

$$T^2 = -tN - I + \eta \circ \xi, \quad NT + nN = 0, \tag{13}$$

$$n^2 = -I - Nt, \quad Tt + tn = 0. \tag{14}$$

3. Pseudo-slant Submanifold of Conformal Sasakian Manifold

Now let us recall some definitions of classes of submanifolds. Let M be a submanifold, then M is said to be

1. Invariant submanifold if N is identically zero in (11), i.e., $\phi X \in TM$, $\forall X \in TM$.
2. Anti-invariant submanifold if T is identically zero in (11), i.e., $\phi X \in T^{\perp}M$, $\forall X \in TM$.
3. Slant submanifold if there exists an angle $\theta(x) \in [0, \pi/2]$ between ϕX and T_xM for all non-zero vector X tangent to M at x called slant angle which is constant.
4. Pseudo-slant submanifold if there exists distributions D_θ and $D^{\perp}$ such that (i) TM admits orthogonal direct composition $TM = D_\theta \oplus D^{\perp} \oplus <\xi>$, (ii) D_θ is a slant distribution with slant angle $\theta \neq \pi/2$ and (iii) $D^{\perp}$ is an anti-invariant distribution [12].

From the above definitions we can note that slant submanifold is the generalization of invariant (if $\theta = 0$) and anti-invariant (if $\theta = \pi/2$) submanifolds. A proper slant submanifold is neither invariant nor anti-invariant submanifold i.e., $\theta \in (0, \pi/2)$. Hence in general we have the following theorem which characterize slant submanifolds of almost contact metric manifolds;

Theorem 3.1 ([6]). *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ such that $\xi \in TM$. Then, M is slant submanifold if and only if there exist a constant $\gamma \in [0,1]$ such that*

$$T^2 = -\gamma(I - \eta \otimes \xi), \tag{15}$$

furthermore, in this case, if θ is the slant angle of M, then $\gamma = \cos^2\theta$.

Corollary 3.2 ([6]). *Let M be a slant submanifold of an almost contact metric manifold $\widetilde{M}$ with slant angle θ. Then for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}, \tag{16}$$

$$\textit{and } g(NX, NY) = \sin^2\theta\{g(X,Y) - \eta(X)\eta(Y)\}. \tag{17}$$

Lemma 3.3. *Let M be a proper pseudo-slant submanifold of conformal Sasakian manifold $\widetilde{M}$. Then*

$$\phi D^{\perp} \perp N D_\theta. \tag{18}$$

Proof. Let $X \in D^{\perp}, Y \in D_\theta$, In view of (3) and (11) we get $g(\phi X, NY) = g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) = 0$. □

Lemma 3.4. *Let M be a pseudo-slant submanifold of conformal Sasakian manifold. Then*

$$TD^{\perp} = \{0\}, \tag{19}$$

$$TD_\theta = D_\theta. \tag{20}$$

Proof. (19) follows from (11). Now for $X \in D^{\perp}$ and $Y \in D_\theta$,

$$g(X, TY) = g(X, \phi Y) = -g(Y, \phi X) = 0.$$

The above equation shows that $TD_\theta \perp D^{\perp}$. Also we have $g(TY, \xi) = 0$ and from the fact $TD_\theta \subseteq TM$ we can infer that $TD_\theta \subseteq D_\theta$. Now for $X \in D_\theta$, from (15)

$$X = \frac{1}{\cos^2\theta}(\cos^2\theta\ X) = \frac{1}{\cos^2\theta}(-T^2X) = -\frac{1}{\cos^2\theta}(T(TX)).$$

Hence we get $D_\theta \subseteq TD_\theta$. Thus we have (20). □

Let M be proper pseudo-slant submanifold of conformal Sasakian manifold $\widetilde{M}$. We take ζ^T and $\zeta^{\perp}$ as tangential and normal part of Lee vector field ζ, i.e.,

$$\zeta = \zeta^T + \zeta^{\perp}. \tag{21}$$

In view of (6), (8), (9), (11), (12) and the above equation we have the following lemma.

Lemma 3.5. *Let M be any submanifold of conformal Sasakian manifold $\widetilde{M}$. Then we have*

$$\begin{aligned}\nabla_X TY - T\nabla_X Y &= A_{NY}X + th(X,Y) + (\exp(f))^{1/2}\{g(X,Y)\xi - \eta(Y)X\} - \frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX \\ &\quad + g(X,Y)T\zeta^T + g(X,Y)t\zeta^{\perp} - g(X,TY)\zeta^T\},\end{aligned} \tag{22}$$

$$\nabla^{\perp}_X NY - N\nabla_X Y = -h(X,TY) + nh(X,Y) - \frac{1}{2}\{-\omega(Y)NX + g(X,Y)N\zeta^T + g(X,Y)n\zeta^{\perp} - g(X,TY)\zeta^{\perp}\}, \tag{23}$$

$$\nabla_X tV - t\nabla^{\perp}_X V = A_{nV}X - TA_V X - \frac{1}{2}\{\omega(\phi V)X - \omega(V)TX - g(X,tV)\zeta^T\}, \tag{24}$$

$$\nabla^{\perp}_X nV - n\nabla^{\perp}_X V = -h(X,tV) - NA_V X + \frac{1}{2}\{\omega(V)NX + g(X,tV)\zeta^{\perp}\}, \tag{25}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$.

Theorem 3.6. *Let M be a submanifold of conformal Sasakian manifold $\widetilde{M}$. Then T is parallel if and only if*

$$A_{NW}Y - A_{NY}W = (\exp(f))^{1/2}\{Y\eta(W) - \eta(Y)W\} - \frac{1}{2}\{\omega(\phi Y)W + \omega(Y)TW + g(T\zeta^T + t\zeta^{\perp}, W)Y - TYg(\zeta^T, W)\}, \tag{26}$$

for any $Y, W \in TM$.

Proof. Let $X, Y \in TM$. From (22) we have

$$0 = A_{NY}X + th(X,Y) + (\exp(f))^{1/2}\{g(X,Y)\xi - \eta(Y)X\} - \frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX + g(X,Y)T\zeta^T + g(X,Y)t\zeta^{\perp} - g(X,TY)\zeta^T\}.$$

Taking inner product of this equation with $W \in TM$, we get

$$\begin{aligned} 0 = &g(A_{NY}X, W) + g(th(X,Y), W) + (\exp(f))^{1/2}\{g(X,Y)\eta(W) - \eta(Y)g(X,W)\} \\ &- \frac{1}{2}\{\omega(\phi Y)g(X,W) - \omega(Y)g(TX,W) + g(X,Y)g(T\zeta^T + t\zeta^\perp, W) - g(X,TY)g(\zeta^T, W)\}. \end{aligned}$$

Thus, using (10) in the above equation we get (26). Converse part is trivial. □

Theorem 3.7. *Let M be a submanifold of conformal Sasakian manifold $\widetilde{M}$. Then covariant derivation of T is skew-symmetric.*

Proof. Let $X, Y, W \in TM$. Using (22), (10), (11), (12) and (21) we have

$$\begin{aligned} g((\nabla_X T)Y, W) =& g(A_{NY}X, W) + g(th(X,Y), W) + (\exp(f))^{1/2}\{g(X,Y)\eta(W) - \eta(Y)g(X,W)\} \\ &- \frac{1}{2}\{\omega(\phi Y)g(X,W) - \omega(Y)g(TX,W) + g(X,Y)g(T\zeta^T + t\zeta^\perp, W) - g(X,TY)g(\zeta^T, W)\} \\ =& g(h(X,W), NY) - g(h(X,Y), NW) - (\exp(f))^{1/2}\{\eta(Y)g(X,W) - g(X,Y)\eta(W)\} \\ &- \frac{1}{2}\{g(\zeta, \phi Y)g(X,W) - g(\zeta, Y)g(TX,W) + g(X,Y)g(T\zeta^T + t\zeta^\perp, W) + g(TX,Y)g(\zeta, W)\} \\ =& - g(th(X,W), Y) - g(A_{NW}X, Y) - (\exp(f))^{1/2}\{\eta(Y)g(X,W) - g(X,Y)\eta(W)\} \\ &- \frac{1}{2}\{-g(T\zeta^T + t\zeta^\perp, Y)g(X,W) - g(\zeta, Y)g(TX,W) + g(X,Y)g(T\zeta^T + t\zeta^\perp, W) + g(TX,Y)g(\zeta, W)\} \\ =& - g(th(X,W) + A_{NW}X + (\exp(f))^{1/2}\{g(X,W)\xi - \eta(W)X\} \\ &- \frac{1}{2}\{(T\zeta^T + t\zeta^\perp)g(X,W) + \zeta^T g(TX,W) + Xg(\zeta, \phi W) - TXg(\zeta, W)\}, Y) = -g((\nabla_X T)W, Y). \end{aligned}$$

□

Theorem 3.8. *Let M be a submanifold of conformal Sasakian manifold $\widetilde{M}$. Then N is parallel if and only if t is parallel.*

Proof. Let $X, Y \in TM$ and $V \in T^\perp M$. In view of (23), (10), (11), (12), (21) and (24), we have

$$\begin{aligned} g((\nabla_X N)Y, V) =& - g(h(X,TY), V) + g(nh(X,Y), V) - \frac{1}{2}\{-\omega(Y)g(NX,V) \\ &+ g(X,Y)g(N\zeta^T, V) + g(X,Y)g(n\zeta^\perp, V) - g(X,TY)g(\zeta^\perp, V)\} \\ =& - g(A_V X, TY) - g(h(X,Y), nV) - \frac{1}{2}\{g(\zeta^\perp, Y)g(X,tV) \\ &- g(X,Y)g(\zeta^T, tV) - g(X,Y)g(\zeta^\perp, nV) + g(TX,Y)g(\zeta, V)\}. \end{aligned}$$

$$\begin{aligned} g((\nabla_X N)Y, V) =& g(TA_V X, Y) - g(A_{nV}X, Y) - \frac{1}{2}\{g(\zeta, Y)g(X,tV) - g(X,Y)g(\zeta, \phi V) + g(TX,Y)\omega(V)\} \\ =& - g(-TA_V X + A_{nV}X - \frac{1}{2}\{-\zeta g(X,tV) + X\omega(\phi V) - TX\omega(V)\}, Y) = g((\nabla_X t)V, Y). \end{aligned} \tag{27}$$

This completes our proof. □

Theorem 3.9. *Let M be a submanifold of conformal Sasakian manifold. Then N is parallel if and only if*

$$A_{nV}Y + A_V TY = \frac{1}{2}\{\omega(\phi V)Y + \omega(V)TY - \omega(Y)tV\}, \quad \textit{for any } Y \in TM \textit{ and } V \in T^\perp M. \tag{28}$$

Proof. Let $X, Y \in TM$, $V \in T^\perp M$ and N is parallel. From (27), we have

$$0 = g((\nabla_X N)Y, V) = -g(A_V TY, X) - g(A_{nV}Y, X) - \frac{1}{2}\{\omega(Y)g(X,tV) - g(X,Y)\omega(\phi V) - g(X,TY)\omega(V)\}.$$

Hence we get

$$0 = -g(A_V TY + A_{nV} Y + \frac{1}{2}\{\omega(Y)tV - \omega(\phi V)Y - \omega(V)TY\}, X).$$

This proves our assertion. □

Theorem 3.10. *Let M be submanifold of conformal Sasakian manifold. Then covariant derivation of n is skew-symmetric.*

Proof. Let $X \in TM$ and $U, V \in T^{\perp}M$. Then from (25), (10), (11), (12) and (21) we have

$$\begin{aligned} g((\nabla_X n)V, U) &= -g(h(X, tV), U) - g(NA_V X, U) + \frac{1}{2}\{g(\zeta, V)g(NX, U) + g(X, tV)g(\zeta^{\perp}, U)\} \\ &= -g(tV, A_U X) + g(h(X, tU), V) + \frac{1}{2}\{-g(\zeta, V)g(X, tU) - g(NX, V)g(\zeta, U)\} \\ &= g(NA_U X + h(X, tU) - \frac{1}{2}\{g(X, tU)\zeta + \omega(U)NX\}, V) \\ &= -g((\nabla_X n)U, V). \end{aligned}$$

□

4. Integrability of the Distributions

In this chapter we discuss the integrability condition of the distributions D_θ and $D^{\perp}$ involved in the definition of pseudo-slant submanifold of conformal Sasakian manifold.

Theorem 4.1. *Anti-invariant distribution $D^{\perp}$ of a pseudo-slant submanifold M of conformal Sasakian manifold $\widetilde{M}$ is integrable if and only if*

$$A_{NY} X - A_{NX} Y = \frac{1}{2}\{\omega(NY)X - \omega(NX)Y\}. \tag{29}$$

Proof. Let $X, Y \in D^{\perp}$. Consider

$$\begin{aligned} g([X, Y], \xi) &= g(\nabla_X Y, \xi) - g(\nabla_Y X, \xi) \\ &= g(\nabla_Y \xi, X) - g(\nabla_X \xi, Y). \end{aligned}$$

From (7), we get

$$(\exp(f))^{1/2}\{g(TX, Y) - g(TY, X)\} - \frac{1}{2}\{\eta(X)g(\zeta^T, Y) - \eta(Y)g(\zeta^T, X)\}.$$

Using (19) in the above equation, we get

$$g([X, Y], \xi) = 0, \text{ for any } X, Y \in D^{\perp}. \tag{30}$$

Further, from (22) and (19) we have

$$-T\nabla_X Y - A_{NY} X - th(X, Y) = (\exp(f))^{1/2}\{g(X, Y)\xi\} - \frac{1}{2}\{\omega(NY)X + g(X, Y)(T\zeta^T + t\zeta^{\perp})\}. \tag{31}$$

By interchanging X and Y we get

$$-T\nabla_Y X - A_{NX} Y - th(X, Y) = (\exp(f))^{1/2}\{g(X, Y)\xi\} - \frac{1}{2}\{\omega(NX)Y + g(X, Y)(T\zeta^T + t\zeta^{\perp})\}. \tag{32}$$

Using (31) and (32) and the fact that h is symmetric we get

$$T[X,Y] + A_{NY}X - A_{NX}Y = \frac{1}{2}\{\omega(NY)X - \omega(NX)Y\}.$$

Thus in view of (30) and (19) one can say that $[X,Y] \in D^{\perp}$ if and only if (29) satisfies. □

Theorem 4.2. *Slant distribution D_θ of a pseudo-slant submanifold M of conformal Sasakian manifold $\widetilde{M}$ is integrable if and only if*

$$\nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X + g(X,TY)\zeta^T \in D_\theta, \ \ \textit{for any } X,Y \in D_\theta. \tag{33}$$

Proof. Let $X,Y \in D_\theta$, from (22) and (20) we get

$$\begin{aligned}\nabla_X TY - T\nabla_X Y =& A_{NY}X + th(X,Y) + (\exp(f))^{1/2}\{g(X,Y)\xi\} - \frac{1}{2}\{\omega(\phi Y)X - \omega(Y)TX \\ &+ g(X,Y)(T\zeta^T + t\zeta^{\perp}) - g(X,TY)\zeta^T\}.\end{aligned} \tag{34}$$

Interchanging X and Y in the above equation we get

$$\begin{aligned}\nabla_Y TX - T\nabla_Y X =& A_{NX}Y + th(X,Y) + (\exp(f))^{1/2}\{g(X,Y)\xi\} - \frac{1}{2}\{\omega(\phi X)Y - \omega(X)TY \\ &+ g(X,Y)(T\zeta^T + t\zeta^{\perp}) - g(Y,TX)\zeta^T\}.\end{aligned} \tag{35}$$

It follows from (34) and (35) that

$$T[X,Y] = \nabla_X TY - \nabla_Y TX + A_{NX}Y - A_{NY}X + \frac{1}{2}\{\omega(\phi X)Y - \omega(\phi Y)X - \omega(X)TY + \omega(Y)TX\} + g(X,TY)\zeta^T. \tag{36}$$

Thus we infer that D_θ is integrable (i.e., for any $X,Y \in D_\theta$, $[X,Y] \in D_\theta$ or $T[X,Y] \in D_\theta$) if and only if (33) satisfies. □

Let M be a pseudo-slant submanifold of conformal Sasakian manifold and $\acute{\nabla}$(respectively $\acute{\nabla}^{\perp}$) be induced Riemannian connection (respectively normal connection) with respect to $\widetilde{\nabla}$ in M (respectively normal bundle $T^{\perp}M$). Then the Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ are given by

$$\widetilde{\nabla}_X Y = \acute{\nabla}_X Y + \acute{h}(X,Y), \tag{37}$$

$$\widetilde{\nabla}_X V = -\acute{A}_V X + \acute{\nabla}^{\perp}_X V, \tag{38}$$

for any $X,Y \in TM$ and $V \in T^{\perp}M$. Here $\acute{h}$ and $\acute{A}$ are the second fundamental form and shape operator with respect to $\widetilde{\nabla}$ and are related by

$$g(\acute{h}(X,Y),V) = g(\acute{A}_V X, Y). \tag{39}$$

Lemma 4.3. *Let M be a pseudo-slant submanifold of a conformal Sasakian manifold $\widetilde{M}$. Then we have*

$$\acute{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\zeta^T\}, \tag{40}$$

$$\acute{h}(X,Y) = h(X,Y) - \frac{1}{2}g(X,Y)\zeta^{\perp}, \tag{41}$$

$$\acute{A}_V X = A_V X - \frac{1}{2}\omega(V)X, \tag{42}$$

$$\acute{\nabla}^{\perp}_X V = \nabla^{\perp}_X V + \frac{1}{2}\omega(X)V, \tag{43}$$

for any $X,Y \in TM$ and $V \in T^{\perp}M$.

Proof. The proof is straightforward from the equations (5), (8), (9), (37) and (38). □

Corollary 4.4. *Let M be a proper pseudo-slant submanifold of conformal Sasakian manifold $\widetilde{M}$. Then the anti-invariant distribution $D^{\perp}$ is integrable if and only if $\acute{A}_{NX}Y = \acute{A}_{NY}X$, for any $X, Y \in D^{\perp}$.*

Proof. We have the proof in view of Theorem 4.1 and (42). □

Lemma 4.5. *Let M be a proper pseudo-slant submanifold of conformal Sasakian manifold $\widetilde{M}$. Then for any $X \in D^{\perp}$ and $Y \in TM$, we have*

$$-T(\acute{\nabla}_X Y) = \acute{A}_{NY}X - \omega(NY) + (\exp(f))^{1/2}\{g(X,Y)\xi\} - g(X,Y)(T\zeta^T + t\omega^{\perp}) + t\acute{h}(X,Y). \tag{44}$$

Proof. Let $X \in TM$ and $Y \in D^{\perp}$. From (5), (6), (8) and (9), we get

$$-A_{NY}X + \nabla^{\perp}_X NY = (\exp(f))^{1/2} g(X,Y)\xi - \frac{1}{2}\{\omega(NY)X - \omega(Y)\phi X + g(X,Y)\phi\zeta\} + \phi(\nabla_X Y + h(X,Y)).$$

Again using Lemma 4.3, (11), (12) and (21) in the above equation we get

$$\begin{aligned}-\acute{A}_{NY}X + \acute{\nabla}^{\perp}_X NY =& \frac{1}{2}\omega(NY)X + (\exp(f))^{1/2} g(X,Y)\xi + \omega(Y)TX + \omega(Y)NX \\ &- (T\zeta^T + N\zeta^T + t\zeta^{\perp} + n\zeta^{\perp})g(X,Y) + T\acute{\nabla}_X Y + N\acute{\nabla}_X Y + t\acute{h}(X,Y) + n\acute{h}(X,Y).\end{aligned}$$

Thus (44) follows from taking the tangential part of the above equation. □

Theorem 4.6. *The anti-invariant distribution $D^{\perp}$ of proper pseudo-slant submanifold is integrable if and only if*

$$\omega(NY)V + g(T\zeta^T + t\zeta^{\perp}, V)Y = 0, \quad \textit{for any } Y, V \in D^{\perp}. \tag{45}$$

Proof. Let $X \in TM$ and $Y, V \in D^{\perp}$. By taking inner product with (44) by V and simplify using (19), (39), we get

$$0 = -g(T\acute{\nabla}_X Y, V) = g(\acute{A}_{NY}V - \acute{A}_{NV}Y, X) - \omega(NY)g(X,V) - g(T\zeta^T + t\zeta^{\perp}, V)g(X,Y).$$

If (45) is true then we get $\acute{A}_{NY}V - \acute{A}_{NV}Y = 0$. Thus our assertion follows in view of Corollary 4.4. □

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