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On k-Near Perfect Numbers

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Abstract: A positive integer n is said to be k-near perfect number, if

$$\sigma(n) = 2n + \sum_{i=1}^{k} d_i$$

where d_i's are proper divisors of n and function $\sigma(n)$ is the sum of all positive divisors of n. In this paper we discuss some results concerning with k-near perfect numbers. Near perfect numbers are nothing but 1-Near Perfect Numbers.

MSC: 05C69, 05C76.

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1. Introduction

A positive integer n is said to be a perfect number if the sum of all its proper divisors is equal to two times the number. For any perfect number n, $\sigma(n) = 2n$. All perfect numbers known are even. The question of existence of an odd perfect number still remains open. Euler proved that even perfect number is of the form $2^{p-1}(2^p - 1)$ [5] where p and $2^p - 1$ are primes. Prime numbers of the form $2^p - 1$ where p is a prime is called Mersenne prime. In the year 2012, P.Pollack and V.Shevelev [2] introduced the notion of near perfect number. It is known that a positive integer n is called near perfect number, if n is the sum of all its proper divisors except for one of them which is termed as redundant divisor. Moreover, a positive integer n is a near perfect number with redundant divisor d if and only if d is a proper divisor of n and $\sigma(n) = 2n + d$. P.Pollack and V.Shevelev [2] have also defined k-near perfect number. A positive integer n is said to be a k-near perfect number, if n is the sum of all its divisors except k numbers of proper divisors. If n is a k-near perfect number with redundant divisors $d_1, ..., d_k$ then we write

$$\sigma(n) = 2n + d_1 + d_2 + ... + d_k$$

Paul Pollack and Vladimir Shevelev [2] introduced the concept of Near perfect numbers in the year 2012 in their paper titled On Pefect and Near Perfect Numbers they have given us the construction of Near Perfect Numbers and have conjectured that there exists infinitely many near pefect numbers with the redundant divisor 2^k and have given an upper bound on the number of near pefect numbers they have also introduced the concept of k-near perfect numbers. Later Bhabesh Das and Helen K Saikia [1] have some results concerning with near perfect numbers from known perfect numbers.

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1.1. Definitions

Definition 1.1. *A divisor function is an arithmetic function that counts the number of positive divisors of an integer.*

Definition 1.2. *A prime number of the form $2^p - 1$ where p is a prime number is called Mersenne prime.*

Definition 1.3. *Prime numbers of the form $2^k + 1$ $(k \geq 0)$ are called Fermat primes.*

Definition 1.4. *A positive integer n is said to be a perfect number if the sum of all its proper divisors is equal to two times the number. For any perfect number n, $\sigma(n) = 2n$.*

Definition 1.5. *A positive integer n is called near perfect number, if n is the sum of all its proper divisors except for one of them which is termed as redundant divisor.*

Definition 1.6. *A positive integer n is said to be a k-near perfect number, if n is the sum of all its divisors except k numbers of proper divisors.*

2. Main Section

Theorem 2.1. *Let $k > 0$ be an odd integer, A be an even perfect number and let $p_1, p_2, ..., p_k$ be distinct odd primes. Then $n = A\prod_{i=1}^{k} p_i$ is not a near perfect number with redundant divisor A.*

Proof. Suppose n is a near perfect number with redundant divisor A then,

$$\sigma(n) = 2A\prod_{i=1}^{k} p_i + A \tag{1}$$

but

$$\sigma(n) = 2A\prod_{i=1}^{k}(p_i + 1) \tag{2}$$

hence by Equation (1) and equation (2)

$$2A\left(\prod_{i=1}^{k}(p_i + 1) - \prod_{i=1}^{k}(p_i)\right) = A$$

this is a contradiction. □

Theorem 2.2. *Let $M = 2^p - 1$ and $N = 2^q - 1$ be Mersenne primes where p and q are distinct odd primes. Then $n = 2^{pq}MN$ is not a near perfect number with redundant divisor 2^{pq}.*

Proof. Suppose n is a near perfect number with redundant divisor 2^{pq} then,

$$\sigma(n) = 2n + 2^{pq} \tag{3}$$

but

$$\sigma(n) = (2^{pq+1} - 1)(M + 1)(N + 1) \tag{4}$$

$$\sigma(n) = (2^{pq+1} - 1)2^p 2^q \tag{5}$$

hence by Equation (3) and Equation (5)

$$(2^{pq+1} - 1)2^p 2^q - (2^{pq+1} - 1)(2^p - 1)(2^q - 1) = 2^{pq}$$

$$\begin{aligned}2^{pq+1}2^p2^q - 2^p2^q - 2^{pq+1}(2^p2^q - 2^p - 2^q + 1) &= 2^{pq}\\ 2^{pq+1}(2^p + 2^q - 1) &= 2^{pq} + 2^p2^q\\ 2^{pq}(2^{p+1} + 2^{q+1} - 3) &= 2^{p+q}\end{aligned}$$

since $2^{p+1} + 2^{q+1} - 3 > 1$ we have

$$2^{pq} < 2^{p+q}$$

this is a contradiction. □

Theorem 2.3. *Let $F_\alpha = 2^\alpha + 1$ and $F_\beta = 2^\beta + 1$ be Fermat primes. Then there is no near perfect number of the form $n = 2^{\alpha\beta}F_\alpha F_\beta$ with redundant divisor $2^{\alpha\beta}$.*

Proof. Suppose n is a near perfect number with redundant divisor $2^{\alpha\beta}$

$$\sigma(n) = 2n + 2^{\alpha\beta} \tag{6}$$

$$\sigma(n) = 2^{\alpha\beta+1}F_\alpha F_\beta + 2^{\alpha\beta} \tag{7}$$

but

$$\sigma(n) = \sigma(2^{\alpha\beta}F_\alpha F_\beta) = (2^{\alpha\beta+1} - 1)(F_\beta + 1)(F_\alpha + 1) \tag{8}$$

hence by equation 7 and equation 8

$$\begin{aligned}(2^{\alpha\beta+1} - 1)(F_\beta + 1)(F_\alpha + 1) &= 2^{\alpha\beta+1}F_\alpha F_\beta + 2^{\alpha\beta}\\ 2^{\alpha\beta+1}(F_\alpha F_\beta + F_\beta + F_\alpha + 1) - (F_\alpha F_\beta + F_\beta + F_\alpha + 1) &= 2^{\alpha\beta+1}F_\alpha F_\beta + 2^{\alpha\beta}\\ 2^{\alpha\beta+1}(2^\beta + 2^\alpha + 3) &= (2^\beta + 1)(2^\alpha + 1) + (2^\beta + 2^\alpha + 3) + 2^{\alpha\beta}\\ 2^{\alpha\beta}(2^{\beta+1} + 2^{\alpha+1} + 6) &= 2^{\alpha+\beta} + 2^\beta + 2^\alpha + 3 + 2^\beta + 2^\alpha + 1 + 2^{\alpha\beta}\\ 2^{\alpha\beta}(2^{\beta+1} + 2^{\alpha+1} + 6) &= 2^{\alpha+\beta} + 2^{\beta+1} + 2^{\alpha+1} + 4 + 2^{\alpha\beta}\end{aligned}$$

add and subtract 2 on the right hand side,

$$2^{\alpha\beta}(2^{\beta+1} + 2^{\alpha+1} + 6) = (2^{\alpha+\beta} - 2 + 2^{\alpha\beta}) + (2^{\beta+1} + 2^{\alpha+1} + 6)$$

by comparing the co-efficient of $(2^{\beta+1} + 2^{\alpha+1} + 6)$, we have, $2^{\alpha\beta} = 1$; $\alpha\beta = 0$ this is a contradiction. □

Theorem 2.4. *If $n = 2^k$ then n is not a k-near perfect number.*

Proof. Suppose that n is a k-near perfect number with redundant divisors $2^{a_1}, ..., 2^{a_k}$, then

$$\sigma(n) = 2^{k+1} + \sum_{i=1}^{k} 2^{a_i} \tag{9}$$

but

$$\sigma(2^k) = 2^{k+1} - 1 \tag{10}$$

from Equation (9) and Equation (10) we have,

$$\sum_{i=1}^{k} 2^{a_i} + 1 = 0$$

this is a contradiction. □

Theorem 2.5. *Let $k > 0$ be an odd integer and let $p_1, p_2, ..., p_k$ be distinct odd primes. Then for any integer $a \geq 1$, $n = 2^a \prod_{i=1}^{k} p_i$ is not a k-near perfect number with redundant divisors $p_1, p_2, ..., p_k$.*

Proof. Suppose n is a k-near perfect number then

$$\sigma(n) = 2n + \sum_{i=1}^{k} p_i \tag{11}$$

but

$$\sigma(n) = (2^{a+1} - 1) \prod_{i=1}^{k} (p_i + 1) \tag{12}$$

hence from Equation (11) and Equation (12) we have,

$$(2^{a+1} - 1) \prod_{i=1}^{k} (p_i + 1) - 2^{a+1} \prod_{i=1}^{k} p_i = \sum_{i=1}^{k} p_i$$

this is a contradiction. □

Theorem 2.6. *Let k be an odd integer and let $p_1, p_2, ..., p_k$ be distinct odd primes. Then $n = \prod_{i=1}^{k} p_i$ is not a k-near perfect number with redundant divisors $p_1, p_2, ..., p_k$.*

Proof. Suppose n is a k-near perfect number

$$\sigma(n) = 2n + \sum_{i=1}^{k} p_i \tag{13}$$

but

$$\sigma(n) = \prod_{i=1}^{k} (p_i + 1) \tag{14}$$

hence by Equation (13) and Equation (14), we have,

$$\prod_{i=1}^{k} (p_i + 1) - 2 \prod_{i=1}^{k} p_i = \sum_{i=1}^{k} p_i$$

this is a contradiction. □

Theorem 2.7. *Let $k > 0$ be an odd integer. If A and B are perfect numbers such that $(A, B) = 1$ then $n = A.B$ is not a k-near perfect number with odd redundant divisors.*

Proof. Suppose $n = A.B$ is a k-near perfect number with redundant divisors $R_1, ..., R_k$ where k is an odd integer. Then

$$\sigma(n) = 2n + \sum_{i=1}^{k} R_i \tag{15}$$

but

$$\sigma(n) = 4AB \tag{16}$$

hence by Equation (15) and Equation (16) we have,

$$4AB - 2AB = \sum_{i=1}^{k} R_i$$

$$2AB = \sum_{i=1}^{k} R_i$$

this is a contradiction. □

Theorem 2.8. *Let $k > 0$ be an odd integer, A be an even perfect number and let $p_1, p_2, ..., p_k$ be distinct odd primes. Then $n = A\prod_{i=1}^{k} p_i$ where A is a perfect number is not a k-near perfect number with redundant divisors $p_1, ..., p_k$.*

Proof. Suppose n is a k-near perfect number

$$\sigma(n) = 2n + \sum_{i=1}^{k} p_i \tag{17}$$

but

$$\sigma(n) = 2A\prod_{i=1}^{k}(p_i + 1) \tag{18}$$

hence by Equation (17) and Equation (18)

$$2A\prod_{i=1}^{k}(p_i + 1) = 2A\prod_{i=1}^{k} p_i + \sum_{i=1}^{k} p_i$$

this is a contradiction. □

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