1. Introduction

The concept of generalized closed sets plays a significant role in topology. In 1970, Levine [6] introduced the concept of generalized closed sets in topological spaces and a class of topological spaces called $T_{1/2}$ space. Extensive research on generalizing closedness was done in recent years by many Mathematicians. Arya and Nour [1], Maki [7], Dontchev and Ganster [3] Tong [11] and Veerakumar [12] introduced generalized semi-closed sets, $\alpha$-generalized closed sets, $\delta$-generalized closed sets and $\tilde{g}$-closed sets in topological spaces. The purpose of this present paper is to define a new class of generalized closed sets called $B\delta g$-closed sets and also we obtain the basic properties of $B\delta g$-closed sets in topological spaces. Applying this set, we obtain a new type of spaces called $B\delta g$-space.

2. Preliminaries

Throughout this paper $(X, \tau)$ (or simply $X$) represent topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of $X$, $cl(A)$, $int(A)$ and $A^c$ denote the closure of $A$, the interior of $A$ and the complement of $A$ respectively. Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset $A$ of a space $(X, \tau)$ is called

(1) a semi-open set [5] if $A \subseteq cl(int(A))$. 
(2). a pre-open set [8] if \( A \subseteq \text{int}(\text{cl}(A)) \).

(3). an \( \alpha \)-open set [9] if \( A \subseteq \text{int}(\text{cl}(\text{int}(A))) \).

(4). a regular open set [10] if \( A = \text{int}(\text{cl}(A)) \).

The complement of a semi-open (respectively a pre-open, an \( \alpha \)-open, a regular) set is called semi-closed (respectively pre-closed, \( \alpha \)-closed, regular closed). The intersection of all semi-closed (respectively \( \alpha \)-closed) sets of \( X \) containing \( A \) is called the semi-closure [2] (respectively \( \alpha \)-closure [9]) of \( A \) and it is denoted by \( \text{scl}(A) \) (respectively \( \text{acl}(A) \)).

**Definition 2.2.** The \( \delta \)-interior [13] of a subset \( A \) of \( X \) is the union of all regular open sets of \( X \) contained in \( A \) and it is denoted by \( \text{Int}_\delta(A) \). A subset \( A \) is called \( \delta \)-open [13] if \( A = \text{Int}_\delta(A) \), i.e., a set is \( \delta \)-open if it is the union of regular open sets. The complement of a \( \delta \)-open set is called \( \delta \)-closed. Alternatively, a set \( A \subseteq (X, \tau) \) is called \( \delta \)-closed [13] if \( A = \text{cl}_\delta(A) \), where \( \text{cl}_\delta(A) = \{ x \in X : \text{int}(\text{cl}(U)) \cap A \neq \emptyset, U \in \tau \text{ and } x \in U \} \).

**Definition 2.3.** A subset \( A \) of \((X, \tau)\) is called

(1). generalized closed (briefly \( g \)-closed) set [6] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(2). generalized semi-closed (briefly \( gs \)-closed) set [1] if \( \text{scl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(3). \( \alpha \)-generalized closed (briefly \( \alpha g \)-closed) set [7] if \( \text{acl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(4). \( \delta \)-generalized closed (briefly \( \delta g \)-closed) set [3] if \( \text{cl}_\delta(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \((X, \tau)\).

(5). \( \hat{g} \)-closed set [12] if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open in \((X, \tau)\).

(6). \( \hat{\delta} \hat{g} \)-closed (briefly \( \delta \hat{g} \)-closed) set [4] if \( \text{cl}_\delta(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \hat{\delta} \hat{g} \)-open in \((X, \tau)\).

The complement of a \( g \)-closed (respectively \( gs \)-closed, \( \alpha g \)-closed, \( \delta g \)-closed, \( \hat{g} \)-closed and \( \delta \hat{g} \)-closed) set is called \( g \)-open (respectively \( gs \)-open, \( \alpha g \)-open, \( \delta g \)-open, \( \hat{g} \)-open and \( \delta \hat{g} \)-open).

**Definition 2.4 ([11]).** A subset \( A \) of a space \((X, \tau)\) is called

(1). a \( t \)-set if \( \text{int}(A) = \text{int}(\text{cl}(A)) \).

(2). a \( B \)-set if \( A = G \cap F \) where \( G \) is open and \( F \) is a \( t \)-set in \( X \).

**Definition 2.5.** A space \((X, \tau)\) is called

(1). \( T_{1/2} \)-space [6] if every \( g \)-closed set in it is closed.

(2). \( T_{3/4} \)-space [3] if every \( \delta g \)-closed set in it is \( \delta \)-closed.

(3). \( \hat{T}_{3/4} \)-space [4] if every \( \delta \hat{g} \)-closed set in it is \( \delta \)-closed.

### 3. \( B\delta g \)-closed Sets

In this section we introduce \( B\delta g \)-closed sets in topological spaces and study some relations between \( B\delta g \)-closed sets and other existing closed sets.

**Definition 3.1.** A subset \( A \) of \((X, \tau)\) is called \( B\delta g \)-closed if \( \text{cl}_\delta(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is a \( B \)-set.
The complement of a $B\delta g$-closed set is called $B\delta g$-open.

**Theorem 3.2.** Every $\delta$-closed set is $B\delta g$-closed.

**Proof.** Let $A$ be a $\delta$-closed set in $X$. Let $U$ be any $B$-set such that $A \subseteq U$. Since $A$ is $\delta$-closed, $\text{cl}_\delta(A) = A$ for every subset $A$ of $(X, \tau)$. Therefore $\text{cl}_\delta(A) \subseteq U$ and hence $A$ is $B\delta g$-closed. \qed

**Remark 3.3.** The converse of Theorem 3.2 need not be true as shown by the following Example.

**Example 3.4.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Then $\{a, c\}$ is $B\delta g$-closed set but not $\delta$-closed.

**Theorem 3.5.** Every $B\delta g$-closed set is $\delta g$-closed.

**Proof.** Let $A$ be a $B\delta g$-closed set in $X$. Let $U$ be any open set containing $A$ in $X$. Since every open set is a $B$-set, $U$ is a $B$-set of $X$. Since $A$ is $B\delta g$-closed, $\text{cl}_\delta(A) \subseteq U$. Hence $A$ is a $\delta g$-closed set of $X$. \qed

**Remark 3.6.** The converse of Theorem 3.5 need not be true as shown by the following Example.

**Example 3.7.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, \{b, c\}, X\}$. Then $\{a, b\}$ is $\delta g$-closed set but not $B\delta g$-closed.

**Theorem 3.8.** Every $B\delta g$-closed set is $g$-closed.

**Proof.** Let $A$ be a $B\delta g$-closed set and $U$ be any open set containing $A$ in $X$. Since every open set is a $B$-set, $\text{cl}_\delta(A) \subseteq U$ for every subset $A$ of $X$. Since $\text{scl}(A) \subseteq \text{cl}_\delta(A) \subseteq U$, $\text{scl}(A) \subseteq U$ and hence $A$ is $g$-closed. \qed

**Remark 3.9.** The converse of Theorem 3.8 need not be true as shown by the following Example.

**Example 3.10.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, X\}$. Then $\{b\}$ is $g$-closed set but not $B\delta g$-closed.

**Theorem 3.11.** Every $B\delta g$-closed set is $a g$-closed.

**Proof.** It is true that $\text{acl}(A) \subseteq \text{cl}_\delta(A)$ for every subset $A$ of $X$. \qed

**Remark 3.12.** The converse of Theorem 3.11 need not be true as shown by the following Example.

**Example 3.13.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{a\}, \{a, c\}, X\}$. Then $\{b, c\}$ is $a g$-closed set but not $B\delta g$-closed.

**Theorem 3.14.** Every $B\delta g$-closed set is $g s$-closed.

**Proof.** Let $A$ be a $B\delta g$-closed set and $U$ be any open set containing $A$ in $X$. Since every open set is a $B$-set, $\text{cl}_\delta(A) \subseteq U$ for every subset $A$ of $X$. Since $\text{scl}(A) \subseteq \text{cl}_\delta(A) \subseteq U$, $\text{scl}(A) \subseteq U$ and hence $A$ is $g s$-closed. \qed

**Remark 3.15.** A $g s$-closed set need not be $B\delta g$-closed as shown by the following Example.

**Example 3.16.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\phi, \{c\}, X\}$. Then $\{b\}$ is $g s$-closed set but not $B\delta g$-closed.

**Remark 3.17.** From the above discussions we summarize the fundamental relationships between several types of generalized closed sets in the following diagram. None of the implications is reversible.
(1) $B\delta g$-closed set  (2) $\alpha g$-closed set  (3) $\delta g$-closed set  (4) $gs$-closed set  (5) $g$-closed set  (6) $\delta$-closed set.

**Remark 3.18.** The following Examples show that the concepts of $B\delta g$-closed set and closed set (respectively semi-closed set, $\hat{g}$-closed set and $\delta \hat{g}$-closed set) are independent.

**Example 3.19.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then $\{a, b\}$ is $B\delta g$-closed set but it is neither closed nor semi-closed. Also $\{a, b\}$ is not $\delta \hat{g}$-closed.

**Example 3.20.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, X\}$. Then $\{b, c\}$ is closed, semi-closed and $\delta \hat{g}$-closed set. But it is not $B\delta g$-closed.

**Example 3.21.** Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{b\}, X\}$. Then $\{a, c\}$ is $\hat{g}$-closed set but not $B\delta g$-closed and $\{a, b\}$ is $B\delta g$-closed set but not $\hat{g}$-closed in.

**Remark 3.22.** From the above discussions we obtain the following diagram.

![Diagram](image)

### 4. Some Topological Properties

**Theorem 4.1.** If $A$ is both $B$-set and $B\delta g$-closed set of $(X, \tau)$, then $A$ is $\delta$-closed.
Proof. Given A is both B-set and $B\delta g$-closed set of $(X, \tau)$. Then $cl_\delta(A) \subseteq A$ whenever A is a B-set and $A \subseteq A$. Therefore we obtain that $A = cl_\delta(A)$ and hence A is $\delta$-closed.

Proposition 4.2. If A and B are $B\delta g$-closed sets, then $A \cup B$ is $B\delta g$-closed.

Proof. Let $A \cup B \subseteq U$, where U is a B-set. Then $A \subseteq U$ and $B \subseteq U$. Since A and B are $B\delta g$-closed sets, $cl_\delta(A) \subseteq U$ and $cl_\delta(B) \subseteq U$, whenever $A \subseteq U$, $B \subseteq U$ and U is a B-set. Therefore $cl_\delta(A \cup B) = cl_\delta(A) \cup cl_\delta(B) \subseteq U$. So we obtain that $A \cup B$ is $B\delta g$-closed set of $(X, \tau)$.

Remark 4.3. The intersection of two $B\delta g$-closed sets need not be a $B\delta g$-closed set.

Example 4.4. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, \{a\}, X\}$. Then $\{a, b\}$ and $\{a, c\}$ are $B\delta g$-closed sets. But $\{a, b\} \cap \{a, c\} = \{a\}$ is not $B\delta g$-closed.

Proposition 4.5. If A is a $B\delta g$-closed set of $(X, \tau)$ such that $A \subseteq B \subseteq cl_\delta(A)$, then B is also a $B\delta g$-closed set of $(X, \tau)$.

Proof. Let $U$ be a B-set of $(X, \tau)$ such that $B \subseteq U$. Since $A \subseteq B$, $A \subseteq U$. Since A is $B\delta g$-closed, we have $cl_\delta(A) \subseteq U$. Now $cl_\delta(B) \subseteq cl_\delta(cl_\delta(A)) = cl_\delta(A) \subseteq U$. Therefore B is also a $B\delta g$-closed set of $(X, \tau)$.

Proposition 4.6. Let A be a $B\delta g$-closed set of $(X, \tau)$, then $cl_\delta(A) - A$ does not contain a non-empty complement of a B-set.

Proof. Suppose that A is $B\delta g$-closed. Let $F$ be the complement of a B-set and $F \subseteq cl_\delta(A) - A$. Since $F \subseteq cl_\delta(A) - A$, $F \subseteq X - A$, $A \subseteq X - F$ and $X - F$ is a B-set. Therefore $cl_\delta(A) \subseteq X - F$ and $F \subseteq X - cl_\delta(A)$. Also $F \subseteq cl_\delta(A)$. Therefore $F \subseteq (cl_\delta(A))^c \cap cl_\delta(A) = \emptyset$. Hence $F = \emptyset$.

Theorem 4.7. Let A be a $B\delta g$-closed set of X. Then A is $\delta$-closed if and only if $cl_\delta(A) - A$ is the complement of a B-set.

Proof. Necessity: Let A be a $\delta$-closed subset of $(X, \tau)$. Then $cl_\delta(A) = A$ and so $cl_\delta(A) - A = \emptyset$ which is the complement of a B-set.

Sufficiency: Let $cl_\delta(A) - A$ be the complement of a B-set. Since A is $B\delta g$-closed, by Proposition 4.6, $cl_\delta(A) - A$ does not contain a non-empty complement of a B-set which implies $cl_\delta(A) - A = \emptyset$. Therefore $cl_\delta(A) = A$. Hence A is $\delta$-closed.

Proposition 4.8. For each $x \in X$ either $\{x\}$ is the complement of a B-set or $\{x\}^c$ is $B\delta g$-closed in X.

Proof. Suppose that $\{x\}$ is not the complement of a B-set in X, then $\{x\}^c$ is not a B-set and the only B-set containing $\{x\}^c$ is the space X itself. That is $\{x\}^c \subseteq X$. Therefore $cl_\delta(\{x\}^c) \subseteq X$ and so $\{x\}^c$ is $B\delta g$-closed.

Definition 4.9. The intersection of all B-sets of X containing A is called the B-kernel of A and is denoted by $B-ker(A)$.

Lemma 4.10. A subset A of $(X, \tau)$ is $B\delta g$-closed iff $cl_\delta(A) \subseteq B-ker(A)$.

Proof. Assume that A is $B\delta g$-closed in X. Then $cl_\delta(A) \subseteq U$ whenever $A \subseteq U$ and U is a B-set in X. Let $x \in cl_\delta(A)$. Suppose $x \notin B-ker(A)$, then there is a B-set U such that $x \notin U$. Since U is a B-set containing A, $x \notin cl_\delta(A)$ which is a contradiction. Hence $x \in B-ker(A)$. Conversely assume that $cl_\delta(A) \subseteq B-ker(A)$. If U is any B-set containing A, then $cl_\delta(A) \subseteq B-ker(A) \subseteq U$. Therefore A is $B\delta g$-closed.

The intersection of all $B\delta g$-closed sets of X containing A is called the $B\delta g$-closure of A and it is denoted by $B\delta g-cl(A)$.

Lemma 4.11. Let A and B be subsets of $(X, \tau)$. Then

1. $B\delta g-cl(\emptyset) = \emptyset$ and $B\delta g-cl(X) = X$. 

(2). If \( A \subseteq B \), then \( B_{\delta g}-cl(A) \subseteq B_{\delta g}-cl(B) \).

(3). \( B_{\delta g}-cl(A) = B_{\delta g}-cl(B_{\delta g}-cl(A)) \).

(4). \( B_{\delta g}-cl(A \cup B) = B_{\delta g}-cl(A) \cup B_{\delta g}-cl(B) \).

(5). \( B_{\delta g}-cl(A \cap B) \subseteq B_{\delta g}-cl(A) \cap B_{\delta g}-cl(B) \).

**Remark 4.12.** If \( A \) is \( B_{\delta g} \)-closed in \( (X, \tau) \), then \( B_{\delta g}-cl(A) = A \) but the converse need not be true as shown by the following Example.

**Example 4.13.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{c\}, X\} \). Let \( A = \{c\} \) then \( B_{\delta g}-cl(A) = \{c\} \). But \( \{c\} \) is not a \( B_{\delta g} \)-closed set.

**Remark 4.14.** In general, \( B_{\delta g}-cl(A) \cap B_{\delta g}-cl(B) \not\subseteq B_{\delta g}-cl(A \cap B) \). This can be shown from the following Example.

**Example 4.15.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\} \). Let \( A = \{a, c\} \) and \( B = \{b, c\} \), then \( B_{\delta g}-cl(A) \cap B_{\delta g}-cl(B) = X \not\subseteq \{c\} = B_{\delta g}-cl(A \cap B) \).

## 5. \( B_{\delta g} \)-open Sets

**Definition 5.1.** A subset \( A \) of \( (X, \tau) \) is called \( B_{\delta g} \)-open if its complement \( A^c \) is \( B_{\delta g} \)-closed in \( (X, \tau) \).

**Theorem 5.2.** If a subset \( A \) of a topological space \( (X, \tau) \) is \( \delta \)-open then it is \( B_{\delta g} \)-open in \( X \).

**Proof.** Let \( A \) be an \( \delta \)-open set in \( X \). Then \( A^c \) is \( \delta \)-closed. By Theorem 3.2, \( A^c \) is \( B_{\delta g} \)-closed in \( (X, \tau) \). Hence \( A \) is \( B_{\delta g} \)-open in \( X \).

**Remark 5.3.** The converse of Theorem 5.2 need not be true as shown by the following Example.

**Example 5.4.** Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{a\}, X\} \). Then \( \{b\} \) is \( B_{\delta g} \)-open set but not \( \delta \)-open in \( (X, \tau) \).

**Proposition 5.5.** Every \( B_{\delta g} \)-open set is \( \delta \)-open (respectively \( g \)-open, \( \alpha g \)-open, \( g_s \)-open).

**Proof.** Let \( A \) be a \( B_{\delta g} \)-open set in \( X \). Then \( A^c \) is \( B_{\delta g} \)-closed. By Theorem 3.5, \( A^c \) is \( \delta \)-closed. Hence \( A \) is \( \delta \)-open in \( X \). (respectively By Theorem 3.8, \( A^c \) is \( g \)-closed. Hence \( A \) is \( g \)-open in \( X \). By Theorem 3.11, \( A^c \) is \( \alpha g \)-closed. Hence \( A \) is \( \alpha g \)-open in \( X \). By Theorem 3.14, \( A^c \) is \( g_s \)-closed. Hence \( A \) is \( g_s \)-open in \( X \).

**Remark 5.6.** For a subset \( A \) of \( X \), \( cl_{\delta}(X-A) = X - int_{\delta}(A) \).

**Theorem 5.7.** A subset \( A \) of a topological space \( (X, \tau) \) is \( B_{\delta g} \)-open if and only if \( G \subseteq int_{\delta}(A) \) whenever \( X - G \) is a \( B \)-set and \( G \subseteq A \).

**Proof.** Necessity: Let \( A \) be \( B_{\delta g} \)-open. Let \( X - G \) be a \( B \)-set and \( G \subseteq A \). Then \( X - A \subseteq X - G \). Since \( X - A \) is \( B_{\delta g} \)-closed, \( cl_{\delta}(X-A) \subseteq X - G \). Hence \( G \subseteq int_{\delta}(A) \).

Sufficiency: Suppose \( X - G \) is a \( B \)-set and \( G \subseteq A \) imply that \( G \subseteq int_{\delta}(A) \). Let \( X - A \subseteq U \) where \( U \) is a \( B \)-set. Then \( X - U \subseteq A \) and \( X - (X - U) \) is a \( B \)-set. By hypothesis \( X - U \subseteq int_{\delta}(A) \). This implies \( X - int_{\delta}(A) \subseteq U \) and \( cl_{\delta}(X-A) \subseteq U \). So \( X - A \) is \( B_{\delta g} \)-closed. Hence \( A \) is \( B_{\delta g} \)-open.

**Proposition 5.8.** If \( A \) is a \( B_{\delta g} \)-open set in \( (X, \tau) \) such that \( int_{\delta}(A) \subseteq B \subseteq A \), then \( B \) is also a \( B_{\delta g} \)-open set of \( (X, \tau) \).
Proof. \( \text{ints}(A) \subseteq B \subseteq A \) implies that \( X - A \subseteq X - B \subseteq X - \text{ints}(A) \). By Remark 5.6, \( X - A \subseteq X - B \subseteq \text{cl}_{\delta}(X - A) \).

Since \( X - A \) is \( B\delta \)-closed, by Proposition 4.5, \( X - B \) is \( B\delta \)-closed and hence \( B \) is \( B\delta \)-open in \( (X, \tau) \).

\[ \square \]

Theorem 5.9. If a set \( A \) is \( B\delta \)-open in \( X \) then \( G = X \) whenever \( G \) is a \( B \)-set and \( \text{ints}(A) \cup A^c \subseteq G \).

Proof. Let \( A \) be a \( B\delta \)-open set, \( G \) be a \( B \)-set and \( \text{ints}(A) \cup A^c \subseteq G \). This implies \( G^c \subseteq \text{ints}(A) \cup A^c = \text{ints}(A) \cap A = \text{ints}(A)^c - A^c = \text{cl}_{\delta}(A^c) - A^c \). Since \( A^c \) is \( B\delta \)-closed and \( G^c \) is the complement of a \( B \)-set, it follows from Proposition 4.6 that \( G^c = \phi \). Hence \( G = X \).

\[ \square \]

Lemma 5.10. Let \( A \) be a subset of \( (X, \tau) \) and \( x \in X \). Then \( x \in \text{B\delta-cl}(A) \) if and only if \( V \cap A \neq \phi \) for every \( \text{B\delta-open} \) set \( V \) containing \( x \).

Proof. Suppose that there exists a \( \text{B\delta-open} \) set \( V \) containing \( x \) such that \( V \cap A = \phi \). Since \( A \subseteq X - V \), \( \text{B\delta-cl}(A) \subseteq X - V \) and then \( x \notin \text{B\delta-cl}(A) \). Conversely, assume that \( x \notin \text{B\delta-cl}(A) \). Then there exists a \( \text{B\delta-closed} \) set \( F \) containing \( A \) such that \( x \notin F \). Since \( x \in X - F \) and \( X - F \) is \( \text{B\delta-open} \), \( (X - F) \cap A = \phi \).

\[ \square \]

6. Applications

Definition 6.1. A space \( X \) is called a \( 3T_{\delta g} \)-space if every \( B\delta \)-closed set in it is \( \delta \)-closed.

Theorem 6.2. Every \( 3_{1/4} \)-space is \( 3T_{\delta g} \)-space.

Proof. Let \( A \) be a \( B\delta \)-closed set in \( X \). Since every \( B\delta \)-set is \( \delta \)-closed by Theorem 3.5, \( A \) is \( \delta \)-closed. Since \( X \) is \( 3_{1/4} \)-space, \( A \) is \( \delta \)-closed. Hence \( X \) is \( 3T_{\delta g} \)-space.

\[ \square \]

Remark 6.3. The converse of Theorem 6.2 need not be true as shown by the following Example.

Example 6.4. Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\} \). Then \( (X, \tau) \) is \( 3T_{\delta g} \)-space but not \( 3_{1/4} \)-space.

Remark 6.5. The concepts of \( 3T_{\delta g} \)-space and \( \tilde{3}_{1/4} \)-space are independent of each another as shown by the following Examples.

Example 6.6. Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \). Then \( (X, \tau) \) is \( \tilde{3}_{1/4} \)-space but not \( 3T_{\delta g} \)-space.

Example 6.7. Let \( X = \{a, b, c\} \) with the topology \( \tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\} \). Then \( (X, \tau) \) is \( 3T_{\delta g} \)-space but not \( \tilde{3}_{1/4} \)-space.

Theorem 6.8. For a topological space \( (X, \tau) \), the following conditions are equivalent.

(1). \( (X, \tau) \) is a \( 3T_{\delta g} \)-space.

(2). Every singleton of \( X \) is either \( \delta \)-open or \( X - \{x\} \) is a \( B \)-set.

Proof. (1) \( \Rightarrow \) (2) Let \( x \in X \). Suppose that \( X - \{x\} \) is not a \( B \)-set of \( (X, \tau) \). Then \( X - \{x\} \) is a \( B\delta \)-closed set of \( (X, \tau) \).

Since \( (X, \tau) \) is \( 3T_{\delta g} \)-space, \( X - \{x\} \) is an \( \delta \)-closed set of \( (X, \tau) \), i.e., \( \{x\} \) is an \( \delta \)-open set of \( (X, \tau) \).

(2) \( \Rightarrow \) (1) Let \( A \) be an \( B\delta \)-closed set of \( (X, \tau) \). Let \( x \in \text{cl}_{\delta}(A) \). By (ii), \( \{x\} \) is either \( \delta \)-open or \( X - \{x\} \) is a \( B \)-set.

Case(a) : Let \( \{x\} \) be \( \delta \)-open. Since \( x \in \text{cl}_{\delta}(A) \), then \( \{x\} \cap A \neq \phi \). This shows that \( x \in A \).

Case(b) : Suppose that \( X - \{x\} \) is a \( B \)-set. If we assume that \( x \notin A \), then we would have \( x \in \text{cl}_{\delta}(A) - A \), which cannot happen according to Proposition 4.6. Hence \( x \in A \). So in both cases we have \( \text{cl}_{\delta}(A) \subseteq A \). Trivially \( A \subseteq \text{cl}_{\delta}(A) \). Therefore \( A = \text{cl}_{\delta}(A) \) or equivalently \( A \) is \( \delta \)-closed. Hence \( (X, \tau) \) is a \( 3T_{\delta g} \)-space.

\[ \square \]
References