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Solution of $x^2 - nx - 1 = 0$ by Continued Fraction Method and Comparison of the Solution by Newton Raphson Method

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Abstract: In this paper we find the solution of the quadratic equation $x^2 - nx - 1 = 0$ using Continued Fraction method and identify the convergence of the solutions with other numerical methods such as Bisection method, False Position method, Iteration method and Newton Raphson method.

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1. Introduction

Any real number has a unique decimal expansion, these expansions can be either finite or infinite [5]. For example in base 10 31/25 has decimal expansion 1.24, 1/3 has decimal expansion 0.3333... and π has decimal expansion 3.14159.... However in base 3 the decimal expansion of 31/25, 1/3 and π are 1.020110221, 0.1 and 10. 0102110122. From the above example we notice that not only the decimal expansions of real numbers change with different bases, but also whether the expansion is finite or infinite. Real numbers have another interesting expansion called a continued fraction expansion. The continued fraction expansion of a real number is base independent. Since these expansions are given by listing non negative integers, when we consider expansions in different bases the only thing that changes is how we represent those integers. Whether or not the expansion is finite or infinite does not change, even if we do change the base [6]. For example, in base 10, 31/25, 1/3 and π have continued fraction expansion [1; 4, 6], [0; 3] and [3; 7, 15, 1,...]. In base 3 the continued fraction expansions of those are [1; 11, 20], [0; 10], and [10; 21, 120, 1,...]. These expansions are unique [7]. An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{b_0}{a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{\ddots}}}}$$

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where a_i, b_i are real or complex numbers is called a continued fraction [1, 2]. An expression of the form

$$\frac{p}{q} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}$$

where $b_i = 1\ \forall i$, and $a_0, a_1, a_2, \ldots$ are each positive integers is called a simple continued fraction [1, 2]. The continued fraction is commonly expressed as $\frac{p}{q} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3+}\ldots$ or simply as $[a_0, a_1, a_2, a_3, \cdots]$. The elements $a_0, a_1, a_2, a_3, \ldots$ are called the partial quotients. If there are finite numbers of partial quotients, we call it finite simple continued fraction, otherwise it is infinite.

Notations:

(1). $\langle a_0, a_1, a_2, a_3, \ldots, a_n\rangle$-Continued fraction expansion.

(2). $[x_1]$-Integral part of the rational number x_1.

2. The Continued Fraction Algorithm [1, 2, 3]

Suppose we wish to find continued fraction expansion of $x \in R$. Let $x_0 = x$ and set $a_0 = [x_0]$. Define $x_1 = \frac{1}{x_0 - [x_0]}$ and set $a_1 = [x_1]$ and $x_2 = \frac{1}{x_1 - [x_1]} \Rightarrow a_2 = [x_2], \ldots, x_k = \frac{1}{x_{k-1} - [x_{k-1}]} \Rightarrow a_k = [x_k], \ldots$. This process is continued infinitely or to some finite stage till an $x_i \in N$ exists such that $a_i = [x_i]$.

Example 2.1.

(1). Continued fraction expansion of 414/283 = 1.4629 is [1; 2, 6, 4, 5].

(2). Continued fraction expansion of $\sqrt{3}$ and $\sqrt{7}$ are [1; 1, 2 , 1, 2, 1, 2,...] and [2; 1, 1, 1, 4, 1, 1, 1, 4,...]. This is known as periodic continued fraction. The above periodic continued fractions are also denoted by $[1; \overline{1, 2}]$ and $[2; \overline{1, 1, 1, 4}]$.

3. Properties and Relations [1,2]

One essential tool in studying the theory of continued fractions is the study of the convergence of a continued fraction.

Definition 3.1. *Let $x = [a_0, a_1, a_2, \ldots a_n]$. The reduced fractions given below are called the convergence of x and are defined by*

$$C_0 = \frac{p_0}{q_0} = a_0,\ C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1},\ C_2 = \frac{p_2}{q_2} = a_0 + \frac{1}{a_1 + \frac{1}{a_2}}, \ldots, C_n = \frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_n}}}}$$

where $\frac{p_n}{q_n}$ denote the n^{th} convergent of x. Let $p_0, p_1, p_2, \ldots, p_n$ denote the numerators of the convergence of x while $q_0, q_1, q_2, \ldots, q_n$ denotes the denominators. We define $p_{-2} = 0$; $p_{-1} = 1$; $q_{-2} = 1$; $q_{-1} = 0$. And define x_i as in the continued fraction algorithm, then the following relations hold.

(1). $p_k = a_k p_{k-1} + p_{k-2},\quad k \geq 0$

$q_k = a_k q_{k-1} + q_{k-2},\quad k \geq 0$

(2). $p_{k-1} q_k - p_k q_{k-1} = (-1)^k\ for\, k \geq -1$

(3). $x = \frac{p_{k-1} x_k + p_{k-2}}{q_{k-1} x_k + q_{k-2}}\ for\, k \geq 1$

(4). $x_k = \frac{p_{k-2} - q_{k-2}x_k}{p_{k-1} - q_{k-1}x_k}$.

Using continued fraction algorithm we find that the continued fraction of Golden mean $\varphi = \frac{1+\sqrt{5}}{2}$ is $[1;\ 1,\ 1,\ 1, ...]$ or $[1;\ \overline{1}]$. From this we observe that the continued fraction of Golden mean is periodic. Also Lagrange proved that every periodic continued fraction represents a quadratic irrational and vice versa, so that the above periodic continued fraction expansion represents the quadratic irrational as $x^2 - x - 1 = 0$. With this idea we try to find the quadratic irrationals of continued fractions of $[2\,;\ \overline{2}]$, $[3\,;\ \overline{3}]$, $[4\,;\ \overline{4}]$,.... The n$^{\text{th}}$ periodic continued fraction in this series is denoted by $T(n) = [n\,;\ \overline{n}]$. So that $T(1) = \varphi = [1\,;\ \overline{1}]$ and $T(n) = n + \frac{1}{n + \frac{1}{n + \ddots}}$ or $T(n) = n + \frac{1}{T(n)}$. This is known as silver mean. A Silver mean is a number $T(n)$ which has the property that it is n more than its reciprocal.

4. Quadratic Irrational of Silver Mean

$T(n) = [n\,;\ \overline{n,\ n}] = [n;\ x]$, where $x = [\overline{n,\ n}]$. Now $x = n + \frac{1}{x} \Rightarrow x = \frac{nx+1}{x} \Rightarrow x^2 - nx - 1 = 0$. Thus the quadratic irrational of $T(n) = [n\,;\ \overline{n}]$ is $x^2 - nx - 1 = 0$.

5. Periodic Continued Fraction of Quadratic Irrational $x^2 - nx - 1 = 0$

Let ξ_0 be the root of $x^2 - nx - 1 = 0$. Therefore $\xi_0 = \frac{n \pm \sqrt{n^2+4}}{2}$. Take $\xi_0 = \frac{m_0 \pm \sqrt{d}}{q_0}$, where $m_0 = n$, $d = n^2 + 4$, $q_0 = 2$ and $q_0 | d - m_0^2$. Set $a_0 = [\xi_0]$. So that $a_0 = n$. Again $\xi_1 = \frac{m_1 \pm \sqrt{d}}{q_1}$, where $m_1 = a_0 q_0 - m_0$ and $q_1 = \frac{d - m_1^2}{q_0}$. Therefore $\xi_1 = \frac{n \pm \sqrt{n^2+4}}{2}$. Again set $a_1 = [\xi_1]$. So that $a_1 = n$. Proceeding in this way we get $a_2 = a_3 = ... = a_n = ... = n$. Thus the continued fraction expansion of the quadratic irrational $x^2 - nx - 1 = 0$ is $[n\,;\ n,\ n,\ n, ...]$ or $[n\,;\ \overline{n,\ n}]$.

$T(1) = [1\,;\ \overline{1,\ 1}]$	$x^2 - x - 1 = 0$	$\frac{1+\sqrt{5}}{2} = 1.618040$
$T(2) = [2\,;\ \overline{2,\ 2}]$	$x^2 - 2x - 1 = 0$	$\frac{2+\sqrt{8}}{2} = 2.414213$
$T(5) = [5\,;\ \overline{5,\ 5}]$	$x^2 - 5x - 1 = 0$	$\frac{5+\sqrt{29}}{2} = 5.192582$
$T(10) = [10\,;\ \overline{10,\ 10}]$	$x^2 - 10x - 1 = 0$	$\frac{10+\sqrt{104}}{2} = 10.099020$
$T(16) = [16\,;\ \overline{16,\ 16}]$	$x^2 - 16x - 1 = 0$	$\frac{16+\sqrt{260}}{2} = 16.062258$
$T(32) = [32\,;\ \overline{32,\ 32}]$	$x^2 - 32x - 1 = 0$	$\frac{32+\sqrt{1028}}{2} = 32.031220$

Table 1. The quadratic irrationals of some Silver mean and their values are given above

In numerical methods there are so many methods available to find the real roots of quadratic irrationals and they have different rate of convergence. The same real roots also have periodic continued fraction expansion and are convergent. In this paper we try to find the roots of quadratic irrational of Silver means by Bisection, False position, Iteration and Newton Raphson method [4] and compare their values of each iterations with the convergence of continued fraction expansion.

6. Illustration

The successive approximations of the quadratic equation $x^2 - 5x - 1 = 0$ under the numerical methods mentioned above and the convergent of continued fraction are given below:

Iterations	Bisection	False position	Iteration	Newton Raphson	Continued fraction
1	5.5	5.166667	5.385165	5	5
2	5.25	5.189189	5.284489	5.2	5.2
3	5.125	5.192140	5.236644	5.192308	5.192308
4	5.1875	5.192525	5.213753	5.192593	5.192593
5	5.21875	5.192575	5.202765	5.192582	5.192582
6	5.203125	5.192531	5.197483		
7	5.195313	5.192576	5.194941		
8	5.191406	5.192582	5.193718		
9	5.193359	5.192582	5.193129		
10	5.192383		5.192846		
11	5.192871		5.192709		
12	5.192627		5.192643		
13	5.192505		5.192612		
14	5.192566		5.192597		
15	5.192596		5.192589		
16	5.192581		5.192585		
17	5.192589				
18	5.192585				
19	5.192583				
20	5.192582				

From the above table we notice that the successive approximations of the Newton Raphson method comes closure to the convergence of the continued fraction where as the successive approximations of the other methods are for away from the convergence of continued fraction . We now try to write the successive approximation of the Newton Raphson method as a continued fractions expansion.

7. Convergent of Continued Fractions

Let $x \in R$ and $x = [a_0, a_1, a_2, ...]$ where $a_0, a_1, a_2, ...$ are partial quotients of x. The first convergent of x is denoted by $C_0 = \frac{p_0}{q_0} = a_0$. The second convergent of x is denoted by $C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}$. In general the n$^{\text{th}}$ convergent of x is denoted by

$$C_n = \frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \frac{1}{a_n}}}$$

8. Identification of Newton Raphson Formula as a Continued Fraction for Silver Mean

Let x_0 be the initial approximation of $f(x) = 0$. The successive approximations of $f(x) = 0$ are denoted by $x_1, x_2, x_3, ...$ and are defined as

$$x_0 - \frac{f(x_0)}{f'(x_0)}, \; x_1 - \frac{f(x_1)}{f'(x_1)}, \; x_2 - \frac{f(x_2)}{f'(x_2)}$$

respectively. The n$^{\text{th}}$ approximation is

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

in which take $x_0 = b_0$ and $\left[-\frac{f'(x_0)}{f(x_0)}\right] = b_1$. Then the first approximation is $x_1 = b_0 + \frac{1}{b_1} = b_2$ (say). Next take

$$\left[-\frac{f'(x_1)}{f(x_1)}\right] = b_3$$

so that the second approximation is $x_2 = b_2 + \frac{1}{b_3} = b_4$ (say) or $x_2 = b_0 + \frac{1}{b_1+\frac{1}{b_3}} = b_4$. Again take

$$\left[-\frac{f'(x_2)}{f(x_2)}\right] = b_3$$

Thus the third approximation for $f(x) = 0$ is

$$x_3 = b_4 + \frac{1}{b_5} = b_6 \ \ (say) \ \ or \ \ x_3 = b_0 + \frac{1}{b_1 + \frac{1}{b_3+\frac{1}{b_5}}} = b_6.$$

Proceeding in this way, the n$^{\text{th}}$ approximation is

$$x_n = b_0 + \cfrac{1}{b_1 + \cfrac{1}{b_3 + \cfrac{1}{b_5 + \ddots + \cfrac{1}{b_{2n-1}}}}} = b_{2n}.$$

Where $b_0 = x_0$; $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$; $b_{2i+1} = \left[-\frac{f'(x_i)}{f(x_i)}\right]$, $i = 0, 1, 2, ...$ and $b_{2i} = x_i$, $i = 1, 2, 3, ...$ Thus the n$^{\text{th}}$ approximation of $f(x) = 0$ can be expressed as the continued fraction expansion

$$[b_0;\ b_1,\ b_3, ...b_{2n-1}].$$

9. Comparison Table

Convergence of Continued Fractions	Successive Iterations of the Newton Raphson Method
$C_0 = \frac{p_0}{q_0} = a_0$	$x_0 = b_0$
$C_1 = \frac{p_1}{q_1} = a_0 + \frac{1}{a_1}$	$x_1 = b_0 + \frac{1}{b_1}$
$C_2 = \frac{p_2}{q_2} = a_0 + \frac{1}{a_1+\frac{1}{a_2}}$	$x_2 = b_0 + \frac{1}{b_1+\frac{1}{b_3}}$
$\vdots$	$\vdots$
$C_n = \frac{p_n}{q_n} = a_0 + \cfrac{1}{a_1+\cfrac{1}{a_2+\ddots+\cfrac{1}{a_n}}}$	$x_n = b_0 + \cfrac{1}{b_1+\cfrac{1}{b_3+\cfrac{1}{b_5+\ddots+\cfrac{1}{b_{2n-1}}}}}$ where $b_{2i+1} = \left[-\frac{f'(x_i)}{f(x_i)}\right]$, $i = 0, 1, 2, ...$ and $b_{2i} = x_i$, $i = 1, 2, 3, ...$

10. Conclusion

It is observed that with reference to the silver mean the solution by Newton Raphson method can be identified with the convergence of the solution by continued fraction method. The convergence of other higher order equations can be studied by the convergence of continued fraction method.

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