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Chromatic Number to the Transformation (G^{---}) of K_n, W_n and F_n

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Abstract: Let $G = (V, E)$ be an undirected simple graph. The transformation graph G^{---} of G is a simple graph with vertex set $V(G) \cup E(G)$ in which adjacency is defined as follows: (a) two elements in $V(G)$ are adjacent if and only if they are non-adjacent in G, (b) two elements in $E(G)$ are adjacent if and only if they are non-adjacent in G, and (c) an element of $V(G)$ and an element of $E(G)$ are adjacent if and only if they are non-incident in G. In this paper, we determine the chromatic number of Transformation graph G^{---} for Complete, Wheel and Friendship graph.

Keywords: Complete Graph, Wheel Graph, Friendship graph, Chromatic Number, Transformation Graph.
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1. Introduction

In this paper, we are concerned with finite, simple graph. Let $G = (V(G), E(G))$ be a graph, if there is an edge e joining any two vertices u and v of G, we say u and v are adjacent. An n-vertex colouring or an n-colouring of a graph $G = (V, E)$ is a mapping $f : V \to S$, where S is a set of n-colours.

Definition 1.1. *A graph G is an ordered pair $(V(G), E(G))$ consisting of a non-empty set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$ of edges together with an incidence function ψ_G that associates with each edge of G is an unordered pair of vertices of G.*

Definition 1.2. *A colouring of a simple connected graph G is colouring the vertices of G such that no two adjacent vertices of G get the same colour. A graph is properly coloured if it is coloured with the minimum possible number of colours.*

Definition 1.3. *The chromatic number of a graph G is the minimum number of colours required to colour G properly and is denoted by $\chi(G)$.*

Definition 1.4. *The total graph $T(G)$ of a graph G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent in T if and only if they are either adjacent or incident in G.*

Definition 1.5. *The complement $\overline{G}$ of a graph G, which has $V(G)$ as it set of points and two points are adjacent in $\overline{G}$ if and only if they are not adjacent in G.*

Definition 1.6. *A wheel graph is a graph formed by connecting a single vertex to all vertices of cycle. A wheel graph with n-vertices is denoted by W_n, that is, $W_n = K_1 + C_{n-1}$, for every $n \geq 3$.*

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Definition 1.7. *A complete graph is a simple graph in which every pair of distinct vertices are connected by a unique edge.*

Definition 1.8. *A friendship graph is a simple graph which consists of n-triangles with a common vertex. It is denoted by F_n.*

In [2] generalized the concept of total graphs to a transformation graph G^{xyz} with $x, y, z; \{-,+\}$, where G^{+++} is the total graph of G, and G^{---} is its complement. Also, G^{--+}, G^{-+-} and G^{-++} are the complement of G^{++-}, G^{+-+} and G^{+--} respectively. Here, we investigate the transformation graph G^{---} of some graphs.

Lemma 1.9. *Let G be any simple graph and G^{---} is the transformation of G, then a colour can be given to three vertices of G^{---} if and only if either they formed a K_2 in G or a pair of edges are incident with a vertex in G.*

Lemma 1.10. *Let G be any path or cycle graph. If its transformation G^{---} has $3k-$vertices, then $\chi\left(G^{---}\right) = k$.*

2. Main Results

Theorem 2.1. *Let G be any simple graph and G^{---} is the transformation of G, then a colour can be assign to more than three vertices of G^{---} if and only if $d(v_i) \geq 3$, for all $v_i \in G$.*

Proof. Let G be any simple graph with n-vertices. Let $V\left(G^{---}\right) = \{v_i, e_j / i = 1, 2, \ldots, n;\ j = 1, 2, \ldots\}$ be the vertex set of G^{---}. Assume that, $d(v_i) \geq 3$, for all $v_i \in G$. Suppose v is a vertex in G and $\{e_j;\ (j = 1, 2, \ldots, k)\}$ are the edges incident with v in G. Clearly, $\{v, e_j;\ (j = 1, 2, \ldots, k)\}$ are independent vertices in G^{---}. Hence, in G^{---} we can give a single colour to the vertex v and the edges incident with v in G. Therefore, a single colour can be given to more than three vertices of G^{---}.
Conversely, assume that, a single colour can be given to more than three vertices of G^{---}.
To prove that, $d(v_i) \geq 3$, for all $v_i \in G$. Suppose, $d\left(v_i\right) = 2$, for all $v_i \in G$. Then the vertices in G^{---} form a pair of edges incident with a vertex in G. Then by Lemma 1.9, we can assign a single colour to exactly three vertices which is a contradiction to our assumption. Therefore, $d(v_i) \geq 3$, for all $v_i \in G$. Hence proved. □

Theorem 2.2. *Let $G = W_n$ be any wheel graph with n-vertices, then $\chi\left(G^{---}\right) = \left\lceil \frac{2(n-1)}{3} \right\rceil + 1$.*

Proof. Let $G = W_n$ be any path graph with n-vertices, whose vertices $\{v_i / i = 1, 2, \ldots, (n-1)\}$ are linear. Its transformation G^{---} has $(3n-2)$-vertices. Let $V\left(G^{---}\right) = \{v,\ v_i,\ e_j /\ \ i = 1, 2, \ldots, (n-1)\,;\ j = 1, 2, \ldots, 2(n-1)\}$ be the vertex set of G^{---}. Now, we divide the vertex set of G^{---} into three sets V_1, V_2 and V_3 such that

(1). $V_1 = \{v_n / n \equiv 1(mod\ 3)\}$

(2). $V_2 = \{v_n / n \equiv 0(mod\ 3)\}$

(3). $V_3 = \{v_n / n \equiv 2(mod\ 3)\}$

Case (1): If $n \equiv 1(mod\,3)$, that is $n = 3k+1$, we have $(9k+1)$-vertices in G^{---}, that is $\left|V(G^{---})\right| = 9k+1 = 6k+(3k+1)$. The $(6k)$-vertices of G^{---} form a cycle C_{n-1} with $(3k)$-vertices in G. By Lemma 1.10, we need $(2k)$-colours to these $(6k)$-vertices of $G^{---} \Rightarrow \left\lceil \frac{6k}{3} \right\rceil = \left\lceil \frac{2(3k)}{3} \right\rceil = \left\lceil \frac{2(n-1)}{3} \right\rceil$-colours. The independent set of $(3k+1)$-vertices in G^{---} are the vertex v and the edges incident with v in G. Since, these $(3k+1)$-vertices are independent and adjacent with the vertices which are coloured by the $\left\lceil \frac{2(n-1)}{3} \right\rceil$-colours. Hence, we need a new colour to colour these $(3k+1)$-vertices of G^{---}. Therefore, we need $\left(\left\lceil \frac{2(n-1)}{3} \right\rceil + 1\right)$-colours to colour the $(9k+1)$-vertices in G^{---}.

Case (2): If $n \equiv 0(mod\,3)$, that is $n = 3k$, we have $(9k-2)$-vertices in G^{---}, that is $\left|V(G^{---})\right| = 9k-2 = (6k-2)+(3k)$. The $(6k-2)$-vertices of G^{---} form a cycle C_{n-1} with $(3k-1)$-vertices in G. By Lemma 1.10, to colour $(6k-3)$-vertices, we need $(2k-1)$-colours. The $(6k-2)^{th}$-vertex of G^{---} is adjacent with the vertices which are coloured by the existing $(2k-1)$-colours. Hence, we need a new colour to colour the $(6k-2)^{th}$-vertex. Therefore, we need $(2k)$-colours to colour these $(6k-2)$-vertices of $C_{n-1} \Rightarrow \left\lceil \frac{6k-2}{3} \right\rceil = \left\lceil \frac{2(3k-1)}{3} \right\rceil = \left\lceil \frac{2(n-1)}{3} \right\rceil$-colours.
The independent set of $(3k)$-vertices in G^{---} are the vertex v and the edges incident with v in G. Since, these $(3k)$-vertices are independent and adjacent with the vertices which are coloured by the $\left\lceil \frac{2(n-1)}{3} \right\rceil$-colours. Hence, we need a new colour to colour these $(3k)$-vertices of G^{---}. Therefore, we need $\left(\left\lceil \frac{2(n-1)}{3} \right\rceil + 1\right)$-colours to colour the $(9k-2)$-vertices in G^{---}.
Case (3): If $n \equiv 2\,(mod\,3)$, that is $n = 3k+2$ and

$$\begin{aligned}\left|V\left(G^{---}\right)\right| &= 9k+4\\ &= (6k+2)+(3k+2)\,.\end{aligned}$$

The $(6k+2)$-vertices of G^{---} form a cycle C_{n-1} with $(3k+1)$-vertices in G. By Lemma 1.10, we need $(2k)$-colours to the $(6k)$-vertices of G^{---}. The $(6k+1)^{th}$ and $(6k+2)^{th}$ vertices of G^{---} are independent and adjacent with the vertices which are coloured by the existing $(2k)$-colours. Hence, we need a new colour to colour these two vertices. Therefore, we need $(2k+1)$-colours to colour these $(6k+2)$-vertices of $C_{n-1} \Rightarrow \left\lceil \frac{6k+2}{3} \right\rceil = \left\lceil \frac{2(3k+1)}{3} \right\rceil = \left\lceil \frac{2(n-1)}{3} \right\rceil$-colours. The independent set of $(3k+2)$-vertices in G^{---} are the vertex v and the edges incident with v in G. Since, these $(3k+2)$-vertices are independent and adjacent with the vertices which are coloured by the $\left\lceil \frac{2(n-1)}{3} \right\rceil$-colours. Hence, we need a new colour to colour these $(3k+2)$-vertices of G^{---}. Therefore, we need $\left(\left\lceil \frac{2(n-1)}{3} \right\rceil + 1\right)$-colours to colour the $(9k+4)$-vertices in G^{---}. Hence, in all the above cases we need $\left(\left\lceil \frac{2(n-1)}{3} \right\rceil + 1\right)$-colours to colour the $(3n-2)$-vertices of G^{---}. Therefore, $\chi\left(G^{---}\right) = \left\lceil \frac{2(n-1)}{3} \right\rceil + 1$. Hence, the theorem is proved. □

Theorem 2.3. *Let $G = F_n$ be the friendship graph with $(2n+1)$-vertices, then $\chi\left(G^{---}\right) = n+1$.*

Proof. Let $G = F_n$ be the friendship graph with $(2n+1)$-vertices. Let v be the vertex adjacent to all the $(2n)$-vertices in G. Hence, $V(G) = \{v, v_{i;\ (i=1,2,\ldots,2n)}\}$ be the vertex set of G and $E(G) = \{e_{j;\ \ (j=1,2,\ldots,3n)}\}$ be the edge set of G. Therefore, $V\left(G^{---}\right) = \{v,\ v_i,\ e_j/\ i = 1,2,\ldots,2n;\ j = 1,2,\ldots,3n\}$ be the vertex set of G^{---} and $\left|V\left(G^{---}\right)\right| = 5n+1$. Fix the vertex v and assign the colour c_0 to it. By the definition of G^{---} and F_n, The $(2n)$-edges incident with v in G are independent in G^{---}, so we can assign the same colour c_0 to these $(2n)$-vertices in G^{---}. The remaining $(3n)$-vertices of G^{---} form n-independent $K_2's$ in G. Therefore, the induced subgraph K_2 formed by the vertices v_{2i-1} and v_{2i} are adjacent with all the vertices and an edge of the remaining $(n-1)-K_2's$. Also, the induced subgraph in G^{---} form by the elements of each K_2 in G are adjacent with at least one vertex of G^{---} which was coloured by the colour c_0. Hence, we need new colours to colour these $(3n)$-vertices of G^{---}. By Lemma 1.9, we need n-colours to colour all the n-independent $K_2's$ of G in G^{---}. Therefore, we need $(n+1)$-colours to colour all the $(5n+1)$-vertices of G^{---}. Hence the proof. □

Theorem 2.4. *Let $G = K_n$ be any complete graph with n-vertices, then $\chi\left(G^{---}\right) = n-1$.*

Proof. Let $G = K_n$ be any complete graph with n-vertices, whose vertices $\{v_i/i = 1,2,\ldots,n\}$ are linear. Its transformation G^{---} has $\left(\frac{n(n+1)}{2}\right)$-vertices. Let $V\left(G^{---}\right) = \left\{v_i,\ e_j/\ i = 1,2,\ldots,n;\ j = 1,2,\ldots,\left(\frac{n(n-1)}{2}\right)\right\}$ be the vertex set of G^{---}. Fix the vertex v_1 in G^{---} and assign the colour c_1 to it. The $(n-1)$-edges incident with v_1 at G are independent in G^{---}, so we can assign the same colour c_1 to all these vertices in G^{---}. Now, choose the vertex v_2. In G^{---}, v_2 is adjacent to at least one of the $(n-1)$-edges incident with v_1 of G, so we can't give the colour c_1 to the vertex v_2. Hence, we need a new colour c_2 to colour the vertex v_2 in G^{---}. All the remaining $(n-2)$-edges incident with v_2 in G are (except the edge incident with v_1 which is already coloured) independent in G^{---}. Therefore, we can assign the same colour c_2 to these $(n-2)$-edges incident with v_2 of G in G^{---}.

Again, choose the vertex v_3. In G^{---}, v_3 is adjacent to at least one of the $(n-1)$-edges incident with v_1 and v_2 of G, so we can't give the colour c_1 and c_2 to the vertex v_3. Hence, we need a new colour c_3 to colour the vertex v_3 in G^{---}. All the remaining $(n-3)$-edges incident with v_3 in G (except the edges incident with v_1 and v_2 which is already coloured) are independent in G^{---}. Therefore, we can assign the same colour c_3 to these $(n-3)$-edges incident with v_3 of G in G^{---}. Repeat the above process to the vertices $\{v_4, v_5, \ldots, v_{n-2}\}$ and the corresponding edges incident with these vertices in G. From the above procedure we can conclude that, to colour the $(n-2)$-vertices of G^{---} we need $(n-2)$-colours. The remaining two vertices $\{v_{n-1},\ v_n\}$ and an edge form a K_2 in G and they are adjacent with all the $(n-2)$-colours (which are already used) in G^{---}. By Lemma 1.9, we need a new colour c_{n-1} to colour this K_2. Hence, we need $(n-1)$-colours to colour all the $\left(\frac{n(n+1)}{2}\right)$-vertices. Therefore, $\chi\left(G^{---}\right) = n-1$. Hence the theorem is proved. □

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