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Oscillation of Third-order Nonlinear Delay Difference Equation

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Abstract: Third-order nonlinear difference equation of the form $\Delta\left(c_n\Delta\left(d_n\Delta x_n\right)\right)+p_n\Delta x_{n+1}+q_nf\left(x_{n-\sigma}\right)=0, n\geq n_0$ are considered. Here, $\{c_n\}$, $\{d_n\}$, $\{p_n\}$, and $\{a_n\}$ are sequence of positive real number for $n_0\in N, f$ is a continuous function such that $f\left(u\right)/u\geq k>0$ for $u\neq 0$. By means of a Riccati transformation technique we obtain some new oscillation criteria. Examples are given to illustrate the importance of the results.

Keywords: Difference equation, Delay, Third order, Oscillation. Nonoscillation, Riccati transformation.
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1. Introduction

Consider the nonlinear delay difference equation

$$\Delta\left(c_n\Delta\left(d_n\Delta x_n\right)\right)+p_n\Delta x_{n+1}+q_nf\left(x_{n-\sigma}\right)=0,\quad n\geq n_0, \tag{1}$$

where $n_0\in N$ is fixed integer, Δ denotes the forward difference operator $\Delta x_n=x_{n+1}-x_n$, and σ is a nonnegative integer. The real sequence $\{c_n\}_{n=n_0}^{\infty}$, $\{d_n\}_{n=n_0}^{\infty}$, $\{p_n\}_{n=n_0}^{\infty}$, $\{q_n\}_{n=n_0}^{\infty}$, and the function f satisfy the following conditions:

$(h1)$ $\{d_n\}_{n=n0}^{\infty}$ is positive, $\lim\limits_{n\to\infty}R_1(n,s)=\infty$, where $R_1(n,s)=\sum\limits_{k=s}^{n}\frac{1}{d_k}$ for $n>s\geq n_0$;

$(h2)$ $\{c_n\}_{n=n_0}^{\infty}$ is positive, $\lim\limits_{n\to\infty}R_2\left(n,s\right)=\infty$, where $R_2\left(n,s\right)=\sum\limits_{k=s}^{n}\frac{1}{c_k}$ for $n>s\geq n_0$;

$(h3)$ $p_n\geq 0$, $q_n\geq 0$ and $q_n\neq 0$ for infinitely many values of $n\in N\left(n_0\right)$;

$(h4)$ $f\in C\left(R,R\right), f\left(u\right)/u\geq K$ for some $k>0$ and for all $u\neq 0$.

By a solution of Equation (1) we mean a nontrivial real sequence $\{x_n\}$ that is defined for $n\geq n_0-\sigma$ and satisfies Equation (1) for all $n\geq n_0$. clearly if $x_n=A_n$ for $n=n_0-\sigma,n_0-\sigma+1,\ldots,n_0-1$ are given, then Equation (1) has a unique solution satisfying the above initial conditions. A solution $\{x_n\}$ of Equation (1) is said to be oscillatory if is neither eventually positive nor eventually negative, and nonoscillatory otherwise. Equation (1) is called nonoscillatory if all its solutions are nonoscillatory.

The oscillation problem for difference equations has been investigated in recent years; for first-order, second-order, and higher-order equations, respectively; see [15,20,8,9,11,19,21]. For general theory of oscillation of difference equations, we refer to [1-3,14], and over 500 refer encescited therein. Compared to the second-order difference equations, the study of

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third-order difference equations has received considerably less attention in the literature even though such equations arises in economics, mathematical biology, and other areas of Mathematics (5). Some recent results on third-order difference equations can be found [10,24-31]. However, it seems that there is much less known regarding the oscillation of Equation (1).

There are many papers dealing with the oscillatory and asymptotic behaviour of solution of difference and differential equations, see for instance Jiang [12], Jiang and Li [13], Li [17], Li and Yeh [16], Li [17], Luo [18], Philos [22], saker [26, 27]. The oscillatory behaviour of solution of difference equations and that of their discrete analogs may be quite different. For instance, the differential equation

$$y^{m} + 8y = 0$$

admits a nonoscillatory solution $y_1(t) = e^{-2t}$ and a pair of oscillatory solutions $y_2(t) = e^t \cos\sqrt{3t}$ and $y_3(t) = e^t \sin\sqrt{3t}$, but the difference equation

$$\Delta^3 x_n + 8x_n = 0$$

Which is a discrete analog of the above difference equation, has there oscillatory solutions $x_n^1 = (-1)^n$, $x_n^2 = \left(\sqrt{7}\right)^n \cos\left[n\left(\arctan\sqrt{3/2}\right)\right]$, and $x_n^3 = \left(\sqrt{7}\right)^n \sin\left[n\left(\arctan\sqrt{3/2}\right)\right]$. We note that Equation (1) may be considered as a discrete analog of the delay differential equation

$$\left(c(t)\left(d(t)x'\right)'\right)' + p(t)x' + q(t)f(x(t-\sigma)) = 0 \tag{2}$$

For some work regarding the oscillation of Equation (2), we refer to Saker [27] ($p(t) \equiv 0$) and Tiryaki and Aktas [32] and the references cited therein. A number of dynamical behaviours of solutions of difference equations are possible; here we will only be concerned with conditions which are sufficient for every solution of Equation (1) to be either oscillatory or convergent to zero as $n \to \infty$. Recently, Saker [26] has established some new conditions which are sufficient for all solution of

$$\Delta\left(c_n \Delta\left(d_n \Delta x_n\right)^\gamma\right) + q_n f\left(x_{n-\sigma}\right) = 0, n \geq n_0, \tag{3}$$

Where $\gamma \geq 1$ is quotient of odd positive integers, to be either oscillatory or tend to zero as $n \to \infty$. Our aim in this paper is to present some new oscillation criteria for Equation (1) by making use of a Riccati type transformation and arguments developed for differential equations in [32]. It should be noted that the results obtained in this paper extend and improve the related ones in [26]. The paper is organized as follows. In section 2, we will present some lemmas which are useful in establishing our main results. In Section 3 we will state and prove the main results and give examples to illustrate them.

2. Preparatory Lemmas

We begin with the following useful lemma.

Lemma 2.1. *Suppose*

$(h5)$ $\Delta\left(c_n \Delta z_n\right) + \frac{p_n}{d_{n+1}} z_{n+1} = 0$ *is nonoscillatory.*

If $\{x_n\}$ is a nonoscillatory solution of Equation (1) for $n \geq n_0$, then there exists a $n_1 \geq n_0$ such that either $x_n\left(d_n \Delta x_n\right) > 0$ or $x_n\left(d_n \Delta x_n\right) < 0$ for all $n \geq n_1$.

Proof. Suppose that $\{x_n\}$ is a nonoscillatory solution of Equation (1) for $n \geq n_0$. Without loss of generality, we may take $x_n > 0$ and $x_{n-\sigma} > 0$, $n \geq n_1 \geq n_0$. We see that $y_n = -d_n \Delta x_n$ is a solution of the order non homogeneous difference equation

$$\Delta\left(c_n \Delta y_n\right)+\frac{p_n}{d_{n+1}} y_{n+1}=q_n f\left(x_{n-\sigma}\right), \quad n \geq n_1 \tag{4}$$

Indeed, since x_n is a oscillatory solution of Equation (1) we have

$$\Delta\left(c_n \Delta\left(-d_n \Delta x_n\right)\right)+\frac{p_n}{d_{n+1}}\left(-d_{n+1} \Delta x_{n+1}\right)=q_n f\left(x_{n-\sigma}\right)$$

and hence

$$\Delta\left(c_n \Delta\left(d_n \Delta x_n\right)\right)+p_n \Delta x_{n+1}+q_n f\left(x_{n-\sigma}\right)=0$$

We claim that solution of (4) are nonoscillatory. We may assume that $z_n>0$ for $n \geq n_1$. Note that $\{-z_n\}$ is also a solution. Let y_n be a oscillatory solution of (4). There exist $n_3>n_2>n_1$ such that $yn_3 \geq 0$, $yn_{3+1} \leq 0$, $yn_2 \leq 0$ and $yn_{2+1} \geq 0$. Summing

$$\Delta\left(c_n\left(y_{n+1} z_n-y_n z_{n+1}\right)\right)=z_{n+1} q_n f\left(x_{n-\sigma}\right)$$

From n_2 to n_3-1, we have

$$c_{n3}\left(y_{n3+1} z_{n3}-z_{n3+1} y_{n3}\right)-c_{n2}\left(y_{n2+1} z_{n2}-z_{n2+1} y_{n2}\right)=\sum_{k=n2}^{n3-1} z_{k+1} q_k f\left(x_{k-\sigma}\right),$$

a contradiction. The proof is complete. □

Definition 2.2. *Let $\{x_n\}$ be a solution of Equation (1). we say that the solution $\{x_n\}$ has property v_2 if there exists $n_* \geq n_0$ such that*

$$x_n \Delta x_n>0, x_n \Delta\left(d_n \Delta x_n\right)>0, x_n \Delta\left(c_n \Delta\left(d_n \Delta x_n\right)\right) \leq 0$$

For every $n \geq n_*$.

Lemma 2.3. *Let the assumption (h2) hold and $\{x_n\}$ be a nonoscillatory solution of Equation (1) such that $x_n\left(d_n \Delta x_n\right) \geq 0$ for every $n \geq n_1 \geq n_0$. Then $\{x_n\}$ has property V_2.*

Proof. Let $\{x_n\}$ be a eventually positive solution of Equation (1) Then there exists an $n_1 \geq n_0$ such that $x_{n-\sigma}>0$ for $n \geq n_1$. Since $x_n\left(d_n \Delta x_n\right)>0$ for every $n \geq n_1 \geq n_0$, we have $\Delta x_n>0$ for $n \geq n_1$. From Equation (1) we have $\Delta\left(c_n \Delta\left(d_n \Delta x_n\right)\right) \leq 0$ for $n \geq n_1$. Then $\Delta\left(d_n \Delta x_n\right)$ is monotone and eventually of one sign. We claim that there is a $n_2 \geq n_1$ such that for $n \geq n_2$, $\Delta\left(d_n \Delta x_n\right)>0$. Suppose to the contrary that $\Delta\left(d_n \Delta x_n\right) \leq 0$ for $n \geq n_2$. Since $c_n>0$ and $c_n \Delta\left(d_n \Delta x_n\right)>0$ is nonincreasing there exists a negative constant C and an $n_3 \geq n_2$ such that $c_n \Delta\left(d_n \Delta x_n\right) \leq C$ for $n \geq n_3$. Dividing both sides by c_n and summing from n_3 to $n-1$, we obtain

$$d_n \Delta x_n \leq d_{n3} \Delta x_{n3}+C \sum_{k=n3}^{n-1} \frac{1}{c_k}.$$

Letting $n \rightarrow \infty$, we see that $d_n \Delta x_n \rightarrow-\infty$ by a contradiction with the fact that $\Delta x_n>0$. Then $\Delta\left(d_n \Delta x_n\right)>0$. The proof is complete. □

Lemma 2.4. *Let $\{x_n\}$ be a solution of Equation (1) and $\{g_n^*\}_{n=n_0}^{\infty}$ be a sequence of integers which satisfies*
(h6) $g_n^* \leq n-\sigma-1$ and $\lim_{n \rightarrow \infty} g_n^*=\infty$.
If $\{x_n\}$ be a property V_2, then there exists an $n_1 \geq n_0$ such that $d_{n-\sigma} \Delta x_{n-\sigma} \geq R_2\left(n-\sigma-1, g_n^*\right) c_n \Delta\left(d_n \Delta x_n\right)$ for $n \geq n_1$.

Proof. Let $\{x_n\}$ be a solution of Equation (1) which has property v_2. Without loss of generality, we may also assume that $x_n > 0$ and $x_{n-\sigma} > 0$, $n \geq n_1 \geq n_0$. Since $\lim_{n\to\infty} g_n^* = \infty$ and $\Delta x_n > 0$, $\Delta(d_n \Delta x_n) > 0$, and $\Delta(c_n \Delta(d_n \Delta x_n)) \leq 0$ for every $n \geq n_2 \geq n_1$,

$$d_{n-\sigma}\Delta x_{n-\sigma} = d_{g_n^*}\Delta x_{g_n^*} + \sum_{k=g_n^*}^{n-\sigma-1} \frac{c_k \Delta(d_k \Delta x_k)}{c_k} \geq R_2(n-\sigma-1, g_n^*)\, c_n \Delta(d_n \Delta x_n)$$

and then we have

$$d_{n-\sigma}\Delta x_{n-\sigma} \geq R_2(n-\sigma-1, g_n^*)\, c_n \Delta(d_n \Delta x_n).$$

The proof is complete. □

Lemma 2.5. *Let μ_n be a positive sequence defined for $n \geq n_0$ and set*

$$\phi_n = d_{n+2}\Delta(c_{n+1}\Delta \mu_n) + \mu_n p_n.$$

Furthermore assume that the following conditions are satisfied:

$(h7)$ $\Delta\mu_n \geq 0, \phi_n \geq 0, \Delta(d_{n+2}\Delta(c_{n+1}\Delta\mu_n)) \geq 0\, (or \Delta(\mu_n p_n) \leq 0)$ *for* $n \geq n_0$*;*

$(h8)$ $\sum_{n=n_0}^{\infty}(K\mu_n q_n - \Delta\phi_n) = \infty$, *where* $K\mu_n q_n - \Delta\phi_n \geq 0$*for* $n \geq n_0$.

If $(h1)$ holds and $\{x_n\}$ is a nonoscillatory solution of Equation (1) which satisfies $x_n(c_n\Delta x_n) \leq 0$ for n sufficiently large, then $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Equation (1). Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for $n \geq n_1 \geq n_0$ for some n_1 sufficiently large. The proof when $\{x_n\}$ is eventually negative is similar, as the substitution $y_n = -x_n$ transforms Equation (1) into an equation of the same form. Since $x_n(c_n\Delta x_n) \leq 0$ for n sufficiently large, Δx_n becomes nonpositive for all $n \geq n_2$ for some $n_2 \geq n_1$. Let $\lim_{n\to\infty} x_n = \lambda \geq 0$. Assume that $\lambda \neq 0$. There exists an $n_3 \geq n_2$ such that $x_n \geq \lambda$ for $n \geq n_3$. Summing Equation (1) from n_3 to $n-1$, we obtain from that

$$\mu_n c_n \Delta(d_n \Delta x_n) \leq C_1 - \lambda \sum_{k=n3}^{n-1}(K\mu_k q_k - \Delta\phi_k),$$

where C_1 is a constant. Employing we see from (2) that $\mu_n c_n \Delta(d_n \Delta x_n)$ must take on negative values for n sufficiently large. By using $(h1)$ we see that x_n must be eventually negative, a contradiction. Hence $\lambda = 0$. This complete the proof. □

3. Oscillation Criteria

In this section we gave the main of our paper.

Theorem 3.1. *Assume that $(h1)-(h8)$ hold, and that there exists a positive sequence $\{\rho_n\}_{n=n_0}^{\infty}$ such that s*

$$\limsup_{n\to\infty} \sum_{k=n_0}^{n}\left[K\rho_k q_k - \frac{d_{k-\sigma}\left(\Delta\rho_k d_{k+1} - \rho_k p_k R_2(k-\sigma-1, g_k^*)\right)^2}{4\rho_k R_2(k-\sigma-1, g_k^*)\, d_{k+1}^2}\right] = \infty. \tag{5}$$

Then every solution $\{x_n\}$ of Equation (1) is either oscillatory or satisfies $x_n \to 0$ as $n \to \infty$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Equation (1) Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ eventually. From Lemma 2.1 it following that $\Delta x_n > 0$ or $\Delta x_n < 0$ for $n \geq n_1 \geq n_0$. If $\Delta x_n > 0$ for $n \geq N \geq n_1$

then $\{x_n\}$ has property V_2 by Lemma 2.2. We define $w_n = \rho_n \frac{c_n \Delta(d_n \Delta x_n)}{x_{n-\sigma}}, n \geq N$. Then, $w_n > 0$ and in view of Equation (1) by employing Lemma 2.3 we have

$$\begin{aligned}
\Delta w_n &= -\frac{f\left(x_{n-\sigma}\right)}{x_{n-\sigma}} \rho_n q_n - \frac{\rho_n}{x_{n-\sigma}} p_n \Delta x_{n+1} + \frac{c_{n+1} \Delta\left(d_{n+1} \Delta x_{n+1}\right) x_{n-\sigma} \Delta \rho_n}{x_{n-\sigma} x_{n-\sigma+1}} - \frac{c_{n+1} \Delta\left(d_{n+1} \Delta x_{n+1}\right) \rho_n \Delta x_{n-\sigma}}{x_{n-\sigma} x_{n-\sigma+1}} \\
&\leq -K \rho_n q_n - \left(w_{n+1}^2\left(\frac{\rho_n R_2\left(n-\sigma-1, g_n^*\right)}{\left(\rho_{n+1}\right)^2 d_{n-\sigma}}\right) - w_{n+1}\left(\frac{\Delta \rho_n}{\rho_{n+1}} - \frac{p_n \rho_n R_2\left(n-\sigma-1, g_n^*\right)}{d_{n+1} \rho_{n+1}}\right)\right) \\
&= -K \rho_n q_n - A_n w_{n+1}^2 + w_{n+1} B_n, \qquad (6)
\end{aligned}$$

Where

$$A_n = \frac{\rho_n R_2\left(n-q-1, g_n^*\right)}{\rho_{n+1}^2 d_{n-\sigma}}, B_n = \frac{\Delta \rho_n}{\rho_{n+1}} - \frac{p_n q_n R_2\left(n-q-1, g_n^*\right)}{d_{n+1} \rho_{n+1}}.$$

Completing the square in (3.2) we obtain

$$\Delta w_n < -\left[K \rho_n q_n - \frac{B_n^2}{4 A_n}\right]. \qquad (7)$$

Summing (7) from N to n, we obtain

$$-w_N < w_{n+1} - w_N < -\sum_{k=N}^{n}\left[K \rho_k q_k - \frac{B_k^2}{4 A_k}\right]$$

which yields

$$\sum_{k=N}^{n}\left[K \rho_k q_k - \frac{B_k^2}{4 A_n}\right] < w_N$$

For all large n and this is contrary to (5). If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4 we have $\lim_{n \to \infty} x_n = 0$. The proof is complete. □

Example 3.2. *Consider the third order delay difference equation*

$$\Delta^3 x_n + \frac{1}{5n^2} \Delta x_{n+1} + \left(1 - \frac{1}{5n^2}\right) x_{n-3} = 0, \quad n \geq 1 \qquad (8)$$

Note that $\Delta^2 z_n + \frac{1}{5n^2} z_{n+1} = 0$ is nonoscillatory by [1]. Taking $\mu_n = \rho_n = 1$ and $g_n^* = n-4$, we have

$$\limsup_{n \to \infty} \sum_{k=n_0}^{n}\left[K \rho_k q_k - \frac{d_{k-\sigma}\left(\Delta \rho_k d_{k+1} - \rho_k p_k R_2\left(k-\sigma-1, g_k^*\right)\right)^2}{4 \rho_k R_2\left(k-\sigma-1, g_k^*\right) d_{k+1}^2}\right] = \sum_{k=1}^{\infty}\left(1 - \frac{1}{5k^2} - \frac{1}{100k^4}\right) = \infty.$$

Thus, condition (5) is satisfied. The other conditions of Theorem 3.1 are also satisfied. Hence every solution $\{x_n\}$ of Equation (8) is either oscillatory or satisfies $x_n \to 0$ as $n \to \infty$. We note that $\left\{\cos \frac{n\pi}{3}\right\}$ is an oscillatory solution of Equation (8).

Example 3.3. *consider the their order difference equation*

$$\Delta^3 x_n + \frac{1}{2^{n+1}} \Delta x_{n+1} + \frac{1}{8}\left(1 + \frac{1}{2^n}\right) x_n = 0, \quad n \geq 1. \qquad (9)$$

Note that $\Delta^3 z_n + \frac{1}{2^{n+1}} z_{n+1} = 0$ is nonoscillatory [1]. Taking $\mu_n = \rho_n = 1$ and $g_n^* = n-1$, condition (5) is satisfied. The other conditions of Theorem 3.1 are also satisfied. Hence, every solution $\{x_n\}$ of Equation (9) is either oscillatory or satisfies $x_n \to 0$ as $n \to \infty$. Indeed, the sequence $\left\{2^{-n}\right\}$ is such a solution of Equation (9).

Theorem 3.4. *Assume that $(h1)-(h8)$ hold. Let $\{\rho_n\}_{n=n_0}^{\infty}$ be a positive sequence and $\{H_{m,n}\}$, $m \geq n \geq n_0$, be a double sequence such that*

(1). $H_{m,m} = 0$ *for* $m \geq n_0$*;*

(2). $H_{m,n} > 0$ *for* $m > n \geq n_0$*;*

(3). $\Delta_2 H_{m,n} = H_{m,n+1} - H_{m,n} \leq 0$ *and* $-\Delta_2 H_{m,n} = h_{m,n}\sqrt{H_{m,n}}$ *for* $m \geq n \geq n_0$*.*

If

$$\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \left[K H_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] = \infty \quad for \quad every \quad m_0 \geq n_0, \tag{10}$$

where

$$Q_{m,n} = h_{m,n} - \left(\frac{\Delta\rho_n}{\rho_{n+1}} - \frac{p_n\rho_n R_2\left(n-\sigma-1, g_n^*\right)}{d_{n+1}\rho_{n+1}} \right)\sqrt{H_{m,n}},\, A_n = \frac{\rho_n R_2\left(n-\sigma-1, g_n^*\right)}{\rho_{n+1}^2 d_{n-\sigma}},$$

Then every solution $\{x_n\}$ *of Equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$*.*

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Equation (1) which may assume to be eventually positive. Proceeding as in the proof of Theorem 3.1 we arrive at the inequality 3.2 Then we see that

$$\begin{aligned} \sum_{n=N}^{m-1} K H_{m,n}\rho_n q_n &\leq \sum_{n=N}^{m-1} H_{m,n}\left(-\Delta w_n + w_{n+1}B_n - A_n w_{n+1}^2\right) \\ &= H_{m,N}w_N + \sum_{n=N}^{m-1}\left\{w_{n+1}\Delta_2 H_{m,n}\left(B_n w_{n+1} - A_n w_{n+1}^2\right)\right\} \\ &= H_{m,N}w_N - \sum_{n=N}^{m-1}\left\{w_{n+1}^2 A_n H_{m,n} + w_{n+1}\left(h_{m,n}\sqrt{H_{m,n}} - H_{m,n}B_n\right)\right\} \\ &\leq H_{m,N}w_N + \sum_{n=N}^{m-1}\frac{(h_{m,n} - B_n\sqrt{H_{m,n}})^2}{4A_n}, \end{aligned} \tag{11}$$

where B_n is as defined in the proof of Theorem 3.1. Thus we obtain

$$\frac{1}{H_{m,N}} \sum_{n=N}^{m-1} \left[K H_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] \leq w_N,$$

Where clearly contradicts (10). If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4, we have $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

As an immediate consequence of Theorem 3.2 we get the following corollary.

Corollary 3.5. *Assume that all the assumption of Theorem 3.2 holds, except that the condition* (10) *is replaced by*

(1). $\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} H_{m,n}\rho_n q_n = \infty$ *for every* $m_0 \geq n_0$*,*

(2). $\limsup_{m\to\infty} \frac{1}{H_{m,m_0}} \sum_{n=m_0}^{m-1} \frac{Q_{m,n}^2}{A_n} < 0$ *for every* $m_0 \geq n_0$*.*

Then, every solution $\{x_n\}$ *of Equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$*.*

Example 3.6. *Consider the third order delay difference equation*

$$\Delta^3 x_n + \frac{9}{2^{n+1}}\Delta x_{n+1} + \frac{27}{32}\left(1 - \frac{1}{2^n}\right)x_{n-2} = 0 \tag{12}$$

Taking $\mu_n = \rho_n = 1, g_n^* = n-3$ *and* $H_{m,n} = m-n$ *condition* (10) *is satisfied. The other condition of Theorem 3.2 are also satisfied. Hence every solution* $\{x_n\}$ *of Equation* (12) *is either oscillatory or satisfies* $x_n \to 0$ *as* $n \to \infty$*. The sequence* $\{(-1/2)^n\}$ *is such a solution of Equation* (12)*.*

Example 3.7. *Consider the third order delay difference equation*

$$\Delta^3 x_n + \frac{27}{32} x_{n-2} = 0, \quad n \geq 1 \tag{13}$$

Taking $\mu_n = \rho_n = 1, g_n^* = n - 3$ *and* $H_{m,n} = m - n$*, we have*

$$\lim_{m\to\infty} \sup \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \left[K H_{m,n} \rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] = \limsup_{m\to\infty} \frac{1}{m-1} \sum_{n=1}^{m-1} \left[\frac{27}{32}(m-n) - \frac{1}{4(m-n)} \right] = \infty.$$

Thus, condition (10) *is satisfied. The other condition of theorem are also satisfied. Hence every solution* $\{x_n\}$ *of Equation is either oscillatory or satisfies* $x \to 0$ *as* $n \to \infty$*. The sequence* $\{2^{-n}\}$ *is a solution of Equation* (13)*.*

Remark 3.8. *One may choose* $\{H_{m,n}\}$ *in appropriate manners, to derive several special oscillation criteria for Equation* (1) *Some choices are*

$$\begin{aligned}
H_{m,n} &= (m-n)^{\lambda}, \lambda \geq 1, m \geq n \geq n_0, \\
H_{m,n} &= \left(\log \frac{m+1}{n+1} \right)^{\lambda}, \lambda \geq 1, m \geq n \geq n_0, \\
H_{m,n} &= (m-n)^{(\lambda)}, \lambda > 2, m \geq n \geq n_0,
\end{aligned}$$

Where $(m-n)^{(\lambda)} = (m-n)(m-n+1)\ldots(m-n+\lambda-1)$*.*

Theorem 3.9. *Let* $\{H_{m,n}\}$ *and* $\{h_{m,n}\}$ *be as in Theorem 3.2 and let*

$$0 < \inf_{n\geq n_0} \left[\liminf_{m\to\infty} \frac{H_{m,n}}{H_{m,n_0}} \right] \leq \infty \tag{14}$$

$$\limsup_{m\to\infty} \frac{1}{H_{m,n_0}} \sum_{n=n}^{m-1} \frac{Q_{m,n}^2}{A_n} < \infty \tag{15}$$

If there is a sequence $\{\Psi_N\}$ *such that*

$$\limsup_{m\to\infty} \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} \left[K H_{m,n} \rho_n q_n - \frac{Q_{m,n}^2}{4A_n} \right] \geq \psi_N \quad for \quad every \quad N \geq n_0 \tag{16}$$

and

$$\sum_{n=n_0}^{\infty} A_n \left[\psi_{n+1}^{+} \right]^2 = \infty, \quad where \quad \psi_{n+1}^{+} = \max\{\psi_{n+1}, 0\} \tag{17}$$

Then every solution $\{x_n\}$ *of Equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$*.*

Proof. As in the proof of Theorem 3.2, we have (11), Then, we have

$$\sum_{n=N}^{m-1} K H_{m,n} \rho_n q_n \leq H_{m,N} w_N + \sum_{n=N}^{m-1} \frac{Q_{m,n}^2}{4A_n} - \sum_{n=N}^{m-1} \left[w_{n+1} \sqrt{A_n H_{m,n}} + \frac{Q_{m,n}}{2\sqrt{A_n}} \right]^2 \tag{18}$$

The remainder of the proof of this case is similar to ones given in [16,18] and hence is omitted. □

Example 3.10. *Consider the third order difference equation*

$$\Delta^2 \left(\frac{1}{n^3} \Delta x_n \right) + \frac{2n+1}{[n(n+1)]^2} \left(x_n + x_n^3 \right) = 0, \quad n \geq 1. \tag{19}$$

We take $\mu_n = n^2$, $\rho_n = 1$, $g_n^* = n-1$, $H_{m,n} = m-n$ *and* $\psi_n = \frac{1}{n^2}$. *In view of* $Q_{m,n} = \frac{1}{\sqrt{m-n}}$ *and* $A_n = n^3$, *we see that*

$$\limsup_{m\to\infty} \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} \frac{Q_{m,n}^2}{A_n} = \limsup_{m\to\infty} \frac{1}{m-1} \sum_{n=1}^{m-1} \frac{1}{(m-n)\, n^3} = 0 < \infty,$$

$$\sum_{n=n_0}^{\infty} A_n \left[\psi_{n+1}^+\right]^2 = \sum_{n=1}^{\infty} \frac{n^3}{(n+1)^4} = \infty$$

and

$$\limsup_{m\to\infty} \frac{1}{H_{m,n}} \sum_{n=N}^{m-1} \left[K H_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n}\right] = \limsup_{m\to\infty} \frac{1}{m-N} \sum_{n=N}^{m-1} \left[(m-n)\left(\frac{1}{n^2} - \frac{1}{(n+1)^2} - \frac{1}{4\,(m-n)\,n^3}\right)\right] = \frac{1}{N^2} = \psi_N.$$

Since the conditions of Theorem 3.3 hold, every solution $\{x_n\}$ *of Equation* (19) *is either oscillatory or satisfies* $x_n \to 0$ *as* $n \to \infty$.

Theorem 3.11. *Let* $\{H_{m,n}\}$ *and* $\{h_{m,n}\}$ *be as in Theorem 3.2 and let* (14) *hold. Suppose that*

$$\liminf_{m\to\infty} \frac{1}{H_{m,n_0}} \sum_{n=n_0}^{m-1} K H_{m,n}\rho_n q_n < \infty, \tag{20}$$

and there is a sequence $\{\psi_N\}$ *satisfying* (17) *and*

$$\liminf_{m\to\infty} \frac{1}{H_{m,N}} \sum_{n=N}^{m-1} \left[K H_{m,n}\rho_n q_n - \frac{Q_{m,n}^2}{4A_n}\right] \geq \psi_N \quad for \quad every \quad N \geq n_0 \tag{21}$$

Then every solution $\{x_n\}$ *of Equation* (1) *either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.3 and hence is omitted. □

Theorem 3.12. *Assume that* (h1)*-*(h8) *hold. Suppose there exists a positive sequence* $\{\rho_n\}_{n=n_0}^{\infty}$ *and a sequence* $\{F_{m,n}\}_{m,n=n_0}^{\infty}$ *such that* $1 + \frac{F_{m,n}}{\rho_{n+1}} + \frac{p_n \rho_n R_2}{d_{n+1}\rho_{n+1}} - \frac{\Delta \rho_n}{\rho_{n+1}} \geq 0$ *and*

$$\limsup_{m\to\infty} \sum_{n=n_0}^{m} \left[\prod_{k-n_0}^{n-1}\left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right]\left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) = \infty, \tag{22}$$

Then every solution $\{x_n\}$ *of Equation* (1) *is either oscillatory or* $\lim_{n\to\infty} x_n = 0$.

Proof. Let $\{x_n\}$ be a nonoscillatory solution of Equation (1) Without loss of generality, we may assume that $x_n > 0$ and $x_{n-\sigma} > 0$ for some $N \geq n_0$. Proceeding as in the proof of Theorem 3.1 we arrive at

$$\Delta w_n \leq -K\rho_n q_n + w_{n+1}B_n - w_{n+1}^2\left(\frac{\rho_n R_2}{\rho_{n+1}^2 d_{n-\sigma}}\right), \quad n \geq N \tag{23}$$

From (23) and Young's inequality, we have

$$\Delta w_n \leq -K\rho_n q_n + w_{n+1}B_n - w_{n+1}^2 \frac{\rho_n R_2}{\rho_{n+1}^2 d_{n-\sigma}} - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2 + \frac{1}{4A_N}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2, \quad n \geq N$$

or

$$w_{n+1} - w_n \leq -K\rho_n q_n + w_{n+1}\left(B_n - \frac{F_{m,n}}{\rho_{n+1}}\right) + \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2, \quad n \geq N$$

It follows that

$$\sum_{n=N}^{m} \left[\prod_{k=N}^{n-1}\left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right]\left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) \leq w_N.$$

Hence

$$\limsup_{m\to\infty} \sum_{n=N}^{m} \left[\prod_{k=N}^{n-1}\left(1 + \frac{F_{m,k}}{\rho_{k+1}} - B_k\right)\right]\left(K\rho_n q_n - \frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right) \leq w_N,$$

Which contradict (22). If $\Delta x_n < 0$ for $n \geq N$, then by Lemma 2.4 we have $\lim_{n\to\infty} x_n = 0$. The proof is complete. □

Example 3.13. *Consider the third order difference equation*

$$\Delta^2\left(\frac{1}{n^5}\Delta x_n\right)+\frac{1}{4n^3}x_n\left(\beta+e^{x_n}\right)=0,\quad n\geq 1, \tag{24}$$

where $\beta>1$ *and* $f(u)=u\left(\beta+e^u\right)$ *with* $K=\beta$. *Taking* $\mu_n=n^2$, $\rho_n=n$, $g_n^*=n-1$, *and* $F_{m,n}=n^2$, *we have*

$$\limsup_{m\to\infty}\sum_{n=1}^{m}\left[\prod_{k=1}^{n-1}\left(1+\frac{F_{m,k}}{\rho_{k+1}}-B_k\right)\right]\left(K\rho_n q_n-\frac{1}{4A_n}\left(\frac{F_{m,n}}{\rho_{n+1}}\right)^2\right)=\frac{\beta-1}{4}\limsup_{m\to\infty}\sum_{n=1}^{\infty}\frac{(n-1)!}{n^2}=\infty$$

Thus, condition (22) is satisfied. The other conditions of Theorem 3.6 are also satisfied. Hence every solution $\{x_n\}$ *of Equation (24) is either oscillatory or satisfies* $x_n\to 0$ *as* $n\to\infty$.

In the proof of following theorem we use a generalized Riccati transformation technique.

Theorem 3.14. *Assume that (h1)-(h8) holds. Let* $\{\rho_n\}_{n=n_0}^{\infty}$ *be a positive sequence. Furthermore, we assume that there exists a double sequence* $-\{H_{m,n}:m\geq n\geq n_0\}$ *such that (i)-(iii). If*

$$\limsup_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}\left[H_{m,n}\psi-\frac{h_{m,n}^2}{4A_n}\right]=\infty\quad for\quad every\quad m_0\geq n_0 \tag{25}$$

where

$$\psi_n=\rho_n\left(Kq_n-\frac{p_n^2 d_{n-\sigma}R_2}{4d_{n+1}^2}-\Delta\left(c_{n-\sigma}\alpha_{n-1}\right)\right),\quad \alpha_n=-\frac{\left(\Delta\rho_n d_{n+1}-p_n\rho_n R_2\right)d_{n-\sigma}}{2d_{n+1}\rho_n R_2 c_{n-\sigma+1}}.$$

Then every solution $\{x_n\}$ *of Equation (1) is either oscillatory or* $\lim_{n\to\infty}x_n=0$.

Proof. We proceed as in Theorem 3.1, take $x_{n-\sigma}>0$ for all $n\geq N$ for some N sufficiently large. Define

$$w_n=\rho_n\left[\frac{c_n\Delta\left(d_n\Delta x_n\right)}{x_{n-\sigma}}+c_{n-\sigma}\alpha_{n-1}\right],\quad n\geq N.$$

Then follows the proof of Theorem 3.1, we obtain

$$\begin{aligned}\Delta w_n &\leq -K\rho_n q_n+\frac{w_{n+1}}{\rho_{n+1}}\left(\Delta\rho_n-\frac{p_n\rho_n R_2}{d_{n+1}}\right)-\left(\frac{\rho_n R_2}{d_{n-\sigma}}\right)\left[\frac{w_{n+1}}{\rho_{n+1}}-c_{n-q+1}\alpha_n\right]^2+\rho_n\Delta\left(c_{n-\sigma}\alpha_{n-1}\right)+\frac{p_n\rho_n R_2}{d_{n+1}}\left(c_{n-\sigma-1}\alpha_n\right)\\ &=-\psi_n-A_n w_{n+1}^2,\quad n\geq N\end{aligned}$$

The remainder of proof is similar to that of the Theorem 3.2 and hence is omitted. If $\Delta x_n<0$ for $n\geq N$, then by Lemma 2.4, we have $\lim_{n\to\infty}x_n=0$. The proof is complete. □

Example 3.15. *Consider the third order difference equation*

$$\Delta^3 x_n+2c^{1-1/2^n}\left(\sqrt{c}-1\right)^3 x_n\left(4-\cos x_n\right)=0,\quad c>1,\quad n\geq 1. \tag{26}$$

Taking $\mu_n=\rho_n=c^{1/2^n}$, $g_n^*=n-1$ *and* $H_{m,n}=m-n$, *we have*

$$\limsup_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}\left[H_{m,n}\psi_n-\frac{h_{m,n}^2}{4A_n}\right]=\limsup_{m\to\infty}\frac{1}{m-m_0}\sum_{n=m_0}^{m-1}\left[6c\left(\sqrt{c}-1\right)^3(m-n)-\frac{1}{4(m-n)}\right]=\infty.$$

Thus, condition (25) is satisfied. The other conditions of Theorem 3.6 are also satisfied. Hence every solution $\{x_n\}$ *of Equation (26) is either oscillatory or* $\lim_{n\to\infty}x_n=0$.

Corollary 3.16. *Assume that all the assumptions of Theorem 3.5 hold, except that the condition (25) is replaced*

(1). $\limsup_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}H_{m,n}\rho_n\left(Kq_n-\frac{p_n^2\rho_n d_{n-\sigma}R_2}{4d_{n+1}^2}-\Delta\left(c_n-\sigma\alpha_{n-1}\right)\right)=\infty$ *for every* $m_0\geq n_0$,

(2). $\lim_{m\to\infty}\frac{1}{H_{m,m_0}}\sum_{n=m_0}^{m-1}\frac{h_{m,n}^2}{A_n}<\infty$ *for every* $m_0\geq n_0$.

Then, every solution $\{x_n\}$ *of Equation (1) is either oscillatory or* $\lim_{n\to\infty}x_n=0$.

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