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On 3-absorbing Hyperideals of Multiplicative Hyperring

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Abstract: Let R be a multiplicative hyperring. In this research, we learn 3-absorbing hyperideal which is an extension of prime hyperideal. A non zero hyperideal J of a multiplicative hyperring R is called a 3-absorbing hyperideal of R if whenever $x, y, z, w \in Q$ and $x \cdot y \cdot z \cdot w \cdot \in J$, then either $x \cdot y \cdot \cdot z \in J$ or $y \cdot z \cdot w \in J$ or $x \cdot z \cdot w \in J$ or $x \cdot y \cdot w \in J$. A number of results concerning 3-absorbing hyperideals and examples of 3-absorbing primary ideals are given.

MSC: 16Y19, 20N20.

Keywords: Hyperideal, 2-absorbing Hyperideals 3-absorbing Hyperideal.

1. Introduction

Marty was the first researcher, how gave the idea of hyperstructure theory. In 1934, he started to study hypergroups [7]. Hyperstructure theory is still developing area of mathematics and many mathematicians have research in this field, See [3, 7, 9]. Hyperstructures have various applications in applied and pure sciences such as Latices, Geometry, Cryptography, Automata and Artificial Intelligence. In the sence of Marty, a hypergroup is a nonempty set H endowed by a hyperstructure $* = H \times H \to P^*(H)$, where $P^*(H)$ is the set of all nonempty subsets of H, which satisfy the associative law and product axiom.

The hyperrings were introduced by Krasner [5]. Krasner hyperrings are a generalization of classical rings in which the multiplication is a binary operation while the addition is a hyperoperation. The another type of hyperrings called Multiplicative hyperring was introduced and studied by Rota in 1982 [9], which was subsequently investigated by many authors [4, 5, 9, 10]. A multiplicative hyperring is a hyperstructure $(R, +, \cdot)$, where $(R, +)$ is an additive abelian group, $(R, \cdot)$ is a semihypergroup and $\cdot$ is distributive over $+$. For nonempty subsets $H, K \subset R$ and $s \in R$, we define $H \cdot K = \cup(h \cdot k)$, where $h \in H, k \in K$ and $K \cdot s = K \cdot \{s\}$.

Dasgupta investigated and studied the prime hyperideal and primary hyperideal of multiplicative hyperring and discussed some basic properties and useful results of primary and prime hyperideal of multiplicative hyperring [4]. In 2014, Ghiasvand generalized the idea of prime hyperideal to 2-absorbing hyperideal in a conference on algebra and its applications [6]. Latter in 2017, a researcher Anborloei extend this idea to 2-absorbing prime and 2-absorbing primary hypperideals of a multiplicative hyperring [2]. This work is the extension of 2-absorbing hyperideals to 3-absorbing hyperideals. In this research, we proved some useful results on 3-absorbing prime hyperideals.

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1.1. Preliminaries

Throughout this paper $(R, +, \cdot)$ denotes the multiplicative hyperring.

Definition 1.1 (Hyperring [9]). *$(R, +, \cdot)$ is called multiplicative hyperring if*

(1). $(Q, +)$ is an abelian hypergroup.

(2). $(R, \cdot)$ is semihypergroup.

(3). $\forall\ x, y, z \in R$, we have $x \cdot (y + z) \subseteq x \cdot y + x \cdot z$.

(4). $\forall\ x, y, z \in R$, we have $(y + z) \cdot x \subseteq y \cdot x + z \cdot x$.

(5). $\forall\ x, y \in R$, we have $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$.

Definition 1.2 (Hyperideal [5]). *A nonempty subset J of a hyperring R is a hyperideal*

(1). If $x, y \in J$, then $x - y \in J$.

(2). If $z \in J$ and $s \in R$ then $z \cdot s \in J$.

Definition 1.3 (Prime Hyperideal [5]). *A hyperideal P of a hyperring R is called a prime hyperideal if whenever $x \cdot y \in P$ then either $a \in P$ or $b \in P$.*

Definition 1.4 (Radical [5]). *Let J be a hyperideal of the R. Then the radical of J is denoted by $\sqrt{J}$, defined as $\sqrt{J} = \{a; a^n \in J\ for\ some\ n \in \mathbb{N}\}$.*

Definition 1.5 (2-absorbing Hyperideal [6]). *A non zero hyperideal of a multiplicative hyperring R is called 2-absorbing if for all $x, y, z \in R$ $x \cdot y \cdot z \subseteq J$, then $x \cdot y \subseteq J$ or $y \cdot z \subseteq J$ or $x \cdot z \subseteq J$.*

2. On 3-absorbing Hyperideal of Multiplicative Hyperring

Definition 2.1. *A non zero hyperideal J of a multiplicative hyperring R is called a 3-absorbing hyperideal of R if for any $x, y, z, w \in R$ and $x \cdot y \cdot z \cdot w \in J$, then either $x \cdot y \cdot z \in J$ or $y \cdot z \cdot w \in J$ or $x \cdot z \cdot w \in J$ or $x \cdot y \cdot w \in J$.*

Remark 2.2. *Every 3-absorbing hyperideal need not to be 2-absorbing hyperideal.*

Theorem 2.3. *Let J be a 3-absorbing hyperideal of hyperring R, then $Rad(J)$ is a 3-absorbing hyperideal of R and $x^3 \in J$ for every $x \in Rad(J)$.*

Proof. Since J is 3-absorbing ideal of R this implies that $x^3 \in J$ for every $x \in Rad(J)$. Now, let $x_1, x_2, x_3, x_4 \in R$ such that $x_1x_2x_3x_4 \in Rad(J)$, then $(x_1x_2x_3x_4)^3 = x_1^3x_2^3x_3^3x_4^3 \in J$ for $x_1, x_2, x_3, x_4 \in Rad(J)$. As J is 3-absorbing, so we can conclude that $x_1^3x_2^3x_3^3 = (x_1x_2x_3)^3, x_1^3x_2^3x_4^3 = (x_1x_2x_4)^3, x_1^3x_3^3x_4^3 = (x_1x_3x_4)^3, x_2^3x_3^3x_4^3 = (x_2x_3x_4)^3 \in J$ this implies that either $x_1x_2x_3 \in Rad(J)$ or $x_1x_2x_4 \in Rad(J)$ or $x_1x_3x_4 \in Rad(J)$ or $x_2x_3x_4 \in Rad(J)$. Hence $Rad(J)$ is a 3-absorbing hyperideal of R. □

Lemma 2.4. *Let K a prime hyperideal of hyperring R and J is a hyperideal of hyperring R where $J \subseteq K$. Then the following points are equivalent.*

(1). K is a minimal prime ideal of J.

(2). For every $x \in K$, there is a $y \in R \setminus K$ and a positive integer n such that $yx^n \in J$.

Theorem 2.5. *Let J be a 3-absorbing hyperideal of a hyperring R, then there are at most 3 prime hyperideal of R minimal over J.*

Proof. Suppose on contrary, there are 4 prime hyperideals J_1, J_2, J_3, J_4 of R, which are minimal over over J. Let $x_1 \in J_1 \setminus J_2 \cup J_3 \cup J_4$, $x_2 \in J_2 \setminus J_1 \cup J_3 \cup J_4$, $x_3 \in J_3 \setminus J_2 \cup J_1 \cup J_4$ and $x_4 \in J_4 \setminus J_2 \cup J_3 \cup J_1$. By lemma there exist $a_1 \in R \setminus J_1$, $a_2 \in R \setminus J_2$, $a_3 \in R \setminus J_3$ and $a_4 \in R \setminus J_4$ such that $a_1 x_1^{n1} \in J$, $a_2 x_2^{n2} \in J$, $a_3 x_3^{n3} \in J$ and $a_4 x_4^{n4} \in J$. Since J is 3-absorbing hyperideal of a hyperring R, $J \subseteq J_4$, $x_1, x_2, x_3 \nsubseteq J_4$ and $a_1 x_1^2$, $a_2 x_2^2$, $a_3 x_3^2 \in J$, hence $(a_1 + a_2 + a_3) x_1^2 x_2^2 x_3^2 \in J$. Since $x_1 \in J_1 \setminus J_2 \cup J_3 \cup J_4$, $x_2 \in J_2 \setminus J_1 \cup J_3 \cup J_4$, $x_3 \in J_3 \setminus J_2 \cup J_1 \cup J_4$ and $x_4 \in J_4 \setminus J_2 \cup J_3 \cup J_1$ and $b_1 x_1^2$, $b_2 x_2^2$, $b_3 x_3^2 \in J \subseteq J_1 \cap J_2 \cap J_3$ this implies that $a_1 \in (J_2 \cap J_3) \setminus J_1$, $a_2 \in (J_1 \cap J_3) \setminus J_2$, $a_3 \in (J_1 \cap J_2) \setminus J_3$, thus $a_1 + a_2 + a_3 \nsubseteq J_1, J_2, J_3$. Hence $(a_1 + a_2 + a_3) x_2^2 x_3^2 \nsubseteq J_1$, $(a_1 + a_2 + a_3) x_1^2 x_3^2 \nsubseteq J_2$, $(a_1 + a_2 + a_3) x_1^2 x_2^2 \nsubseteq J_3$, so $(a_1 + a_2 + a_3) x_1^2 x_2^2$,$(a_1 + a_2 + a_3) x_2^2 x_3^2$,$(a_1 + a_2 + a_3) x_1^2 x_3^2 \nsubseteq J$, it means $x_1^2 x_2^2 x_3^2 \in J \subseteq J_4$. Since J is a 3-absorbing hyperideal of R, but then $x_1 x_2 x_3 \subseteq J_4$, which is a contradiction, Hence there are at most 3 prime ideals of R over J. □

Theorem 2.6. *Suppose that J is a 3-absorbing hyperideal of hyperring R, then following statements hold.*

(1). $Rad(J) = K$ is a prime hyperideal of R such that $K^3 \subseteq J$.

(2). $Rad(J) = K_1 \cap K_2 \cap K_3$, $Rad(J)^3 \subseteq J$ and $K_1 K_2 K_3 \subseteq J$, where K_1, K_2 and K_3 are prime hyperideals of hyperring R which are distinct and minimal over J.

Proof.

(1). Let $Rad(J) = K$ be a prime hyperideal of R and $x_1, x_2, x_3 \in K$ then by theorem 1, we have $x_1^3, x_2^3, x_3^3 \in J$. Let $x_1 x_2 x_3 (x_1 + x_2 + x_3) \in J$, since J is 3-absorbing hyperideal, then either $x_1 x_2 (x_1 + x_2 + x_3) \in J$ or $x_1 x_3 (x_1 + x_2 + x_3) \in J$ or $x_2 x_3 (x_1 + x_2 + x_3) \in J$ or $x_1 x_2 x_3 \in J$. Hence $x_1 x_2 x_3 \in J$ this implies $K^3 \subseteq J$.

(2). Suppose that $Rad(J) = K_1 \cap K_2 \cap K_3$, where K_1, K_2, K_3 are distinct prime hyperideals which are minimal over J. Let $x_1, x_2, x_3 \in Rad(J)$, then $x_1 x_2 x_3 \in J$ by the same argument given above, hence $Rad(J) \subseteq J$.

Now, we show that $K_1 K_2 K_3 \subseteq J$. For each $m \in Rad(J)$, $m^3 \in J$ then by theorem 1. Let $y \in Rad(J)$ and $x_1 \in K_1 \setminus K_2 \cup K_3$, $x_2 \in K_2 \setminus K_1 \cup K_3$, $x_3 \in K_3 \setminus K_1 \cup K_2$ then by theorem 3 $x_1 x_2 x_3 \in J$ and $x_1 + y \in K_1 \setminus K_2 \cup K_3$. Thus $x_2 (x_1 + y) x_3 = x_1 x_2 x_3 + y x_2 x_3 \in J$, hence $x_1 x_2 x_3 \in J$ and $K_1 K_2 K_3 \subseteq J$. □

Theorem 2.7. *Let J be a 3-absorbing ideal and $Rad(J) = K$ is a prime ideal of R such that $J \neq K$, then $J_x = \{y \in R | yx \in J\}$ is a 2-absorbing hyperideal of R containing K for each $x \in K \setminus J$.*

Proof. Let $x \in K \setminus J$, since $K^3 \subseteq J$ (by theorem 4) this implies that $K \subseteq J_x$. Suppose that $K \neq J_x$ and $x_1 x_2 x_3 \in J_x$ for some $x_1, x_2, x_3 \in R$. Since $K \subseteq J_x$,let $x_1, x_2, x_3 \nsubseteq K$ this implies $x_1 x_2 x_3 \nsubseteq J$. Since $x_1 x_2 x_3 \in J_x$, so we have $y x_1 x_2 x_3 \in J$. As J is 3-absorbing hyperideal of R and $x_1 x_2 x_3 \nsubseteq J$ this implies that either $y x_1 x_2 \in J$ or $y x_2 x_3 \in J$ or $y x_1 x_3 \in J$ and $x_1 x_2 \in J_x$ or $x_2 x_3 \in J_x$ or $x_1 x_3 \in J_x$. Hence J_x is 2-absorbing hyperideal of R containing V. □

Theorem 2.8. *Let J be a 3-absorbing ideal of R such that $J \neq Rad(J) = K_1 \cap K_2 \cap K_3$, where K_1, K_2, K_3 are non zero prime hyperideals of hyperring R which are distinct and minimal over J, then $J_x = \{y \in R \mid xy \in J\}$ is 2-absorbing hyperideal of R containing K_1, K_2, K_3 for each $x \in Rad(J) \setminus J$.*

Proof. Let $x \in Rad(J) \setminus J$ and $K_1K_2K_3 \subseteq J$ (Theorem 4). We can conclude that $xJ_1 \subseteq J$, $xJ_2 \subseteq J$ and $xJ_3 \subseteq J$. Thus $K_1, K_2, K_3 \subseteq J_x$. Suppose $x_1x_2x_3 \in J_x$ for some $x_1, x_2, x_3 \in R$. Since $K_1, K_2, K_3 \subseteq J_x$, we may assume that $x_1x_2x_3 \nsubseteq J$. As $x_1x_2x_3 \in J_x$ this implies that $x_1x_2x_3x \in J$, J is 3-absorbing hyperideal and $x_1x_2x_3 \nsubseteq J$. We come to an end that $x_1x_2x \in J$ or $x_1x_3x \in J$ or $x_2x_3x \in J$ from this we conclude that either $x_1x_2 \in J_x$ or $x_1x_3 \in J_x$ or $x_2x_3 \in J_x$. Hence J_x is 2-absorbing hyperideal of R. □

Theorem 2.9. *Suppose that J is a hyperideal of hyperring R such that $J \neq Rad(J) = K_1 \cap K_2 \cap K_3$ where K_1, K_2, K_3 are non zero prime hyperideals of hyperring R which are distinct and minimal over J, if $J_x = \{y \in R \mid yx \in J\}$ is 2-absorbing hyperideal of R for $x \in (K_1 \cup K_2 \cup K_3 \setminus J$, then J is 3-absorbing hyperideal of R.*

Proof. Let $xx_1x_2x_3 \in J$. Assume that $x \in (K_1 \cup K_2 \cup K_3) \setminus J$, thus $x_1x_2x_3 \in J_x$, since J_x is 2-absorbing hyperideal of hyperring R (bytheorem 5), we come to an end that either $xx_1x_2 \in J$ or $xx_2x_3 \in J$ or $xx_1x_3 \in J$. Hence J is a 3-absorbing hyperideal of hyperring R. □

Theorem 2.10. *Let J be a 3-absorbing hyperideal of R then $J_x = \{y \in R | yx \in J\}$, where $x \in R \setminus J$ is 3-absorbing hyperideal of R containing J.*

Proof. $x_1x_2x_3x_4 \in J_x$ for $x_1,\ x_2,\ x_3,\ x_4 \in R$, then $(xx_1)x_2x_3x_4 \in J$, so either $(xx_1)x_2x_3 \in J$ or $(xx_1)x_2x_4 \in J$ or $x_2x_3x_4 \in J$. Hence J_x is 3-absorbing hyperideal of R containing J. □

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