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Generalization of an Inequality with Sum of Fractions

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Abstract: Cvetkovski introduced the inequality: $\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\geq a+b+c$ in his book *Inequalities: Theorems, Techniques and Selected Problems*. Starting from there we generalize this inequality to the case with non-negative real number exponents: $\frac{a^m+b^m}{a^n+b^n}+\frac{b^m+c^m}{b^n+c^n}+\frac{c^m+a^m}{c^n+a^n}\geq a^{m-n}+b^{m-n}+c^{m-n}$ for $m\geq n$. Moreover, we also discuss the order of the sum of fractions when the exponents in the fractions change, and prove the inequality $\frac{a^m+b^m}{a^n+b^n}+\frac{b^m+c^m}{b^n+c^n}+\frac{c^m+a^m}{c^n+a^n}\geq\frac{a^{m-l}+b^{m-l}}{a^{n-l}+b^{n-l}}+\frac{b^{m-l}+c^{m-l}}{b^{n-l}+c^{n-l}}+\frac{c^{m-l}+a^{m-l}}{c^{n-l}+a^{n-l}}$ for real numbers $m\geq n\geq l\geq 0$. Versions of the two inequalities with k-variables are also examined.

MSC: 26D15.

Keywords: Chebyshev's Sum Inequality, Rearrangement Inequality, Fibonacci Number.

1. Introduction

Mathematical Inequality is an important topic to study. It provides insights, and sometimes shortcuts as well, into many optimization or extreme value problems. There are more and more research focusing on it, and more and more books discussing this area. Hardy, Littlewood, and Pólya's classical work: *Inequalities* [3] is a great example. Recently, Cvetkovski also composed a book (see [2]) using problem-based structure to introduce many useful inequalities and problem-solving techniques applying these inequalities. This is the book I studied, and chose to use in my class. The inspiration of the research in this paper also comes from it. In 2018, Lai and Risher proved the following inequalities (see [5]):

Theorem 1.1. *Let a,b,c be positive real numbers, and let $m\geq n$ be non-negative real numbers. Then*

$$\frac{a^m}{b^n}+\frac{b^m}{c^n}+\frac{c^m}{a^n}\geq a^{m-n}+b^{m-n}+c^{m-n}.$$

Theorem 1.2. *Let a,b,c be positive real numbers, and let $m\geq n\geq p$ be non-negative real numbers. Then*

$$\frac{a^m}{b^n}+\frac{b^m}{c^n}+\frac{c^m}{a^n}\geq\frac{a^{m-p}}{b^{n-p}}+\frac{b^{m-p}}{c^{n-p}}+\frac{c^{m-p}}{a^{n-p}}.$$

While studying Cvetkovski's book, we notice that he introduced a seemingly similar inequality as an exercise problem (see [2]).

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Example 1.3. *Let a, b, c be positive real numbers. Prove the inequality*

$$\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a} \geq a+b+c.$$

Naturally, we start to think about generalizing it to a result similar to Theorem 1.1: $\frac{a^m+b^m}{a^n+b^n}+\frac{b^m+c^m}{b^n+c^n}+\frac{c^m+a^m}{c^n+a^n} \geq a^{m-n}+b^{m-n}+c^{m-n}$, for real numbers $m \geq n \geq 0$. However, the proof of the exercise provided by the author applied Bergströms Inequality (in the book it is listed as a corollary of Cauchy-Schwartz Inequality, see [1] for information about Bergströms Inequality), which can not be used to prove the generalization when the exponents are real numbers due to the restriction of the method itself. We therefore use another technique, Chebyshev's Sum Inequality, to prove this generalization, and extend the result to a k variable case. We also conjecture that an inequality similar to Theorem 1.2 may also be true: $\frac{a^m+b^m}{a^n+b^n}+\frac{b^m+c^m}{b^n+c^n}+\frac{c^m+a^m}{c^n+a^n} \geq \frac{a^{m-l}+b^{m-l}}{a^{n-l}+b^{n-l}}+\frac{b^{m-l}+c^{m-l}}{b^{n-l}+c^{n-l}}+\frac{c^{m-l}+a^{m-l}}{c^{n-l}+a^{n-l}}$, for real numbers $m \geq n \geq l \geq 0$. After further examination, we confirm our conjecture, and then extend the result to a k-variable case as well. Since we will be using these two inequalities in our proofs, we would like to state Chebyshev's Sum Inequality and Rearrangement Inequality here. Interested readers may check [2] or [3] for their proofs and applications.

Theorem 1.4 (Chebyshev's Sum Inequality)**.** *If the real number sequences $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ have the same increasing or decreasing order, then*

$$\frac{1}{n}\sum_{i=1}^{n}(a_i \cdot b_i) \geq \left(\frac{1}{n}\sum_{i=1}^{n}a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}b_i\right).$$

If the sequences have the opposite order, then the inequality is reversed.

Theorem 1.5 (Rearrangement Inequality)**.** *Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers. For any permutation $(x_1, x_2, \cdots, x_n)$ of $(a_1, a_2, \cdots, a_n)$ we have the following inequalities:*

$$a_1b_1+a_2b_2+\cdots+a_nb_n \geq x_1b_1+x_2b_2+\cdots+x_nb_n \geq a_nb_1+a_{n-1}b_2+\cdots+a_1b_n.$$

We also refer to [2] as our main source of notations.

2. Main Result

As mentioned earlier, we first generalize the exercise to the following result.

Theorem 2.1. *Let a, b, c be positive real numbers, and m, n be non-negative real numbers such that $m \geq n$. Then*

$$\frac{a^m+b^m}{a^n+b^n}+\frac{b^m+c^m}{b^n+c^n}+\frac{c^m+a^m}{c^n+a^n} \geq a^{m-n}+b^{m-n}+c^{m-n}.$$

Proof. Since $m \geq n \geq 0$, $\{a^n, b^n\}$ and $\left\{a^{m-n}, b^{m-n}\right\}$ have the same increasing or decreasing order. Applying Chebyshev's Sum Inequality we have

$$2\left(a^m+b^m\right) \geq \left(a^{m-n}+b^{m-n}\right)\cdot\left(a^n+b^n\right),$$

or equivalently,

$$\frac{a^m+b^m}{a^n+b^n} \geq \frac{a^{m-n}+b^{m-n}}{2}.$$

Similarly,

$$\frac{b^m+c^m}{b^n+c^n} \geq \frac{b^{m-n}+c^{m-n}}{2}, \qquad \frac{c^m+a^m}{c^n+a^n} \geq \frac{c^{m-n}+a^{m-n}}{2}.$$

The claimed inequality is then achieved by summing the above three inequalities. □

Notice that if m and n are both negative real numbers (still with $m \geq n$), or if $m \geq 0$ and $n \leq 0$, the inequality is reversed. Here we show an interesting example involving Fibonacci numbers.

Example 2.2. *Prove that*

$$\frac{F_k^5 + F_{k+1}^5}{F_k + F_{k+1}} + \frac{F_{k+1}^5 + F_{k+2}^5}{F_{k+1} + F_{k+2}} + \frac{F_{k+2}^5 + F_k^5}{F_{k+2} + F_k} \geq \frac{1}{2}\left(F_k^2 + F_{k+1}^2 + F_{k+2}^2\right)^2,$$

where F_i indicates the i^{th} Fibonacci number.

Proof. Applying Theorem 2.1, we know that

$$\frac{F_k^5 + F_{k+1}^5}{F_k + F_{k+1}} + \frac{F_{k+1}^5 + F_{k+2}^5}{F_{k+1} + F_{k+2}} + \frac{F_{k+2}^5 + F_k^5}{F_{k+2} + F_k} \geq F_k^4 + F_{k+1}^4 + F_{k+2}^4.$$

According to Candido's identity (see [4], p.88): $2\left(F_k^4 + F_{k+1}^4 + F_{k+2}^4\right) = \left(F_k^2 + F_{k+1}^2 + F_{k+2}^2\right)^2$, the claimed inequality is then proved. □

Using a similar method shown in the proof of Theorem 2.1, we can easily generalize this result to a k-variable version.

Theorem 2.3. *Let $a_1, \cdots, a_k$ be positive real numbers, and m, n be non-negative real numbers such that $m \geq n$. Then for a fix natural number $p < k$,*

$$\frac{a_1^m + \cdots + a_p^m}{a_1^n + \cdots + a_p^n} + \cdots + \frac{a_k^m + a_1^m + \cdots + a_{p-1}^m}{a_k^n + a_1^n + \cdots + a_{p-1}^n} \geq a_1^{m-n} + \cdots + a_k^{m-n}.$$

Notice that the fractions at the left side of the above inequality is not symmetric. It is cyclically chosen throughout $a_1, \cdots, a_k$. We now discuss our conjecture with only three variables.

Theorem 2.4. *Let a, b, c be positive real numbers, and m, n, l be non-negative real numbers such that $m \geq n \geq l$. Then*

$$\frac{a^m + b^m}{a^n + b^n} + \frac{b^m + c^m}{b^n + c^n} + \frac{c^m + a^m}{c^n + a^n} \geq \frac{a^{m-l} + b^{m-l}}{a^{n-l} + b^{n-l}} + \frac{b^{m-l} + c^{m-l}}{b^{n-l} + c^{n-l}} + \frac{c^{m-l} + a^{m-l}}{c^{n-l} + a^{n-l}}.$$

Proof. It suffices to prove just

$$\frac{a^m + b^m}{a^n + b^n} \geq \frac{a^{m-l} + b^{m-l}}{a^{n-l} + b^{n-l}},$$

or equivalently

$$(a^m + b^m) \cdot \left(a^{n-l} + b^{n-l}\right) \geq (a^n + b^n) \cdot \left(a^{m-l} + b^{m-l}\right).$$

After we develop the products at both sides and simplify the whole expression, the above inequality is equivalent to

$$a^{m-n+l} + b^{m-n+l} \geq a^l b^{m-n} + a^{m-n} b^l,$$

which is apparently true due to Rearrangement Inequality. □

Before we generalize this result to a k-variable case, we would also like to show an interesting example first.

Example 2.5. *For positive real numbers a, b, c, prove that*

$$\frac{\sqrt{a^3} + \sqrt{b^3}}{a + b} + \frac{\sqrt{b^3} + \sqrt{c^3}}{b + c} + \frac{\sqrt{c^3} + \sqrt{a^3}}{c + a} \geq \frac{a + b}{\sqrt{a} + \sqrt{b}} + \frac{b + c}{\sqrt{b} + \sqrt{c}} + \frac{c + a}{\sqrt{c} + \sqrt{a}}.$$

Proof. The inequality is true when $m = \frac{3}{2}$, $n = 1$, and $l = \frac{1}{2}$ in Theorem 2.4. □

Theorem 2.6. *Let $a_1, \cdots, a_k$ be positive real numbers, and m, n, l be real numbers such that $m \geq n \geq l$. Then for a fix natural number $p < k$,*

$$\frac{a_1^m + \cdots + a_p^m}{a_1^n + \cdots + a_p^n} + \cdots + \frac{a_k^m + a_1^m + \cdots + a_{p-1}^m}{a_k^n + a_1^n + \cdots + a_{p-1}^n} \geq \frac{a_1^{m-l} + \cdots + a_p^{m-l}}{a_1^{n-l} + \cdots + a_p^{n-l}} + \cdots + \frac{a_k^{m-l} + a_1^{m-l} + \cdots + a_{p-1}^{m-l}}{a_k^{n-l} + a_1^{n-l} + \cdots + a_{p-1}^{n-l}}.$$

Proof. Just like the proof of Theorem 2.4, we only need to prove that

$$\frac{a_1^m + \cdots + a_p^m}{a_1^n + \cdots + a_p^n} \geq \frac{a_1^{m-l} + \cdots + a_p^{m-l}}{a_1^{n-l} + \cdots + a_p^{n-l}},$$

or equivalently,

$$\left(a_1^m + \cdots + a_p^m\right) \cdot \left(a_1^{n-l} + \cdots + a_p^{n-l}\right) \geq \left(a_1^n + \cdots + a_p^n\right) \cdot \left(a_1^{m-l} + \cdots + a_p^{m-l}\right).$$

We notice that

$$\left(a_i^m a_j^{n-l} + a_j^m a_i^{n-l}\right) - \left(a_i^n a_j^{m-l} + a_j^n a_i^{m-l}\right) = a_i^{n-l} a_j^{n-l} \left(a_i^l - a_j^l\right) \left(a_i^{m-n} - a_j^{m-n}\right) \geq 0.$$

Thus considering all $i, j \leq p$,

$$\left(a_1^m + \cdots + a_p^m\right) \cdot \left(a_1^{n-l} + \cdots + a_p^{n-l}\right) - \left(a_1^n + \cdots + a_p^n\right) \cdot \left(a_1^{m-l} + \cdots + a_p^{m-l}\right) \geq 0,$$

hence completing the proof. □

References

[1] B.M.Bătineţu-Giurgiu and O.T.Pop, *A Generalization of Radon's Inequality*, Creative Math. & Inf., 19(2)(2010), 116-121.

[2] Z. Cvetkovski, *Inequalities: Theorems, Techniques and Selected Problems*, Springer, (2012).

[3] G. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, (1952).

[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, (2001).

[5] W.-K. Lai and J. Risher, *Generalization of a class of cyclic inequalities*, International Mathematical Forum, 13(3)(2018), 137-145.