Int. J. Math. And Appl., 6(3)(2018), 231–236
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Oscillation of Third Order Quasilinear Advanced Differential Equation with "Maxima"

G. Ayyappan[1,*], P. Gayathri[1] and V. Mahadevan[2]

1 Department of Mathematics, Periyar University College of Arts and Science, Pappireddipatti, Tamilnadu, India.
2 Department of Mathematics, Government College of Engineering, Salem, Tamilnadu, India.

Abstract: This paper study the asymptotic properties of third order advanced differential equation with "maxima" of the form

$$\left(a(t)(x''(t))^{\gamma}\right)' + q(t)\max_{[t,\tau(t)]} x^{\gamma}(s) = 0, \quad t \geq t_0,$$

where γ is ratio of odd positive integers. Give an example to illustrate the main results.

MSC: 34C10, 34K11.

Keywords: Third order, delay differential equation, oscillation, quasilinear.

Accepted on: 30.07.2018

1. Introduction

In this paper is to study asymptotic properties of the third order advanced differential equation with maxima of the form

$$\left(a(t)(x''(t))^{\gamma}\right)' + q(t)\max_{[t,\tau(t)]} x^{\gamma}(s) = 0, \quad t \geq t_0, \tag{1}$$

subject to the following conditions:

(C_1) $a(t),\ q(t) \in C([t_0,\infty))$, $a(t),\ q(t)$ are positive, $\tau(t) \in C([t_0,\infty))$, $\tau(t) \geq t$;

(C_2) γ is a quotient of odd positive integers.

Whenever, it is assumed

$$R(t) = \int_{t_0}^{\infty} \frac{1}{a^{1/\gamma}(s)} ds \to \infty \quad \text{as} \quad t \to \infty. \tag{2}$$

By a solution of equation (1), we mean a function $x(t) \in C^2[T_x,\infty)$, $T_x \geq t_0$ which has the property $a(t)(x''(t))^{\gamma} \in C^1[T_x,\infty)$ and satisfies equation (1) on the interval $[T_x,\infty)$. We consider only those solutions x of equation (1) which satisfy condition $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that equation (1) possesses such a solution. A solution of equation (1) is called oscillatory if it has infinitely many zeros on $[T_x,\infty)$; otherwise it is said to be nonoscillatory. In the last few years,

* *E-mail: ayyapmath@gmail.com*

the oscillation and asymptotic behavior of differential equations with "maxima" received considerable attention because of the fact that they appear in the study of systems with automatic regulation, and automatic control of various technical systems. It often occurs that the law of regulation depends on maximum values of some regulated state parameter over certain intervals, see [1, 5, 8]. The qualitative theory of these equations is relatively little developed compared to functional differential equations without maxima. The oscillatory behavior of functional differential equations with maxima are studied in [2, 3, 5, 6, 9]. Therefore in this paper we present some new criteria that essentially utilize the value of the advanced argument, that is, the criteria obtained involves advances argument $\tau(t)$ explicity. The results established in this paper extend that of in [4, 7] for equation without maxima. In Section 2, we obtain some sufficient conditions for the asymptotic behavior of solutions of third order advanced differential equation (1). And give an example to illustrate the main results.

2. Main Results

In this section, we first start with the classification of possible nonoscillatory solutions of (1).

Lemma 2.1. *Let $x(t)$ be a nonoscillatory solution of equation (1). Then $x(t)$ satisfies one of the following conditions*

(I). $x(t)x'(t) < 0,\ x(t)x''(t) > 0,\ x(t)\left(a(t)(x''(t))^{\gamma}\right)' < 0;$

(II). $x(t)x'(t) > 0,\ x(t)x''(t) > 0,\ x(t)\left(a(t)(x''(t))^{\gamma}\right)' < 0;$

eventually.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1), say $x(t) > 0$ for $t \geq t_0$. It follows from (1) that $(a(t)(x''(t))^{\gamma})' < 0$, eventually. Thus $a(t)(x''(t))^{\gamma}$ is decreasing and of fixed sign eventually. If $(a(t)(x''(t))^{\gamma})' < 0$, then if follows from hypothesis (C_1) that $x'(t) < 0$, which implies $x(t) < 0$. A contradiction and we conclude that $a(t)(x''(t))^{\gamma} > 0$, eventually. Consequently $x'(t)$ is of fixed sign for all t large enough. Therefore, either Case (I) or Case (II) holds. □

Lemma 2.2. *Assume $x(t) > 0,\ x'(t) > 0,\ x''(t) > 0$, eventually. Then for arbitrary $k_0 \in (0,1)$*

$$x(\tau(t)) \geq k_0 \frac{\tau(t)}{t} x(t), \tag{3}$$

eventually.

Proof. It follows from the monotonicity of $x'(t)$ that

$$x(\tau(t)) - x(t) = \int_t^{\tau(t)} x'(s)ds \geq x'(t)(\tau(t) - t).$$

That is,

$$x(\tau(t)) \geq x(t) + x'(t)(\tau(t) - t)$$

or

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \frac{x'(t)}{x(t)}(\tau(t) - t). \tag{4}$$

On the otherhand, since $x(t) \to \infty$ as $t \to \infty$, then for any $k_0 \in (0,1)$ there exists a t_1 large enough, such that

$$k_0 x(t) \leq x(t) - x(t_1) = \int_{t_1}^{t} x'(s)ds \leq x'(t)(t - t_1) \leq x'(t)t,$$

or equivalently

$$\frac{x'(t)}{x(t)} \geq \frac{k_0}{t}. \tag{5}$$

Using (5) in (4), we obtain

$$\frac{x(\tau(t))}{x(t)} \geq 1 + \frac{k_0}{t}(\tau(t) - t) \geq k_0\frac{\tau(t)}{t}.$$

The proof is complete. □

Lemma 2.3. *The function $x(t)$ is a negative solution of equation (1) if and only if $-x(t)$ is a positive solution of the equation*

$$\left(a(t)(x''(t))^{\gamma}\right)' + q(t)\max_{[t,\tau(t)]} x^{\gamma}(s) = 0, \quad t \geq t_0.$$

Proof. The proof is obvious. □

Definition 2.4. *We say that (1) enjoys property (A) if every its nonoscillatory solution satisfies Case(I) of Lemma 2.1.*

Now, we are prepared to offer our main results. For our further references we set

$$Q(t) = \int_t^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} ds.$$

Theorem 2.5. *If*

$$\lim_{t\to\infty} inf \frac{1}{Q(t)}\int_t^{\infty} R(s)Q^{1+1/\gamma}(s)ds > \frac{1}{(\gamma+1)^{1+1/\gamma}}, \tag{6}$$

then (1) has property (A).

Proof. Assume the contrary, let $x(t)$ be an eventually positive solution of (1) satisfying Case(II) from Lemma 2.1. By (6), it is easy to see that there exists some $k \in (0,1)$ such that

$$\lim_{t\to\infty} inf \frac{k^{1+1/\alpha}}{Q(t)}\int_t^{\infty} R(s)Q^{1+1/\gamma}(s)ds > \frac{1}{(\gamma+1)^{1+1/\gamma}}. \tag{7}$$

From equation (1) and Lemma 2.2 implies

$$\begin{aligned}\left(a(t)(x''(t))^{\gamma}\right)' &= -q(t)\max_{[t,\tau(t)]} x^{\gamma}(s)\\ &\leq -q(t)x^{\gamma}(\tau(t))\\ &\leq -q(t)k_0^{\gamma}\frac{\tau^{\gamma}(t)}{t^{\gamma}}x^{\gamma}(t).\end{aligned}$$

Set $k_0 = k^{1/\gamma}$, the above inequality

$$\left(a(t)(x''(t))^{\gamma}\right)' \leq -q(t)k\frac{\tau^{\gamma}(t)}{t^{\gamma}}x^{\gamma}(t).$$

We define

$$w(t) = \frac{a(t)(x''(t))^{\gamma}}{x^{\gamma}(t)}. \tag{8}$$

The monotonicity of $w(t) > 0$ and differentiating (8), we get

$$\begin{aligned}w'(t) &= \frac{(a(t)(x''(t))^{\gamma})'}{x^{\gamma}(t)} - \gamma\frac{a(t)(x''(t))^{\gamma}}{x^{\gamma}(t)}\frac{x'(t)}{x(t)}\\ &\leq -kq(t)\frac{\tau^{\gamma}(t)}{t^{\gamma}} - \gamma w(t)\frac{x'(t)}{x(t)}. \end{aligned} \tag{9}$$

On the otherhand, using the monotonicity of $a(t)(x''(t))^\gamma$, we have

$$\begin{aligned} x'(t) &\geq \int_{t_1}^t x''(s)ds = \int_{t_1}^t \left(a(s)(x''(s))^\gamma\right)^{1/\gamma} a^{-1/\gamma}(s)ds \\ &\geq \left(a(t)(x''(t))^\gamma\right)^{1/\gamma} \int_{t_1}^t \frac{1}{a^{1/\gamma}(s)}ds \\ &\geq a^{1/\gamma}(t)x''(t)kR(t), \end{aligned} \tag{10}$$

eventually, let say $t \geq t_2$. Setting the last inequality in (9), we obtain

$$\begin{aligned} w'(t) &\leq -kq(t)\left(\frac{\tau(t)}{t}\right)^\gamma - \gamma w(t)\frac{a^{1/\gamma}(t)x''(t)}{x'(t)}kR(t) \\ &\leq -k\left(q(t)\left(\frac{\tau(t)}{t}\right)^\gamma + \gamma w^{1+1/\gamma}(t)R(t)\right). \end{aligned}$$

Integrating the last inequality from $t(\geq t_2)$ to ∞, we get

$$\begin{aligned} -w(t) &\leq -k\left(\int_t^\infty q(s)\left(\frac{\tau(s)}{s}\right)^\gamma ds + \int_t^\infty \gamma w^{1+1/\gamma}(s)R(s)ds\right) \\ &\leq -k\left(Q(t) + \gamma\int_t^\infty w^{1+1/\gamma}(s)R(s)ds\right) \end{aligned}$$

or

$$w(t) \geq k\left(Q(t) + \gamma\int_t^\infty w^{1+1/\gamma}(s)R(s)ds\right) \tag{11}$$

or

$$\begin{aligned} \frac{w(t)}{kQ(t)} &\geq 1 + \frac{\gamma}{Q(t)}\int_t^\infty w^{1+1/\gamma}(s)R(s)ds \\ &= 1 + \frac{\gamma k^{1+1/\gamma}}{Q(t)}\int_t^\infty Q^{1+1/\gamma}(s)R(s)\left(\frac{w(s)}{kQ(s)}\right)^{1+1/\gamma} ds. \end{aligned}$$

Since $w(t) > kQ(t)$, then

$$\inf_{t\geq t_1}\frac{w(t)}{kQ(t)} = \lambda \geq 1.$$

Thus

$$\frac{w(t)}{kQ(t)} \geq 1 + \frac{\gamma(k\lambda)^{1+1/\gamma}}{Q(t)}\int_t^\infty R(s)Q^{1+1/\gamma}(s)ds. \tag{12}$$

From (7), we see that there exists some positive η, such that

$$\frac{k^{1+1/\gamma}}{Q(t)}\int_t^\infty R(s)Q^{1+1/\gamma}(s)ds > \eta > \frac{1}{(\gamma+1)^{1+1/\gamma}}. \tag{13}$$

Combining (12) together with (13), we have

$$\frac{w(t)}{kQ(t)} \geq 1 + \gamma\lambda^{1+1/\gamma}\eta.$$

Therefore

$$\lambda \geq 1 + \gamma\lambda^{1+1/\gamma}\eta > 1 + \gamma\lambda^{1+1/\gamma}\frac{1}{(\gamma+1)^{1+1/\gamma}} = 1 + \gamma\left(\frac{\lambda}{\gamma+1}\right)^{1+1/\gamma},$$

or

$$0 > \frac{1}{\gamma+1} + \frac{\gamma}{\gamma+1}\left(\frac{\lambda}{\gamma+1}\right)^{1+1/\gamma} - \frac{\lambda}{\gamma+1}.$$

This contradicts the fact, that the function

$$f(\alpha) = \frac{1}{\gamma+1} + \frac{\gamma}{\gamma+1}\alpha^{1+1/\gamma} - \alpha$$

is positive for all $\alpha > 0$. □

Now we present some corollaries.

Corollary 2.6. *If*

$$\int_{t_0}^{\infty} q(s)\left(\frac{\tau(s)}{s}\right)^{\gamma} ds = \infty, \tag{14}$$

then (1) *has property (A).*

Corollary 2.7. *If*

$$\int_{t_0}^{\infty} R(s)Q^{1+1/\gamma}(s)ds = \infty, \tag{15}$$

then (1) *has property (A).*

Proof. It follows from (11) and $w(t) > kQ(t)$ that

$$w(t_1) \geq k\left(Q(t_1) + k^{1+1/\gamma}\int_{t_1}^{\infty} \gamma Q^{1+1/\gamma}(s)R(s)ds\right)$$

which contradicts our assumption. □

Theorem 2.8. *Assume that* (1) *has property (A). If moreover,*

$$\int_{t_0}^{\infty}\int_{v}^{\infty} \frac{1}{a^{1/\gamma}(u)}\left(\int_{u}^{\infty} q(s)ds\right)^{1/\gamma} dudv = \infty, \tag{16}$$

then every nonoscillatory solution $x(t)$ of (1) *tends to zero as $t \to \infty$.*

Proof. Property (A) of (1) implies that an eventually positive solution $x(t)$ of (1) satisfies Case(I) of Lemma 2.1. Then there exists a finite $\lim_{t\to\infty} x(t) = \ell$. We claim that $\ell = 0$. Assume that $\ell > 0$.

Integrating (1) from t to ∞ and using $x(\tau(t)) > \ell$, we obtain

$$\begin{aligned} -a(t)(x''(t))^{\gamma} &= -\int_{t}^{\infty} q(s) \max_{[s,\tau(s)]} x^{\gamma}(u)ds \\ &= -\int_{t}^{\infty} q(s)x^{\gamma}(s)ds \\ &\leq -\int_{t}^{\infty} q(s)x^{\gamma}(\tau(s))ds, \end{aligned}$$

or

$$a(t)(x''(t))^{\gamma} \geq \int_{t}^{\infty} q(s)x^{\gamma}(\tau(s))ds \geq \ell^{\gamma}\int_{t}^{\infty} q(s)ds,$$

which implies

$$x''(t) \geq \frac{\ell}{a^{1\gamma}(t)}\left(\int_{t}^{\infty} q(s)ds\right)^{1/\gamma}.$$

Integrating the last inequality from t to ∞, we get

$$-x'(t) \geq \ell\int_{t}^{\infty} \frac{1}{a^{1/\gamma}(u)}\left(\int_{u}^{\infty} q(s)ds\right)^{1/\gamma} du.$$

Now integrating from t_1 to ∞, we arrive at

$$x(t_1) \geq \ell\int_{t_1}^{\infty}\int_{v}^{\infty} \frac{1}{a^{1/\gamma}(u)}\left(\int_{u}^{\infty} q(s)ds\right)^{1/\gamma} dudv.$$

A contradiction with (16) and so we have verified that $\lim_{t\to\infty} x(t) = 0$. □

We conclude this paper with the following example.

Example 2.9. *Consider the third order nonlinear differential equation with maxima*

$$\left(t(x''(t))^3\right)' + \frac{8}{t^6} \max_{[t,\ \lambda t]} x^3(s) = 0, \quad t \geq 0, \tag{17}$$

where $a(t) = t,\ q(t) = \frac{8}{t^6},\ \tau(t) = \lambda t,$ *and* $\gamma = 3$. *By taking* $\lambda \geq 1$. *Now, one can easily verify that*

$$Q(t) = \int_t^\infty q(s) \left(\frac{\tau(s)}{s}\right)^3 ds = \frac{\lambda^3 8}{5t^5}$$

and equation (6) *reduces to*

$$\lambda > \frac{1}{3}\left(\frac{5}{4}\right)^{4/3},$$

which, by Theorem 2.5 guarantees property (A) of (17). *On the otherhand*

$$\int_{t_0}^\infty \int_v^\infty \frac{1}{a^{1/3}(u)} \left(\int_u^\infty q(s)ds\right)^{1/3} dudv = \frac{2}{5^{1/3}} \int_{t_0}^\infty \frac{dv}{v} = \infty.$$

Therefore (16) *holds, so every nonoscillatory solution of equation* (17) *tends to zero as* $t \to \infty$.

References

[1] M. Arolska and D. Bainov, *Periodic solutions of differential equations with “maxima”*, Plovdiv. Univ. Nauchn Trud., 18(1980), 143-153.

[2] R. Arul and M. Mani, *On the oscillatio of third order quasilinear delay differential equations with “maxima”*, Malaya J. Mathematik, 2(2014), 489-496.

[3] G. Ayyappan, *Oscillation of third order nonlinear differential equations with “maxima”*, J. of Modern Sci., 8(1)(2018), 80-86.

[4] B. Baculikova and J. Dzurina, *Oscillation of third order nonlinear differential equations*, Appl. Math. Lett., 24(2011), 446-470.

[5] D. Bainov and S. G. Hristova, *Differential Equations with “Maxima”*, CRC Press, New York, (2011).

[6] D. Bainov and A. Zahariev, *Oscillation and asymptotic properties of a class of functional differential equations with “maxima”*, Czech. Math. J., 34(1984), 247-251.

[7] J. Dzurina and B. Baculikova, *Oscillation of third order quasilinear advanced differential equations*, Differ. Equ. Appl., 4(3)(2012), 411-421.

[8] E. P. Popov, *Automatic Regulation and Control*, Nauka, Moscow, (1996).

[9] B. Zhang and G. Zhang, *Qualitative properties of functional differential equations with “maxima”*, Rocky Mountain J. Math., 29(2000), 357-369.