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New Fuzzy Soft Topologies Via Fuzzy Soft Ideals

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Abstract: The aim of this study is to define fuzzy soft topology in Lowen's sense and to introduce the notion of fuzzy soft ideal in fuzzy soft set theory. The concept of fuzzy soft local function is also introduced and we obtain some of its properties. The basic structure for generated fuzzy soft topologies are also studied here. These concepts are discussed with a view to find new fuzzy soft topologies from original one via fuzzy soft ideals.

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1. Introduction and Preliminaries

Fuzzy set theory was first introduced by L.A.Zadeh[7] in 1965 which is a generalization of classical or crisp sets. R.Lowen[4] proposed the definition of fuzzy topology. The concept of soft sets was first initiated by Molodtsov [5] in 1999 to deal with problems of incomplete information. The notion of soft topological space, soft interior, soft closure, soft ideal, soft local function, * - soft topology and compatible soft ideal was investigated in [3]. Some structures of soft topology was found in [2]. Some structural properties of fuzzy soft topological spaces was discussed in [6]. In this paper we define fuzzy soft topological space in Lowen's sense, fuzzy soft ideal and fuzzy soft local function etc. we investigate some theorems related with those concepts. Finally we obtain some new fuzzy soft topologies from old one through fuzzy soft ideals. The following definitions and theorems are in[1], which are needed for our study. Throughout this paper, X be an initial universe and E be the set of all parameters for X, I^X is the set of all fuzzy sets on X (where, $I = [0, 1]$ and for $\lambda \in$[0,1], $\bar{\lambda}(x) = \lambda$, for all $x \in X$).

Definition 1.1. *Let $A \subseteq E$. f_A is called a fuzzy soft set on X, where f is a mapping from E into I^X. i.e., $f_e \triangleq f(e) \triangleq f_A(e)$ is a fuzzy set on X for each $e \in A$ and $f_e = \bar{0}$ if $e \notin A$, where $\bar{0}$ is zero function on X. f_e, for each $e \in E$, is called an element of the fuzzy soft set f_A. $FS(X, E)$ denotes the collection of all fuzzy soft sets on X and is called a fuzzy soft universe. If $f_D \in FS(X, E)$ then we understand that $D \subseteq E$.*

In this paper to each parameter $e \in E$, f_e is equivalent to $f_e(x)$ for all $x \in X$.

Definition 1.2. *For two fuzzy soft sets f_A and g_B on X, we say that f_A is a fuzzy soft subset of g_B and write $f_A \sqsubseteq g_B$ if $f_e \leq g_e$, for each $e \in E$.*

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Definition 1.3. *Two fuzzy soft sets* f_A *and* g_B *on* X *are called equal if* $f_A \sqsupseteq g_B$ *and* $g_B \sqsupseteq f_A$.

Definition 1.4. *Union of two fuzzy soft sets* f_A *and* g_B *on* X *is the fuzzy soft set* $h_C = f_A \sqcup g_B$, *where* $C = A \cup B$ *and* $h_e = f_e \bigvee g_e$, *for each* $e \in E$. *That is,* $h_e = f_e \bigvee \bar{0} = f_e$ *for each* $e \in A - B$, $h_e = \bar{0} \bigvee g_e = g_e$ *for each* $e \in B - A$ *and* $h_e = f_e \bigvee g_e$, *for each* $e \in A \cup B$.

Definition 1.5. *Intersection of two fuzzy soft sets* f_A *and* g_B *on* X *is the fuzzy soft set* $h_C = f_A \sqcap g_B$, *where* $C = A \cap B$ *and* $h_e = f_e \bigwedge g_e$, *for each* $e \in E$.

Definition 1.6. *The complement of a fuzzy soft set* f_A *is denoted by* f_A^c, *where* $f^c : E \to I^X$ *is a mapping given by* $f_e^c = \bar{1} - f_e$, *for each* $e \in E$. *Clearly* $(f_A^c)^c = f_A$.

Definition 1.7. *A fuzzy soft set* f_E *on* X *is called a null fuzzy soft set and denoted by* $\emptyset$, *if* $f_e = \bar{0}$, *for each* $e \in E$.

Definition 1.8. *A fuzzy soft set* f_E *on* X *is called an absolute fuzzy soft set and denoted by* $\tilde{E}$, *if* $f_e = \bar{1}$, *for each* $e \in E$. *Clearly* $(\tilde{E})^c = \emptyset$ *and* $\emptyset^c = \tilde{E}$.

Definition 1.9. *A fuzzy soft set* f_E *on* X *is called a* λ*-absolute fuzzy soft set and denoted by* $\tilde{E}^\lambda$, *if* $f_e = \bar{\lambda}$, *for each* $e \in E$. *Clearly,* $(\tilde{E}^\lambda)^c = \tilde{E}^{1-\lambda}$.

In this paper we write $\tilde{E}^\lambda$ as $\tilde{\lambda}_E$.

Theorem 1.10. *Let* $\triangle$ *be an index set and* f_A, g_B, h_C, $(f_A)_i \triangleq (f_i)_{A_i}, (g_B)_i \triangleq (g_i)_{B_i} \in FS(X,E)$, $\forall\ i \in \triangle$, *then we have the following properties:*

(1). $f_A \sqcap f_A = f_A$, $f_A \sqcup f_A = f_A$.

(2). $f_A \sqcap g_B = g_B \sqcap f_A$, $f_A \sqcup g_B = g_B \sqcup f_A$.

(3). $f_A \sqcup (g_B \sqcup h_C) = (f_A \sqcup g_B) \sqcup h_C$.

(4). $f_A = f_A \sqcap (f_A \sqcup g_B)$, $f_A = f_A \sqcup (f_A \sqcap g_B)$.

(5). $f_A \sqcap (\underset{i\in\triangle}{\sqcup} (g_B)_i) = (\underset{i\in\triangle}{\sqcup} (f_A \sqcap (g_B)_i))$.

(6). $f_A \sqcup (\underset{i\in\triangle}{\sqcap} (g_B)_i) = (\underset{i\in\triangle}{\sqcap} (f_A \sqcup (g_B)_i))$.

(7). $\emptyset \sqsubseteq f_A \sqsubseteq \tilde{E}$.

(8). $(f_A^c)^c = f_A$.

(9). $(\underset{i\in\triangle}{\sqcap} (f_A)_i)^c = \underset{i\in\triangle}{\sqcup} (f_A)_i)^c$.

(10). $(\underset{i\in\triangle}{\sqcup} (f_A)_i)^c = \underset{i\in\triangle}{\sqcap} (f_A)_i)^c$.

(11). If $f_A \sqsubseteq g_B$, *then* $g_B^c \sqsubseteq f_A^c$.

2. Fuzzy Soft Topological Spaces

Definition 2.1. *The support of a fuzzy soft set* f_D *where* $D \subseteq E$ *on* X *is defined as* $S(f_D) = \{e \in E / f_e > \bar{0}\}$. *A fuzzy soft set* f_B *where* $B \subseteq E$ *is said to be finite fuzzy soft set of* X *iff* $S(f_B)$ *is a finite parameter set. A fuzzy soft set* f_B *is said to be countable fuzzy soft set of* X *iff* $S(f_B)$ *is a countable parameter set.*

Definition 2.2. *Let $A, B \subseteq E$. A non empty family $\mathscr{I} \subseteq FS(X,E)$ of fuzzy soft sets is called fuzzy soft ideal on X if*

(1). $f_A \in \mathscr{I}$, $g_B \sqsubseteq f_A$ implies that $g_B \in \mathscr{I}$.

(2). $f_A \in \mathscr{I}$, $g_B \in \mathscr{I}$ implies that $f_A \sqcup g_B \in \mathscr{I}$. (As $\mathscr{I}$ is not empty, $\emptyset \in \mathscr{I}$).

Example 2.3.

(1). $\mathscr{I}_f$ - is the fuzzy soft ideal of fuzzy soft sets of X with finite support.

(2). $\mathscr{I}_c$ - is the fuzzy soft ideal of fuzzy soft sets of X with countable support.

(3). Let $A, B \subseteq E$ and f_A be a fixed fuzzy soft set in X. Then $\mathscr{I}(f_A) = \{g_B \in FS(X,E)/g_B \sqsubseteq f_A\}$ is a fuzzy soft ideal.

Definition 2.4. *Let f_A and g_B are two fuzzy soft sets of X with $A, B \subseteq E$. Then f_A "intersection" g_B is redefined as follows: $f_A \tilde{\sqcap} g_B = \max(\bar{0}, f_e + g_e - \bar{1})$ for each $e \in E$.*

Remark 2.5.

(1). $f_A \tilde{\sqcap} g_B = \emptyset$ iff $g_e \leq \bar{1} - f_e$ for each $e \in E$. That is $g_B \sqsubseteq f_A^c$.

(2). $f_A \tilde{\sqcap} g_B \sqsubseteq f_A \sqcap g_B$.

Now we prove a lemma which will be useful in the following sequel.

Lemma 2.6. *If A,B, and C are the subsets of the parameter set E, then $f_C \tilde{\sqcap} (f_A \sqcup f_B) = (f_C \tilde{\sqcap} f_A) \sqcup (f_C \tilde{\sqcap} f_B)$.*

Proof. For all $e \in E$,

$$\begin{aligned} f_C \tilde{\sqcap} (f_A \sqcup f_B)(e) &= \max\{\bar{0}, f_C(e) + (f_A \sqcup f_B)(e) - \bar{1}\} \\ &= \max\{\bar{0}, f_C(e) + f_A(e) - \bar{1}, f_C(e) + f_B(e) - \bar{1}\} \\ &= \max\{\max\{\bar{0}, f_C(e) + f_A(e) - \bar{1}\}, \max\{\bar{0}, f_C(e) + f_B(e) - \bar{1}\}\} \\ &= \max\{(f_C \tilde{\sqcap} f_A)(e), (f_C \tilde{\sqcap} f_B)(e)\} \\ &= ((f_C \tilde{\sqcap} f_A) \sqcup (f_C \tilde{\sqcap} f_B))(e). \end{aligned}$$

□

Definition 2.7. *Let C, D and $P \subseteq E$. A family $\tau \subseteq FS(X,E)$ is called a fuzzy soft topology for X, if it satisfies the following axioms.*

(1). For all $\lambda \in [0,1]$, $\tilde{\lambda}_E \in \tau$.

(2). f_C, $g_D \in \tau$ implies that $f_C \sqcap g_D \in \tau$.

(3). If $\{f_{i_P}\}_{i \in \triangle}$ is an indexed subfamily of τ, then $\bigsqcup_{i \in \triangle} f_{i_P} \in \tau$.

The pair (X, τ) is called a fuzzy soft topological space. The members of τ are called fuzzy soft open sets.

2.1. Examples for fuzzy soft topological spaces

(1). $\tau = \{\lambda$ - absolute fuzzy soft sets/ $0 \leq \lambda \leq 1\}$ is a fuzzy soft topology and is called indiscrete fuzzy soft topology on X.

(2). If τ equals $FS(X, E)$ then τ is called discrete fuzzy soft topology on X.

(3). Let $B \subseteq E$. Consider the collection $\tau = \{f_B^c \in FS(X, E)/f_B = \tilde{E}$ (or) to each $\bar{\lambda}$ such that $f_{e_0} < \bar{\lambda} < \bar{1}$ for some $e_0 \in E$, the set $\{e \in E/f_e > \bar{\lambda}\}$ is a finite set$\}$. Then τ is a fuzzy soft topology on X. We call it as cofinite fuzzy soft topology on X. It is enough to show that $\tau\prime = \{g_A \in FS(X, E)/g_A = f_B^c$ and $f_B \in \tau$ where $A, B \subseteq E\}$ satisfies.

(a). $\tilde{\lambda}_E \in \tau\prime$, for all $\lambda \in [0, 1]$.

(b). $g_{i_A} \in \tau\prime$, for all $i \in \triangle \Rightarrow \sqcap g_{i_A} \in \tau\prime$.

(c). $g_{1_A}, g_{2_A} \in \tau\prime \Rightarrow g_{1_A} \sqcup g_{2_A} \in \tau\prime$.

Obviously $\tilde{\lambda}_E \in \tau\prime$ for all $\lambda \in [0, 1]$. Consider a collection $\{g_{i_A}\}_{i \in \triangle}$ such that each $g_{i_A} \in \tau\prime$. We may Assume that atleast one of $g_{i_A} \neq \tilde{E}$. Let $\bar{\lambda}$ be such that $(\bigwedge g_{i_{e_0}}) < \bar{\lambda} < \bar{1}$ for some e_0 in E. Then $g_{k_{e_0}} < \bar{\lambda}$ for some $k \in \triangle$. That is $\{e \in E/g_{k_e} > \bar{\lambda}\}$ is a finite set. That is $\{e \in E/(\bigwedge g_{i_e}) > \bar{\lambda}\}$ is a finite set. Hence $\sqcap g_{i_A} \in \tau\prime$. If $g_{1_A} \sqcup g_{2_A} = \tilde{E}$ then $g_{1_A} \sqcup g_{2_A} \in \tau\prime$. If $(g_1 \bigvee g_2)_{e_0} < \bar{\lambda} < \bar{1}$ for some $e_0 \in E$, then $B_1 = \{e \in E/g_{1_e} > \bar{\lambda}\}$ and $B_2 = \{e \in E/g_{2_e} > \bar{\lambda}\}$ are finite sets. Therefore $\{e \in E/(g_1 \bigvee g_2)_e > \bar{\lambda}\} = \{e \in E/g_{1_e} > \bar{\lambda}\} \cup \{e \in E/g_{2_e} > \bar{\lambda}\}$ is a finite set. Hence $g_{1_A} \sqcup g_{2_A} \in \tau\prime$ for all $g_{1_A}, g_{2_A} \in \tau\prime$. Thus τ is a fuzzy soft topology on X.

(4). Let $A \subseteq E$ with $|A| > 1$. Then $\tau = \{g_A \in FS(X, E)/f_e = \bar{0}$ for each $e \in A$ (or) $f_e \neq \bar{0}$ for all $e \in A\}$ is a fuzzy soft topology on X. If $\tilde{\lambda}_E = \emptyset$, then $\tilde{\lambda}_e = \bar{0}$ for all $e \in A$. This implies that $\emptyset \in \tau$. For each $\lambda \in (0, 1]$, $\tilde{\lambda}_e = \bar{\lambda} \neq \bar{0}$ for all $e \in A$. Therefore $\tilde{\lambda}_E \in \tau$. Let $\{g_{j_A}\}_{j \in \triangle}$ be a collection of elements of τ. If $\bigvee\limits_{j \in \triangle} g_{j_{e_0}} \neq \bar{0}$ for some $e_0 \in A$, then $g_{k_{e_o}} \neq \bar{0}$ for atleast one k (say) $\in \triangle$. That is $g_{k_{e_o}} \neq \bar{0}$ for all $e \in A$. Then $\bigvee g_{k_{e_0}} \neq \bar{0}$ for all $e \in A$. Therefore $\bigsqcup\limits_{j \in \triangle} g_{j_A} \in \tau$. Let $f_C, g_D \in \tau$ where $C, D \subseteq E$, then $g_e = \bar{0}$ for all $e \in E$ (or) $f_e \neq \bar{0}$ for all $e \in E$. Similarly we have $g_e = \bar{0}$ for all $e \in E$ (or) $g_e \neq \bar{0}$ for all $e \in E$. If either $f_e = \bar{0}$ (or) $g_e = \bar{0}$ for all $e \in E$ then $(f \bigwedge g)_e = \bar{0} \ \forall \ e \in E$. If $f_e \neq \bar{0}$ (or) $g_e \neq \bar{0} \ \forall \ e \in E$ then $(f \bigwedge g)_e \neq \bar{0} \ \forall \ e \in E$. Therefore $f_C, g_D \in \tau \Rightarrow f_e \bigwedge g_e \in \tau$. Hence τ is a fuzzy soft topology on X.

(5). Let us take the parameter set E by the set of all natural numbers, for all $m \in E$, consider the subset $B_m = \{2m-1, 2m\}$ of E. Obviously by (4) to each $m \in E$, $\tau_m = \{f_C \in FS(X, E)/f_e = \bar{0} \ \forall \ e \in B_m$ (or) $f_e \neq \bar{0} \ \forall \ e \in B_m$ where $C \subseteq E\}$ is a fuzzy soft topology on X. Hence $\bigcap\limits_{m=1}^{\infty} \tau_m$ is a fuzzy soft topology on X. Therefore $\tau = \{f_C \in FS(X, E)/$ for all $m \in E, f_{2m} = \bar{0}$ if and only if $f_{2m-1} = \bar{0}\}$.

Definition 2.8. *The closure of a fuzzy soft set f_E can be defined as usual way: $\bar{f_A} = \sqcap\{f_B/f_A \sqsubseteq f_B$ and $f_B^c \in \tau\}$ where $A, B \subseteq E$.*

We now give a modified definition for $\bar{f_A}$.

Definition 2.9. *Let (X, τ) be a fuzzy soft topological space. the closure of a fuzzy soft set f_A, denoted as $cl(f_A)$, is a fuzzy soft set defined by $cl(f_A) = \sqcup\{\tilde{\lambda}_E/f_B \in \tau, f_B \sqsupset \tilde{\lambda}_A^c \Rightarrow f_A \tilde{\sqcap} f_B \neq \emptyset\}$.*

Now we try to prove the equality of $\bar{f_A}$ and $cl(f_A)$.

Theorem 2.10. *Let (X, τ) be a fuzzy soft topological space and f_A be a fuzzy soft set on X where $A \subseteq E$. Then $\bar{f_A} = cl(f_A)$.*

Proof. Let $B \subseteq E$. Take $f_B = \bar{f}_A^c$. Then $f_B \in \tau$ and to each $e \in E$, $f_B = \bar{f}_A^c \sqsupset \tilde{\lambda}_E^c$ for all $\tilde{\lambda}_E \sqsupset \bar{f}_A$. Since $f_A \sqsubseteq \bar{f}_A$ and $\bar{f}_A \tilde{\sqcap} f_B = \emptyset$, we get that $f_A \tilde{\sqcap} f_B = \emptyset$. That is $cl(f_A) \sqsubset \tilde{\lambda}_E$ for all $\tilde{\lambda}_E \sqsupset \bar{f}_A$. Therefore

$$cl(f_A) \sqsubseteq \bar{f}_A \tag{1}$$

Next we prove that $\bar{f}_A \sqsubseteq cl(f_A)$. Let $\bar{\mu} = cl(f_A)_e$ for all $e \in E$. Let $\tilde{\lambda}_E \sqsupset \tilde{\mu}_E$. There exists $f_B \in \tau$ such that $f_B \sqsupset \tilde{\lambda}_E^c$ and $f_A \tilde{\sqcap} f_B = \emptyset$. Take $f_K = f_B^c$ where $K \subseteq E$. Now $f_A \sqsubseteq f_K$. Since $f_A \sqsubseteq f_K$ and $f_K^c = f_B \in \tau$, we have that $\bar{f}_A \sqsubseteq f_K$. That is $\bar{f}_A \sqsubseteq f_B^c \sqsubset \tilde{\lambda}_E$. This implies that $\bar{f}_A \sqsubset \tilde{\lambda}_E$ for all $\tilde{\lambda}_E \sqsupset cl(f_A)$. Therefore

$$\bar{f}_A \sqsubseteq cl(f_A) \tag{2}$$

From (1) and (2) we obtain that $\bar{f}_A = cl(f_A)$. □

3. Fuzzy Soft Local Function

Definition 3.1. *Let (Y, τ) be a fuzzy soft topological space with a fuzzy soft ideal $\mathscr{J}$ on Y. Let f_A be a fuzzy soft set on Y where $A \subseteq E$. The fuzzy soft local function $f_A^* = \sqcup\{\tilde{\lambda}_E / f_B \in \tau$ where $B \subseteq E, f_B \sqsupset \tilde{\lambda}_E^c \Rightarrow f_A \tilde{\sqcap} f_B \notin \mathscr{J}\}$.*

Example 3.2. *We consider the fuzzy soft topology τ given in the Example 2.8.5. and fuzzy soft ideal $\mathscr{I}_f$ of fuzzy soft sets with finite support. Then we show that for any fuzzy soft set f_A, $f_A^*(\mathscr{I}_f) = \emptyset$. Let $e \in E$. If $f_e^*(\mathscr{I}_f) \neq \bar{0}$, then $f_e^*(\mathscr{I}_f) = \bar{\mu}$ for atleast one $\bar{0} < \bar{\mu} \leq \bar{1}$. By Definition of f_A^*, whenever $f_B \in \tau$ with $f_B \sqsupset \tilde{\mu}_E^c$ where $e \in B \subseteq E$, then $f_A \tilde{\sqcap} f_B \notin \mathscr{I}_f$. That is $S(f_A \tilde{\sqcap} f_B)$ is not a finite subset of E. But this is not possible, because $e_1 \in B_m$ for exactly one $m \in E$. Let e_2 be the other element of B_m. Consider a fuzzy soft set g_K where $k \subseteq E$ such that $g_{e_2} = g_{e_1} = \bar{1}$ and $g_e = \bar{0}$ for all $e \neq e_1, e_2$. Obviously $g_K \in \tau$ and $S(f_A \tilde{\sqcap} g_K)$ is a finite subset of E. Hence $f_A^*(\mathscr{I}_f) = \emptyset$.*

Example 3.3. *Let (Y, τ) be a fuzzy soft topological space with indiscrete fuzzy soft topology given in the Example 2.8.1. Then for all $\lambda \in (0, 1], \tilde{\lambda}_E^* = \tilde{\lambda}_E$*

$$\begin{aligned}
\tilde{\lambda}_E^* &= \sqcup\{\tilde{\alpha}_E / \tilde{\beta}_E \in \tau \quad with \quad \tilde{\beta}_E \sqsupset \tilde{\alpha}_E^c \Rightarrow \tilde{\alpha}_E \tilde{\sqcap} \tilde{\beta}_E \notin \mathscr{I}\} \\
&= \sqcup\{\tilde{\alpha}_E / \tilde{\beta}_E \sqsupset \tilde{\alpha}_E^c \Rightarrow \tilde{\beta}_E \sqsupset \tilde{\lambda}_E^c\} \\
&= \sqcup\{\tilde{\alpha}_E / \tilde{\alpha}_E \sqsubseteq \tilde{\lambda}_E\} \\
&= \tilde{\lambda}_E
\end{aligned}$$

So $\tilde{\lambda}_E^* = \tilde{\lambda}_E$.

Remark 3.4.

(1). If $\mathscr{I} = \{\emptyset\}$, then $f_A \sqsubseteq f_A^*$. Let $f_A^* = \tilde{\mu}_E$. If $f_B \in \tau$ and $f_B \sqsupset \tilde{\mu}_E^c$ then $f_B \tilde{\sqcap} \tilde{\mu}_E \neq \emptyset$. Hence $f_B \tilde{\sqcap} f_E \neq \emptyset$. So $f_B \tilde{\sqcap} f_E \notin \mathscr{I}$ as $\mathscr{I} = \{\emptyset\}$. Therefore $f_A^* \sqsupseteq \tilde{\mu}_E$ and $f_A \sqsubseteq f_A^*$.

(2). If $f_A \in \mathscr{I}$, then $f_A^* = \emptyset$ as $\forall \ f_B \in \tau$, $f_A \tilde{\sqcap} f_B \sqsubseteq f_A \in \mathscr{I}$. Therefore there is no $f_B \in \tau$ such that $f_A \tilde{\sqcap} f_B \notin \mathscr{I}$. Thus $f_A^* = \emptyset$.

(3). Let $\tilde{\Lambda}_e = \{\bar{\lambda} / f_B \in \tau, f_e > \bar{1} - \bar{\lambda}$ where $e \in B \Rightarrow f_A \tilde{\sqcap} f_B \notin \mathscr{I}\}$. If $f_A^* = \tilde{\mu}_E$, then $\bar{\mu} \in \tilde{\Lambda}_e$. If $f_A^* = \tilde{\mu}_E$, then $\tilde{\mu}_E = \sqcup\{\tilde{\lambda}_E / \bar{\lambda} \in \tilde{\Lambda}_e\}$. If $f_B \in \tau$ and $f_e > \bar{1} - \bar{\mu}$ where $e \in B$, find $\bar{\lambda} \in \tilde{\Lambda}_e$ such that $\bar{\mu} > \bar{\lambda} > \bar{1} - f_e$ where $e \in B$. Since $f_B \sqsupset \tilde{\lambda}_E^c, f_B \tilde{\sqcap} f_A \notin \mathscr{I}$. Therefore $\bar{\mu} \in \tilde{\Lambda}_e$.

Theorem 3.5. *Let (X, τ) be a fuzzy soft topological space with $\mathscr{I}$ and $\mathscr{L}$ are fuzzy soft ideals on X. Then.*

(a). $f_C \sqsubseteq f_D$ *implies* $f^*_C(\mathscr{I}) \sqsubseteq f^*_D(\mathscr{I})$.

(b). $\mathscr{I} \subseteq \mathscr{L}$ *implies* $f^*_C(\mathscr{L}) \sqsubseteq f^*_C(\mathscr{I})$.

(c). $f^*_C = cl(f^*_C) \sqsubseteq cl(f_C)$.

(d). $(f^*_C)^* \sqsubseteq f^*_C$.

(e). $(f_C \sqcup f_D)^* = f^*_C \sqcup f^*_D$.

(f). $\tilde{\alpha}^*_E \sqsubseteq \tilde{\alpha}_E$ *for all* $\alpha \in [0, 1]$.

Proof.

(a). Let $f^*_C(\mathscr{I}) = \tilde{\mu}_E$. Then for all $f_A \in \tau$ with $f_A \sqsupset \tilde{\mu}^c_E$ implies that $f_A \tilde{\sqcap} f_C \notin \mathscr{I}$. This implies that $f_A \tilde{\sqcap} f_D \notin \mathscr{I}$. [As $f_C \sqsubseteq f_D, f_A \tilde{\sqcap} f_C \sqsubseteq f_A \tilde{\sqcap} f_D$ and $f_A \tilde{\sqcap} f_C \notin \mathscr{I}$ gives that $f_A \tilde{\sqcap} f_D \notin \mathscr{I}$]. Hence $\tilde{\mu}_E \sqsubseteq f^*_D(\mathscr{I})$. Therefore $f^*_C(\mathscr{I}) \sqsubseteq f^*_D(\mathscr{I})$.

(b). $f^*_C(\mathscr{L}) = \sqcup\{\tilde{\lambda}_E / f_B \in \tau$ and $f_B \sqsupset \tilde{\lambda}^c_E \Rightarrow f^*_C \tilde{\sqcap} f_B \notin \mathscr{L}\} \sqsubseteq \sqcup\{\tilde{\lambda}_E / f_B \in \tau$ and $f_B \sqsupset \tilde{\lambda}^c_E \Rightarrow f^*_C \tilde{\sqcap} f_B \notin \mathscr{I}\} = f^*_C(\mathscr{I})$. Thus $\mathscr{I} \subseteq \mathscr{L} \Rightarrow f^*_C(\mathscr{L}) \sqsubseteq f^*_C(\mathscr{I})$.

(c). Clearly

$$f^*_C \sqsubseteq cl(f^*_C) \tag{3}$$

If $\tilde{\mu}_E = cl(f^*_C)$ then by the Remark 3.4., to each $f_D \in \tau$ with $f_D \sqsupset \tilde{\mu}^c_E$, there exists $e \in C \cap D$ such that $f^*_C \tilde{\sqcap} f^*_D \neq \emptyset$. Let $\bar{\lambda} = f^*_e$ where $e \in C$. Then $f^*_D \sqsupset \tilde{\lambda}^c_E$ and hence by the definition of $f^*_C, f_C \tilde{\sqcap} f_D \notin \mathscr{I}$. Thus $f_D \in \tau$ and $f_D \sqsupset \tilde{\mu}^c_E \Rightarrow f_C \tilde{\sqcap} f_D \notin \mathscr{I}$. So $\tilde{\mu}_E \sqsubseteq cl(f^*_C)$. Therefore

$$cl(f^*_C) \sqsubseteq f^*_C \tag{4}$$

From (3) and (4),

$$f^*_C = cl(f^*_C) \tag{5}$$

If $f^*_C = \tilde{\lambda}_E$, then $f_D \in \tau$, $f_D \sqsupset \tilde{\lambda}_E \Rightarrow f_C \tilde{\sqcap} f_D \notin \mathscr{I}$. This implies that $f_D \in \tau, f_D \sqsupset \tilde{\lambda}_E \Rightarrow f_C \tilde{\sqcap} f_D \neq \emptyset$. This implies that $\tilde{\lambda}_E \sqsubseteq cl(f_C)$. This gives that

$$f^*_C \sqsubseteq cl(f_C) \tag{6}$$

From (5) and (6), we obtain that $f^*_C = cl(f^*_C) \sqsubseteq cl(f_C)$.

(d). Let $\tilde{\mu}_E = (f^*_C)^*$. Then for any $f_B \in \tau$ with $f_B \sqsupset \tilde{\mu}^c_E$, we have $f_B \tilde{\sqcap} f^*_C \notin \mathscr{I}$. This implies that $f_B \tilde{\sqcap} f^*_C \neq \emptyset$. Let $e \in B \subseteq E$ such that $f_B \tilde{\sqcap} f^*_C \neq \emptyset$. Let $\bar{\lambda} = f^*_e$. Then $f^*_e \neq \bar{0}$ and $f_B \sqsupset \tilde{\lambda}^c_E$. That is $f_B \tilde{\sqcap} f_C \notin \mathscr{I}$. Thus if $\tilde{\mu}_E = (f^*_C)^*$ then $f_B \in \tau$ and $f_B \sqsupset \tilde{\mu}^c_E \Rightarrow f_B \tilde{\sqcap} f_C \notin \mathscr{I}$. Therefore $\tilde{\mu}_E \sqsubseteq f^*_C$. Hence $(f^*_C)^* \sqsubseteq f^*_C$.

(e). Clearly

$$f^*_C \sqcup f^*_D \sqsubseteq (f_C \sqcup f_D)^* \tag{7}$$

Let $\tilde{\mu}_E = (f_C \sqcup f_D)^*$. Assume that $\tilde{\mu}_E \sqsupset f^*_C$ and $\tilde{\mu}_E \sqsupset f^*_D$. Then there exists $f_{C_1}, f_{C_2} \in \tau$ such that $f_{C_1} \sqsupset \tilde{\mu}^c_E$ and $f_{C_1} \tilde{\sqcap} f_C \in \mathscr{I}$; $f_{C_2} \sqsupset \tilde{\mu}^c_E$ and $f_{C_2} \tilde{\sqcap} f_D \in \mathscr{I}$. This implies that $(f_{C_1} \tilde{\sqcap} f_C) \sqcup (f_{C_2} \tilde{\sqcap} f_D) \in \mathscr{I}$. As $f_{C_1} \sqsupset \tilde{\mu}^c_E$, there exists $e \in E$ such that $f_{C_1} \tilde{\sqcap} (f_C \sqcup f_D) \neq \emptyset$. So $f_{C_1} \tilde{\sqcap} f_C \neq \emptyset$ (or) $f_{C_1} \tilde{\sqcap} f_D \neq \emptyset$. By Lemma 2.6. $f_{C_1} \tilde{\sqcap} (f_C \sqcup f_D) =$

$(f_{C_1}\tilde{\sqcap}f_C)\sqcup(f_{C_1}\tilde{\sqcap}f_D)\in\mathscr{I}$. Similarly $f_{C_2}\tilde{\sqcap}(f_C\sqcup f_D)=(f_{C_2}\tilde{\sqcap}f_C)\sqcup(f_{C_2}\tilde{\sqcap}f_D)\in\mathscr{I}$. Let $f_G=f_{C_1}\sqcap f_{C_2}$. Then $f_G\in\tau$ and $f_G\sqsupset\tilde{\mu}^c_E$, we get $f_G\tilde{\sqcap}(f_C\sqcup f_D)\notin\mathscr{I}$. But $f_G\tilde{\sqcap}(f_C\sqcup f_D)=(f_G\tilde{\sqcap}f_C)\sqcup(f_G\tilde{\sqcap}f_D)\sqsubseteq(f_{C_1}\tilde{\sqcap}f_C)\sqcup(f_{C_2}\tilde{\sqcap}f_D)$ and $f_G\tilde{\sqcap}(f_C\sqcup f_D)\in\mathscr{I}$. Which is a contradiction. Therefore either $f^*_C\sqsupset\tilde{\mu}_E$ (or) $f^*_D\sqsupset\tilde{\mu}_E$. Hence

$$(f_C\sqcup f_D)^*\sqsubseteq f^*_C\sqcup f^*_D \tag{8}$$

From (7) and (8) we get $(f_C\sqcup f_D)^*=f^*_C\sqcup f^*_D$.

(f). Let $f_C=\tilde{\alpha}_E$. Then $f_e=\bar{\alpha}$ for all $e\in E$. Let $\tilde{\alpha}_{1_E}\sqsupset\tilde{\alpha}_E$. Select $\tilde{\alpha}_{2_E}$ such that $\tilde{\alpha}_E\sqsupset\tilde{\alpha}_{2_E}\sqsupset\tilde{\alpha}_{1_E}$. Then $\tilde{\alpha}^c_{1_E}\sqsubset\tilde{\alpha}^c_{2_E}\sqsubset\tilde{\alpha}^c_E$. Let $f_B=\tilde{\alpha}^c_{2_E}$. Then $f_B\in\tau$. Also $f_B=\tilde{\alpha}^c_{2_E}\sqsupset\tilde{\alpha}^c_{1_E}$. But

$$\begin{aligned}f_B\tilde{\sqcap}f_C&=\max(\bar{0},\bar{\alpha}^c_{2_e}+\bar{\alpha}-\bar{1})\\&=\max(\bar{0},\bar{\alpha}-\bar{\alpha_2})\\&=\emptyset\ \ as\ \ \tilde{\alpha}_E\sqsupset\tilde{\alpha}_{2_E}.\end{aligned}$$

Therefore $f_B\tilde{\sqcap}f_C\in\mathscr{I}$. So $\tilde{\alpha}_{1_E}\notin\{\tilde{\lambda}_E/f_B\in\tau,f_B\sqsupset\tilde{\lambda}^c_E\Rightarrow f_C\tilde{\sqcap}f_B\notin\mathscr{I}\}$. Therefore $f^*_C\sqsubseteq\tilde{\alpha}_E$. That is $\tilde{\alpha}^*_E\sqsubseteq\tilde{\alpha}_E$ for all $\alpha\in[0,1]$. □

Theorem 3.6. *If $\varphi:FS(X,E)\to FS(X,E)$ is a function satisfying*

(1). $\varphi(\tilde{\alpha}_E)\sqsubseteq\tilde{\alpha}_E$ for all $\alpha\in[0,1]$.

(2). $\varphi(f_C\sqcup f_D)=\varphi(f_C)\sqcup\varphi(f_D)$ for all $f_C,f_D\in FS(X,E)$ and

(3). $\varphi(\varphi(f_C))\sqsubseteq\varphi(f_C)$. Then $\psi:FS(X,E)\to FS(X,E)$ defined by $\psi(f_C)=f_C\sqcup\varphi(f_C)$ is a Kuratowski fuzzy soft closure operator on $FS(X,E)$.

Proof. We prove that the function ψ satisfies all the conditions in Kuratowski fuzzy soft closure operator.

(i). $\psi(\tilde{\alpha}_E)=\tilde{\alpha}_E\sqcup\varphi(\tilde{\alpha}_E)=\tilde{\alpha}_E$ for all $\alpha\in[0,1]$ (by (1)).

(ii). As $\psi(f_C)=f_C\sqcup\varphi(f_C)$ for each $f_C\in FS(X,E)$, $f_C\sqsubseteq\psi(f_C)$.

(iii). Let $f_C\in FS(X,E)$ and $f_D\in FS(X,E)$. Now

$$\begin{aligned}\psi(f_C\sqcup f_D)&=(f_C\sqcup f_D)\sqcup\varphi(f_C\sqcup f_D).\\&=(f_C\sqcup f_D)\sqcup(\varphi(f_C)\sqcup\varphi(f_D))\ \ (by\ \ (2)).\\&=(f_C\sqcup\varphi(f_C))\sqcup(f_D\sqcup\varphi(f_D))\\&=\psi(f_C)\sqcup\psi(f_D).\end{aligned}$$

Therefore $\psi(f_C\sqcup f_D)=\psi(f_C)\sqcup\psi(f_D)$.

(iv). Now

$$\begin{aligned}\psi(\psi(f_C))&=\psi(f_C\sqcup\varphi(f_C))\\&=(f_C\sqcup\varphi(f_C))\sqcup\varphi(f_C\sqcup\varphi(f_C))\\&=(f_C\sqcup\varphi(f_C))\sqcup(\varphi(f_C)\sqcup\varphi(\varphi(f_C)))\ \ (by\ \ (2))\end{aligned}$$

$$= (f_C \sqcup \varphi(f_C)) \sqcup \varphi(f_C) \quad as \quad \varphi(\varphi(f_C)) \sqsubseteq \varphi(f_C) \quad [by \quad (3)]$$
$$= f_C \sqcup \varphi(f_C)$$
$$= \psi(f_C).$$

Therefore $\psi(\psi(f_C)) = \psi(f_C)$. □

4. Fuzzy Soft Topology $\tau^*(\mathscr{I})$

Definition 4.1. *Let (X,τ) be a fuzzy soft topological space with fuzzy soft ideal $\mathscr{I}$ on X. We define a fuzzy soft topology $\tau^*(\mathscr{I})$ induced by τ and $\mathscr{I}$ as $\tau^*(\mathscr{I}) = \{f_U \in FS(X,E) : \psi(f_U^c) = f_U^c\}$ where $\psi(f_C) = f_C \sqcup f_C^* : \psi(f_C)$ is denoted as $cl^*(f_C)$.*

Theorem 4.2. *The fuzzy soft topology $\tau^*(\mathscr{I})$ is finer than the fuzzy soft topology τ.*

Proof. Let (X,τ) be a fuzzy soft topological space and $\mathscr{I}$ be a fuzzy soft ideal. We prove that $\tau \subseteq \tau^*(\mathscr{I})$. Let f_C be a fuzzy soft closed set in τ. Then $f_C^* \sqsubseteq cl(f_C) = f_C$. Therefore $f_C^* \sqsubseteq f_C$. and $f_C = f_C \sqcup f_C^*$. f_C is fuzzy soft closed set in $\tau^*(\mathscr{I})$. So every τ- soft closed set $f_C is \tau^*$- soft closed set and hence the fuzzy soft topology τ^* is finer than the fuzzy soft topology τ. □

Remark 4.3. *We have investigated that if $\mathscr{I} = \{\emptyset\}$, then $f_C^* = cl(f_C)$ for all f_C and hence in this case $\psi(f_C) = cl(f_C)$, that is we have $\tau^* = \tau$. If $\mathscr{I} = FS(X,E)$, then $f_C^* = \emptyset$, for every fuzzy soft set f_C in X. In this case*

$$\tau^* = \{f_C^c : \psi(f_C) = f_C\}$$
$$= \{all\ fuzzy\ soft\ sets\ in\ X\}$$
$$= FS(X,E).$$

Thus $\tau^*(FS(X,E))$ is the discrete fuzzy soft topology on X. The fuzzy soft ideals $\{\emptyset\}$ and $FS(X,E)$ exhibit the extreme cases where $\tau^* = \tau$ and $\tau^* =$ discrete fuzzy soft topology respectively. Since for every fuzzy soft ideal $\mathscr{I}$ on X, we have $\{\emptyset\} \subseteq \mathscr{I} \subseteq FS(X,E)$, it follows from Theorem 4.2. that $\tau \subseteq \tau^* \subseteq$ discrete fuzzy soft topology. More particularly, if $\mathscr{I}$ and $\mathscr{J}$ are fuzzy soft ideals on (X,τ) such that $\mathscr{I} \subseteq \mathscr{J}$ then $\tau^*(\mathscr{I}) \subseteq \tau^*(\mathscr{J})$.

Now we obtain some new fuzzy soft topologies through old fuzzy soft topologies by using fuzzy soft ideals.

Example 4.4. *Let (X,τ) be a fuzzy soft topological space where τ is the indiscrete fuzzy soft topology given in Example 2.8 and the parameter set E be an infinite set. Consider the fuzzy soft ideal $\mathscr{I}_f$ given in example 2.3. Note that $\tau^*(\mathscr{I}_f) = \{f_E^c \in FS(X,E)/f_E = cl^*(f_E)\}$. Now $f_E^* \sqsubseteq f_E \Rightarrow$ for all $e \in E$, $f_e^* \leq f_e$. If $f_E \neq \tilde{E}$, fix e_0 in E, such that $f_{e_0} \neq 1$. Then $f_{e_0}^* \leq f_{e_0} < \bar{1}$. Take $\bar{\lambda}$ such that $f_{e_0} < \bar{\lambda} < \bar{1}$. As $\bar{\lambda} > f_{e_0}^*$, there exists $f_B \in \tau$ with $e_0 \in B$ and $f_B(e_0) > \bar{1} - \bar{\lambda}$ and $f_B \tilde{\sqcap} f_E \in \mathscr{I}_f$. As $f_B \in \tau$, $f_B = \tilde{\lambda}_E$ where $\bar{\lambda} = f_B(e_0)$. Now $f_B \tilde{\sqcap} f_E \in \mathscr{I}_f$ implies that $S(f_B \tilde{\sqcap} f_E)$ is finite. That is $S(f_B \tilde{\sqcap} f_E) = \{e \in E/f_B(e) + f_e > \bar{1}\} = \{e \in E/f_e > \bar{1} - f_B(e_0)\}$ is a finite subset of E. As $\bar{\lambda} > \bar{1} - f_B(e_0)$, the set $\{e \in E/f_e > \bar{\lambda}\}$ is a finite set. Therefore $\tau^* = \{f_E^c \in FS(X,E)/f_E = \tilde{E}$ (or) to each $\bar{\lambda}$ such that $f_{e_0} < \bar{\lambda} < \bar{1}$ for some $e_0 \in E$, the set $\{e \in E/f_e > \bar{\lambda}\}$ is a finite set $\}$. Thus τ^* is a cofinite fuzzy soft topology on X.*

Example 4.5. *Let (X,τ) be a fuzzy soft topological space with indiscrete fuzzy soft topology. Let f_E be a fixed fuzzy soft set on X. Consider the fuzzy soft ideal $\mathscr{I}(f_E) = \{g_E \in FS(X,E)/g_E \sqsubseteq f_E\}$. Let $f_A \neq \tilde{E}$. $f_A = cl^*(f_A) \Rightarrow$ to each $e \in E, f_A^*(e) \leq f_A(e)$. If $\bar{\lambda} > f_A(e_0)$ for some $e_0 \in E$, there exists $\bar{\mu} > \bar{1} - \bar{\lambda}$ such that $f_A \tilde{\sqcap} \tilde{\mu}_E \in \mathscr{I}$. This implies that if*

$\bar{\lambda} > f_A(e_0)$, there exists a $\bar{\mu}$ such that $\bar{\mu}+\bar{\lambda} > \bar{1}$ and $f_A(e)+\bar{\mu}-\bar{1} \leq f_E(e)$ for all $e \in E$. That is $f_A(e) \leq f_e+\bar{1}-\bar{\mu} < f_e+\bar{\lambda}$. Therefore $f_A(e) \leq f_e+\bar{\lambda}$ for all e and for all $\bar{\lambda} > f_A(e_0)$. Therefore $f_A(e) \leq f_e+f_A(e_0)$ for all $e \in E$. This is true for any e_0. Thus $f_A(e) \leq f_e+f_A(e_0)$ for all e, $e_0 \in E$. So $f_A(e) \leq f_e+\inf_{e_o} f_A(e_0)$ for all $e \in E$.

Conversely, let $f_A(e)-\bar{m} \leq f_E(e)$ for all $e \in E$, where $\bar{m} = \inf_e f_A(e)$. Fix one $e_0 \in E$ such that $f_A(e_0) \neq \bar{1}$. Let $\bar{1} \geq \bar{\lambda} > f_A(e_0)$. Take $\bar{\mu}$ such that $\bar{1}-\bar{\lambda} < \bar{\mu} < \bar{1}-\bar{m}$. Then $-\bar{\lambda} < \bar{\mu}-\bar{1} < -\bar{m}$ and $f_A(e)-\bar{\lambda} < f_A(e)+\bar{\mu}-\bar{1} < f_A(e)-\bar{m} \leq f_A(e)$. Therefore $f_A(e)+\bar{\mu}-\bar{1} \leq f_A(e)$ for all $e \in E$. Therefore $f_A \tilde{\sqcap} \tilde{\mu}_E \in \mathscr{I}$. As $\bar{\mu} > \bar{1}-\bar{\lambda}, \bar{\lambda} > f_A(e_0)$ and $f_A \tilde{\sqcap} \tilde{\mu}_E \in \mathscr{I}$, we get $f_A^*(e_0) < \bar{\lambda}$. This is true for all $\bar{\lambda} > f_A(e_0)$. Therefore $f_A^*(e_0) \leq f_A(e_0)$. So $f_A^*(e) \leq f_A(e)$ for all $e \in E$ and $cl^*(f_A) = f_A$ Hence $\tau^*(\mathscr{I}(f_E)) = \{\tilde{f}_A^c / f_A \in FS(X,E)$ and $f_A(e) \leq f_e+\bar{m}$ where $\bar{m} = \inf_e f_A(e)\}$ is a fuzzy soft topology on X.

5. Conclusion

In the present work, we have defined and studied the important properties of fuzzy soft topological spaces and fuzzy soft local functions. We have generated new fuzzy soft topologies using fuzzy soft ideals. These findings will strengthen the foundation of fuzzy soft topological spaces. We hope the researchers working on fuzzy soft topological structures will be benefited.

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