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Triple Mixed Quadrature for Analytic Functions Through Extrapolation

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Abstract: This paper deals with the mixed quadrature rule of degree of precision eleven for the numerical evaluation of analytic functions through Richardson extrapolation. The numerical results are convergent to the exact results which are tested by taking some suitable texts and also error bound is determined.

MSC: 65D30, 65D32.

Keywords: Mixed quadrature, Analytic functions, Degree of precision, Maclaurin's theorem, Richardson extrapolation, Error bound.

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1. Introduction

Here we mixed up Birkhoff-Young modified rule which is obtained using Richardson extrapolation and Gauss-Legendre-4 point transformed quadrature rule each of precision seven to obtain a rule of precision nine [1, 2, 4, 5, 7, 8]. Again this rule is mixed with Gauss-Legendre-5 point rule to form a triple mixed quadrature rule $R_{RM\frac{BY}{2}GL4GL5}(f)$ for evaluating an integral of type

$$I(f) = \int_L f(z)\,dz \tag{1}$$

Where L is a directed line segment from the point $(z_0 - h)$ to $(z_0 + h)$ in the complex plane and $f(z)$ is analytic in certain domain Ω containing the line segment L. Lether (1976), [6] using the transformation $z = (z_0 + th)$, where $t \in [-1, 1]$ transform the integral (1) to the integral

$$h\int_{-1}^{1} f(z_0 + th)\,dt \tag{2}$$

The paper is organized as follows. Section 2 contains the mixed quadrature rule of modified Birkhoff-Young rule by Richardson extrapolation with Gauss-Legendre-4 point transformed rule $R_{RM\frac{BY}{2}GL4}(f)$ and its error $E_{RM\frac{BY}{2}GL4}(f)$. In Section 3 we formulate the triple mixed quadrature rule $R_{RM\frac{BY}{2}GL4GL5}(f)$ by the convex combination of $R_{RM\frac{BY}{2}GL4}(f)$ and Gauss-Legendre-5 point rule $R_{GL5}(f)$. In Section 4 the error analysis is done. The numerical verification of our purposed rule is obtained in Table 1 which is also compared with the numerical results of Table 2 for different researchers [4, 5, 7, 9] in literature in Section 5. The Section 6 contains conclusion.

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2. Modified Birkhoff-Young Rule with the Help of Richardson Extrapolation

In order to obtain that we have to analyze the Birkhoff-Young rule for analytic functions. Where,

$$R_{BY}(f) = \frac{h}{15}\left[24f(z_0) + 4\{f(z_0+h) + f(z_0-h)\} - \{f(z_0+ih) + f(z_0-ih)\}\right] \tag{3}$$

Applying Maclaurin's theorem in equation (3) we have,

$$R_{BY}(f) = 2h\begin{bmatrix} f(z_0) + \frac{h^2}{3!}f^{ii}(z_0) + \frac{h^4}{5!}f^{iv}(z_0) + \frac{h^6}{3\times 6!}f^{vi}(z_0) + \\ \frac{h^8}{5\times 8!}f^{viii}(z_0) + \frac{h^{10}}{3\times 10!}f^{x}(z_0) + \frac{h^{12}}{5\times 12!}f^{xii}(z_0) + \ldots \end{bmatrix} \tag{4}$$

Error in Birkhoff-Young rule ($E_{BY}(f)$):

$$I(f) = R_{BY}(f) + E_{BY}(f) \tag{5}$$

Using equation (1) and equation (4) in equation (5) we have,

$$E_{BY}(f) = -\frac{8h^7}{21\times 6!}f^{vi}(z_0) - \frac{8h^9}{45\times 8!}f^{viii}(z_0) - \frac{16h^{11}}{33\times 10!}f^{x}(z_0) - \frac{16h^{13}}{65\times 12!}f^{xii}(z_0)\ldots \tag{6}$$

Richardson extrapolation for modified Birkhoff-Young rule $\left(R_{RM\frac{BY}{2}}(f)\right)$:

$$E_{RBY}(f) = I(f) - I_n(f) \tag{7}$$

Where,

$$I_n(f) = R_{BY}(f)$$

$$E_{R\frac{BY}{2}}(f) = I(f) - I_{\frac{n}{2}}(f) \tag{8}$$

Where, $I_{\frac{n}{2}}(f) = R_{\frac{BY}{2}}(f)$ and n is number of intervals. From equation (6),

$$E_{RBY}(f) = -\frac{h^7}{1890}f^{vi}(z_0) - \frac{h^9}{226800}f^{viii}(z_0) - \frac{h^{11}}{7484400}f^{x}(z_0) - \frac{h^{13}}{1945944000}f^{xii}(z_0)\ldots \tag{9}$$

Again from equation (6),

$$E_{R\frac{BY}{2}}(f) = -\frac{2^7h^7}{1890}f^{vi}(z_0) - \frac{2^9h^9}{226800}f^{viii}(z_0) - \frac{2^{11}h^{11}}{7484400}f^{x}(z_0) - \frac{2^{13}h^{13}}{1945944000}f^{xii}(z_0)\ldots \tag{10}$$

Now multiplying $\left(2^7\right)$ with equation (7) and subtracting it from equation (8) we have,

$$\left\{I(f) - I_{\frac{n}{2}}(f)\right\} - 2^7\{I(f) - I_n(f)\} = E_{R\frac{BY}{2}}(f) - 2^7E_{RBY}(f) \tag{11}$$

Using equation (9) and equation (10) in equation (11) we have,

$$I(f) = \left[\frac{128I_n(f) - I_{\frac{n}{2}}(f)}{127}\right] + \begin{bmatrix} \frac{2^7h^9}{127\times 5\times 15120}f^{viii}(z_0) + \frac{2^7h^{11}}{33\times 127\times 15120}f^{x}(z_0) + \\ \frac{2^7h^{13}}{125\times 16\times 127\times 108\times 143}f^{xii}(z_0)\ldots \end{bmatrix} \tag{12}$$

$$I(f) = R_{RM\frac{BY}{2}}(f) + E_{RM\frac{BY}{2}}(f) \tag{13}$$

Where,

$$R_{RM\frac{BY}{2}}(f) = \left[\frac{128 I_n(f) - I_{\frac{n}{2}}(f)}{127}\right] \tag{14}$$

and

$$E_{RM\frac{BY}{2}}(f) = \left[\begin{array}{l} \frac{2^7 h^9}{127\times 5\times 15120} f^{viii}(z_0) + \frac{2^7 h^{11}}{33\times 127\times 15120} f^{x}(z_0) + \\ \frac{2^7 h^{13}}{125\times 16\times 127\times 108\times 143} f^{xii}(z_0)\ldots \end{array}\right] \tag{15}$$

Where equation (14) and equation (15) are called Modified Birkhoff-Young rule due to Richardson extrapolation and Error in Modified Birkhoff-Young rule respectively.

Gauss-Legendre-4 point Transformed rule: The Gauss-Legendre-4 point transformed rule is,

$$R_{GL4}(f) = \frac{h}{36}\left[\begin{array}{l} \left(18+\sqrt{30}\right)\{f(z_0-\alpha h) + f(z_0+\alpha h)\} + \\ \left(18-\sqrt{30}\right)\{f(z_0-\beta h) + f(z_0+\beta h)\} \end{array}\right] \tag{16}$$

Where,

$$\alpha = \sqrt{\frac{3-2\sqrt{\frac{6}{5}}}{7}}, \quad \beta = \sqrt{\frac{3+2\sqrt{\frac{6}{5}}}{7}}$$

Applying Maclaurin's theorem in equation (16) we have,

$$R_{GL4}(f) = 2h\left[\begin{array}{l} f(z_0) + \frac{h^2}{3!} f^{ii}(z_0) + \frac{h^4}{5!} f^{iv}(z_0) + \frac{h^6}{7!} f^{vi}(z_0) + \frac{6321}{5^2\times 7^4\times 8!} h^8 f^{viii}(z_0) + \\ \frac{32781}{5^2\times 7^5\times 10!} h^{10} f^{x}(z_0) + \frac{850689}{5^3\times 7^6\times 12!} h^{12} f^{xii}(z_0)\ldots \end{array}\right] \tag{17}$$

Error in Gauss-Legendre-4 point Transformed rule ($E_{GL4}(f)$):

$$I(f) = R_{GL4}(f) + E_{GL4}(f) \tag{18}$$

Using equation (1) and equation (17)

$$E_{GL4}(f) = \left(\frac{6272}{540225\times 8!}\right).h^{9} f^{viii}(z_0) + \left(\frac{119168}{4621925\times 10!}\right).h^{11} f^{x}(z_0) + \left(\frac{7294336}{191179625\times 12!}\right).h^{13} f^{xii}(z_0)\ldots \tag{19}$$

Mixed quadrature rule $\left(R_{RM\frac{BY}{2}GL4}(f)\right)$: Mixed quadrature rule for modified Birkhoff-Young rule using Richardson extrapolation and Gauss-Legendre-4 point transformed rule is obtained as follows. Multiplying equation (13) and equation (18) by $\left(\frac{1}{5880}\right)$ and $\left(-\frac{1}{127}\right)$ respectively and adding them we have,

$$I(f) = \frac{1}{5753}\left(5880 R_{GL4}(f) - 127 R_{RM\frac{BY}{2}}(f)\right) + \frac{1}{5753}\left(5880 E_{GL4}(f) - 127 E_{RM\frac{BY}{2}}(f)\right)$$

$$I(f) = R_{RM\frac{BY}{2}GL4}(f) + E_{RM\frac{BY}{2}GL4}(f) \tag{20}$$

Where,

$$R_{RM\frac{BY}{2}GL4}(f) = \frac{1}{5753}\left(5880 R_{GL4}(f) - 127 R_{RM\frac{BY}{2}}(f)\right) \tag{21}$$

$$E_{RM\frac{BY}{2}GL4}(f) = \frac{1}{5753}\left(5880 E_{GL4}(f) - 127 E_{RM\frac{BY}{2}}(f)\right) \tag{22}$$

Where equation (23) and equation (**??**) are called mixed quadrature of modified Birkhoff-Young and Gauss-Legendre-4 point rule due to Richardson extrapolation and error due to mixed quadrature rule respectively.

Error Bound for $R_{RM\frac{BY}{2}GL4}(f)$: Using equation (15) and equation (19) in equation (**??**) we have,

$$E_{RM\frac{BY}{2}GL4}(f) = -\left(\frac{253152}{135\times 5753\times 11\times 10!}\right).h^{11} f^{x}(z_0) - \left(\frac{481134551}{49\times 3\times 143\times 5753\times 13\times 12!}\right).h^{13} f^{xii}(z_0)\ldots \tag{23}$$

3. Triple Mixed Quadrature Rule $R_{RM\frac{BY}{2}GL4GL5}(f)$ of Degree of Precision Eleven

The Gauss-Legendre-5 point rule is

$$I(f) = R_{GL5}(f) \cong \int_{-1}^{1} f(z)\,dz$$
$$= \frac{h}{900}\begin{bmatrix} \left(322+13\sqrt{70}\right)\{f(z_0-\alpha h)+f(z_0+\alpha h)\}+ \\ \left(322-13\sqrt{70}\right)\{f(z_0-\beta h)+f(z_0+\beta h)\}+512f(0) \end{bmatrix} \tag{24}$$

Where,

$$\alpha = \sqrt{\frac{5-2\sqrt{\frac{10}{7}}}{9}},\quad \beta = \sqrt{\frac{5+2\sqrt{\frac{10}{7}}}{9}}$$

Applying Maclaurin's series in equation (24)

$$R_{GL5}(f) = 2h\begin{bmatrix} f(z_0)+\frac{h^2}{3!}f^{ii}(z_0)+\frac{h^4}{5!}f^{iv}(z_0)+\frac{h^6}{7!}f^{vi}(z_0)+\frac{h^8}{9!}f^{viii}(z_0)+ \\ \frac{355}{7^2\times 9^2\times 10!}h^{10}f^{x}(z_0)+\frac{1899045}{7^2\times 9^6\times 12!}h^{12}f^{xii}(z_0)\ldots \end{bmatrix} \tag{25}$$

Error in Gauss-Legendre-5 point rule ($E_{GL5}(f)$):

$$I(f) = R_{GL5}(f) + E_{GL5}(f) \tag{26}$$

Using equation (1) and equation (25) we get

$$E_{GL5}(f) = \left(\frac{128}{7^2\times 9^2\times 11\times 10!}\right)h^{11}f^{x}(z_0) + \left(\frac{2706048}{7^2\times 9^6\times 13\times 12!}\right)h^{13}f^{xii}(z_0)\ldots \tag{27}$$

Mixed quadrature rule $\left(R_{RM\frac{BY}{2}GL4GL5}(f)\right)$: Multiplying equation (20) and equation (26) by $\left(\frac{4}{3969}\right)$ and $\left(\frac{293}{28765}\right)$ respectively and adding them we have,

$$I(f) = \frac{1}{1277977}\left(115060R_{RM\frac{BY}{2}GL4}(f)+1162917R_{GL5}(f)\right)+\frac{1}{1277977}\left(115060E_{RM\frac{BY}{2}GL4}(f)+1162917E_{GL5}(f)\right) \tag{28}$$

$$I(f) = R_{RM\frac{BY}{2}GL4GL5}(f) + E_{RM\frac{BY}{2}GL4GL5}(f) \tag{29}$$

Where,

$$R_{RM\frac{BY}{2}GL4GL5}(f) = \frac{1}{1277977}\left(115060R_{RM\frac{BY}{2}GL4}(f)+1162917R_{GL5}(f)\right) \tag{30}$$

and

$$E_{RM\frac{BY}{2}GL4GL5}(f) = \frac{1}{1277977}\left(115060E_{RM\frac{BY}{2}GL4}(f)+1162917E_{GL5}(f)\right) \tag{31}$$

From equation (31),

$$\left|E_{RM\frac{BY}{2}GL4GL5}(f)\right| = \frac{3872292677073}{17058113254065640\times 12!}h^{13}f^{xii}(z_0) \tag{32}$$

4. Error Analysis

An asymptotic error estimate and an error bound of the rule (30) are given in Theorem 4.1 and Theorem 4.2.

Theorem 4.1. *Let $f(z)$ be a sufficiently differentiable function in the closed interval $[-1,1]$. Then the error $E_{RM\frac{BY}{2}GL4GL5}(f)$ associated with the rule $R_{RM\frac{BY}{2}GL4GL5}(f)$ is given by,*

$$\left|E_{RM\frac{BY}{2}GL4GL5}(f)\right| \cong \frac{3872292677073}{17058113254065640\times 12!}h^{13}\left|f^{xii}(z_0)\ldots\right|$$

Proof. From equation (31),

$$E_{RM\frac{BY}{2}GL4GL5}(f) \cong \frac{3872292677073}{17058113254065640\times 12\,!}h^{13}f^{xii}(z_0)\ldots$$

So,

$$\left|E_{RM\frac{BY}{2}GL4GL5}(f)\right| \cong \frac{3872292677073}{17058113254065640\times 12\,!}h^{13}\left|f^{xii}(z_0)\right|\ldots.$$

□

Theorem 4.2. *The bound for the truncation error*

$$E_{RM\frac{BY}{2}GL4GL5}(f) = I(f) - R_{RM\frac{BY}{2}GL4GL5}(f)$$

is given by,

$$\left|E_{RM\frac{BY}{2}GL4GL5}(f)\right| \leq \frac{37504\,M}{14057747\times 10\,!}\left|\eta_2-\eta_1\right|,\ \ \eta_1,\,\eta_2\in[-1,1]$$

Where,

$$M = \underset{-1\leq x\leq 1}{Max}\left|f^{xi}(z_0)\right|$$

Proof. We have,

$$E_{RM\frac{BY}{2}GL4}(f) \cong \frac{-253152}{135\times 5753\times 11\times 10\,!}h^{11}f^{x}(\eta_1)\,\eta_1\in[-1,1]$$

And

$$E_{GL5}(f) \cong \frac{128}{7^2\times 9^2\times 11\times 10!}h^{11}f^{x}(\eta_2)\,,\quad \eta_2\in[-1,1]$$

Hence,

$$\begin{aligned}
E_{RM\frac{BY}{2}GL4GL5}(f) &\cong \frac{1}{1277977}\Big(115060E_{RM\frac{BY}{2}GL4}(f)+1162917E_{GL5}(f)\Big)\\
&= \frac{1}{1277977}\left[\begin{array}{l}115060\left\{\frac{-253152}{135\times5753\times11\times10\,!}.h^{11}f^{x}(\eta_1)\right\}+\\ 1162917\left\{\frac{128}{7^2\times9^2\times11\times10\,!}.h^{11}f^{x}(\eta_2)\right\}\end{array}\right]\\
&= \frac{2^5}{11\times1277977\times10\,!}h^{11}\left[1172f^{x}(\eta_2)-1172f^{x}(\eta_1)\right]\\
&= \frac{32\times1172}{11\times1277977\times10\,!}h^{11}\left[f^{x}(\eta_2)-f^{x}(\eta_1)\right]\\
&= \frac{37504}{14057747\times10!}h^{11}\left[f^{x}(\eta_2)-f^{x}(\eta_1)\right]\\
&= \frac{37504}{14057747\times10\,!}h^{11}\int_{\eta_1}^{\eta_2}f^{xi}(z_0)\,dz\ \ (\text{Assuming }\eta_1<\eta_2)\\
\left|E_{RM\frac{BY}{2}GL4GL5}(f)\right| &= \left|\frac{37504}{14057747\times10!}h^{11}\int_{\eta_1}^{\eta_2}f^{xi}(z_0)\,dz\right|
\end{aligned}$$

$$\leq \frac{37504}{14057747 \times 10\,!} h^{11} \int_{\eta_1}^{\eta_2} \left| f^{xi}(z_0) \right| dz$$

$$\left| E_{RM\frac{BY}{2}GL4GL5}(f) \right| \leq \frac{37504\,M}{14057747 \times 10\,!} h^{11} \left| \eta_2 - \eta_1 \right|$$

Which gives only a theoretical error bound as η_1 and η_2 are known points in $[-1, 1]$. From the equation (1) it is evident that the error will be less if the points η_1 and η_2 are closer to each other. □

Corollary 4.3. *The error bound for the truncation error is*

$$\left| E_{RM\frac{BY}{2}GL4GL5}(f) \right| \leq \frac{75008\,M}{14057747 \times 10\,!} h^{11}$$

Proof. We know from Theorem 4.2,

$$\left| E_{RM\frac{BY}{2}GL4GL5}(f) \right| \leq \frac{37504\,M}{14057747 \times 10\,!} h^{11} \left| \eta_2 - \eta_1 \right|, \quad \eta_1,\, \eta_2 \in [-1,\, 1]$$

Where,

$$M = \underset{-1 \leq x \leq 1}{Max} \left| f^{xi}(z_0) \right|$$

By Conte and Boor [3]; $|\eta_2 - \eta_1| \leq 2$. So we have,

$$\left| E_{RM\frac{BY}{2}GL4GL5}(f) \right| \leq \frac{75008\,M}{14057747 \times 10\,!} h^{11}$$

□

5. Numerical Verification

Here some texts are

$$I_1 = \int_{-i}^{i} e^z dz,$$

$$I_2 = \int_{-i}^{i} \cos\, zdz,$$

$$I_3 = \int_{\frac{-i}{3}}^{\frac{i}{3}} \cosh zdz$$

Quadrature rule	Approximation value I_1	Approximation value I_2	Approximation value I_3
$R_{BY}(f)$	1.682417145154309 i	2.350936031119045 i	0.654389151885734 i
$R_{RM\frac{BY}{2}}(f)$	1.682930690528399 i	2.350387003041248 i	0.654389391594494 i
$R_{GL4}(f)$	1.682941688695974 i	2.350402092156376 i	0.654389393577715 i
$R_{RM\frac{BY}{2}GL4}(f)$	1.682941931485350 i	2.350402425255215 i	0.654389393621495 i
$R_{GL5}(f)$	1.682941970407192 i	2.350402386462826 i	0.654389393592309 i
$R_{RM\frac{BY}{2}GL4GL5}(f)$	1.682941966902945 i	2.350402389955419 i	0.654389393594937 i
Exact Value	1.682941969615793 i	2.350402387287603 i	0.654389393592304 i
$\left\| E_{RM\frac{BY}{2}GL4GL5}(f) \right\|$	0.000000002712847 i	0.000000002667815 i	0.000000000002633 i

Table 1.

From Table 1 it is evident that,

$$\left| E_{RM\frac{BY}{2}GL4GL5}(f) \right| \leq \left| E_{RM\frac{BY}{2}GL4}(f) \right| \leq \left| E_{RM\frac{BY}{2}}(f) \right|$$

From the result of different researchers, $R_{DJ}(f)$ [4], $R_{JD}(f)$ [5], $R_{MD}(f)$ [7], $R_{SAA}(f)$ [9] the respective approximate value of I_1 is represented in Table 2.

$$I_1 = \int_{-i}^{i} e^z dz$$

Quadrature rule	Approximation value I_1
$R_{DJ}(f)$	1.6829423 i
$R_{JD}(f)$	**1.682941652305530 i**
$R_{MD}(f)$	1.682944 i
$R_{SAA}(f)$	1.6829419140 i
Exact Value	1.682941969615793 i

Table 2. Comparison with result of Table 1

6. Conclusion

From Table 1 and Table 2 it is found that our mixed quadrature rule $R_{RM\frac{BY}{2}GL4GL5}(f)$ of degree of precision eleven is more efficient than all the mixed quadrature rules. Our mixed quadrature rule numerically integrates more accurately to the exact result of I_1.

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