Int. J. Math. And Appl., 6(3)(2018), 69–75
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

Common Best Proximity Point Theorem of Two Non-self Mappings Satisfying a Generalized Weak Contractive Condition

Vinita Dewangan[1,*] and Amitabh Banerjee[2]

1 Department of Mathematics, Government J.Y. Chhattishgarh College, Raipur, Chhattishgarh, India.

2 Department of Mathematics, Government S.N. College, Nagari, Dhamtari, Chhattishgarh, India.

Abstract: In this paper, we use the notion of weakly contractive non-self mappings and establish the existence of common best proximity points for two mappings which satisfy a generalized weak contractive condition in metric spaces. We also provide an example to illustrate our result.

MSC: 46J10, 47H10, 54H25, 41A65, 90C30.

Keywords: Common fixed point, weakly contractive non-self mappings, best proximity points, P-property.

 Accepted on: 01.05.2018

1. Introduction and Preliminaries

Let A and B be nonempty subsets of a metric space (X, d). An operator $T : A \to B$ is said to be contractive if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$ for any $x, y \in A$. The well known Banach contraction principle says: Let (x, d) be a complete metric space and let $T : X \to X$ be a contraction of X into itself then T has a unique fixed point in X. The Banach contraction has many generalizations. One of the generalization is weakly contractive mappings. In 1997, Alber and Guerre [1] introduced the notion of weakly contractive self mapping.

Definition 1.1 ([1]). *Let X be a metric space and A be a nonempty subset of X. A map $T : A \to A$ is said to be a weakly contractive self-map if*

$$d(Tx, Ty) = d(x, y) - \psi(d(x, y)), \quad for\ all\ x, y \in A,$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ψ is positive on $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. If A is bounded, then the infinity condition can be omitted.

Alber and Guerre [1] futher proved that if $T : A \to A$ is a weakly contractive self-map, where A is a closed convex subset of a Hilbert space, then T has a unique fixed point in A. Later, in [2], Rhoades proved that the existence of a unique fixed point for a weakly contractive self-map could be achieved even in a complete metric space setting. He proved the following result.

* *E-mail: vinitadewangan12@gmail.com*

Theorem 1.2 ([2]). *Let (X, d) be a complete metric space and $T : X \to X$ be a weakly contractive self-map. Then T has a unique fixed point in X.*

Dutta and Choudhury [6] gave a generalization of Banach contraction principle, which in turn generalize [[2], Theorem 1] and corresponding result of [1]. Let $T : A \to B$, where A, B are two nonempty subsets of a metric space (X, d). Note that if $A \cap B = \emptyset$, the equation Tx = x might have no solution. Under this circumstance it is meaningful to find a point $x \in A$ such that $d(x, Tx)$ is minimum. Essentially, if $d(x, Tx) = dist(A, B) = \min\{d(x, y) : x \in A, y \in B\}$, d(x, Tx) is the global minimal value and hence x is an approximate solution of the equation Tx = x with the least possible error. Such a solution is known as the best proximity point of the mapping T. A point $x \in A$ is called the best proximity point of T if

$$d(x, Tx) = dist(A, B) = \min\{d(x, y) : x \in A, y \in B\}.$$

It is easy to verify that if $A \cap B \neq \emptyset$, the best proximity point is the fixed point of T. Best proximity point theory of non-self function was initiated by Fan [3] and Kirk et al. [4]; see also [8–10]. The aim of this paper is to present common best proximity point theorem for non-self maps satisfying a generalized weak contractive condition and give an example. Before proceeding further let us fix the following notations: Let A and B are nonempty subsets of a metric space X,

$$d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},$$
$$A_0 = \{x \in A : d(x, y) = d(A, B) \quad for\ some\ y \in B\}$$
$$B_0 = \{y \in B : d(x, y) = d(A, B) \quad for\ some\ x \in A\}.$$

In [4], the authors discussed sufficient conditions which guarantee the non emptiness of A_0 and B_0. Also, in [12], the authors proved that A_0 is contained in the boundary of A. Further, a common fixed point theorem for commuting self-mappings is a special case of common best proximity point theorem.

Definition 1.3 ([11]). *Let X be a set, and f, g self maps of X. A point x in X is called a coincidence point of f and g if and only if $fx = gx$. We will call $w = fx = gx$ a point of coincidence of f and g.*

Definition 1.4 ([11]). *Two maps f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 1.5 ([7]). *Let $f_1, f_2, ..., f_n : A \to B$ are non-self mappings and an element $x \in A$ is said to common best proximity point if*

$$d(x, f_1x) = d(x, f_2x) = ... = d(x, f_nx) = dist(A, B).$$

Definition 1.6 ([13]). *The mappings $f : A \to B$ and $g : A \to B$ are said to commute proximally if they satisfy the condition*

$$d(u, fx) = dist(A, B) \quad and \quad d(v, gx) = dist(A, B) \ \Rightarrow fv = gu,$$

for all $x, u, v \in A$.

It is clear that the proximal commutativity of self mappings is just commutativity of the non-self mappings. The notion of P-property was introduced in [5] as follows.

Definition 1.7 ([5]). *Let (A, B) be a pair of nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if*

$$d(x_1, y_1) = dist(A, B) \quad d(x_2, y_2) = dist(A, B) \quad \Rightarrow \ d(x_1, x_2) = d(y_1, y_2),$$

where $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$.

By using the P-property, a best proximity point theorem were proved in [5].

Theorem 1.8 ([5]). *Let (A, B) be a pair of non-empty closed subsets of a complete metric space X such that A_0 is non-empty. Let $T : A \to B$ be a weakly contractive non-self mapping; that is,*

$$d(Tx, Ty) = d(x, y) - \phi(d(x, y)), \; for\ all\ x, y \in A,$$

where $\phi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that ϕ is positive on $(0, \infty)$, $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. Assume that the pair (A, B) has the P-property and $T(A_0) \subseteq B_0$. Then T has a unique best proximity point.

2. A Common Best Proximity Point Theorem

Set $G = \{\psi : [0, \infty) \to [0, \infty) : \psi\ is\ continuous\ and\ nondecreasing\ mapping\ with\ \psi(t) = 0\ if\ and\ only\ if\ t = 0\}$. Following is the main result of this paper.

Theorem 2.1. *Let A and B be a pair of two non-empty closed subsets of a complete metric space X such that A_0 is non-empty. Let $f, g : A \to B$ be two non-self mappings satisfying the following conditions:*

(1). f and g commute proximally;

(2). the pair (A, B) has the P-property;

(3). f and g are continuous;

(4). $f(A_0) \subseteq B_0$, $g(A_0) \subseteq B_0$ and $f(A_0) \subseteq g(A_0)$;

(5). f and g satisfy the condition; $\psi\big(d(fx, fy)\big) \leq \psi\big(d(gx, gy)\big) - \varphi\big(d(gx, gy)\big)$, for all $x, y \in A$ where $\psi, \varphi \in G$.

Then f and g have a unique common best proximity point.

Proof. Let $x_0 \in A_0$. Since $f(A_0) \subseteq g(A_0)$, then there exists $x_1 \in A_0$ such that $f(x_0) = g(x_1)$. Similarly, a point $x_2 \in A_0$ can be chosen such that $f(x_1) = g(x_2)$. Continuing this process, having chosen $x_n \in A_0$, we obtain $x_{n+1} \in A_0$ such that $f(x_n) = g(x_{n+1})$, $n = 0, 1, 2 \cdots$. Since $f(A_0) \subseteq B_0$ and $g(A_0) \subseteq B_0$ there exists $\{x_n\} \in A_0$ such that

$$d(x_{n+1}, gx_{n+1}) = d(A, B), \tag{1}$$

$$d(x_n, gx_n) = d(A, B). \tag{2}$$

Since (A, B) satisfy the P-property, by (1) and (2) we conclude that

$$\begin{aligned} \psi\big(d(x_n, x_{n+1})\big) &= \psi\big(d(gx_{n+1}, gx_n)\big) \\ &= \psi\big(d(fx_n, fx_{n-1})\big) \\ &\leq \psi\big(d(gx_n, gx_{n-1})\big) - \varphi\big(d(gx_n, gx_{n-1})\big) \\ &< \psi\big(d(gx_n, gx_{n-1})\big). \end{aligned}$$

That is, $\psi\big(d(gx_{n+1}, gx_n)\big) < \psi\big(d(gx_n, gx_{n-1})\big)$ and hence

$$d(gx_{n+1}, gx_n) \leq d(gx_n, gx_{n-1}),$$

it follows that $\{d(gx_{n+1}, gx_n)\}$ is nondecreasing sequence numbers and consequently there exists $r \geq 0$ such that

$$d(gx_{n+1}, gx_n) \to r \ as \ n \to \infty.$$

Suppose that $r > 0$ then

$$\begin{aligned} 0 < \psi(r) &\leq \psi\big(d(gx_{n+1}, gx_n)\big) \\ &\leq \psi\big(d(fx_n, fx_{n-1})\big) \\ &\leq \psi\big(d(gx_n, gx_{n-1})\big) - \varphi\big(d(gx_n, gx_{n-1})\big), \end{aligned}$$

which on taking limit as $n \to \infty$ yields.

$$\psi(r) \leq \psi(r) - \varphi(r) < \psi(r),$$

which is a contradiction. Therefore $r = 0$. Now we prove that $\{gx_n\}$ is a cauchy sequence. If not, then there exists $\epsilon > 0$ and subsequence $\{g(x_{n_k})\}$ and $\{g(x_{m_k})\}$ of $\{g(x_n)\}$ with $k < n_k < m_k$ such that $d(gx_{n_k}, gx_{m_k}) \geq 3\epsilon$ for each k. As $d\big(g(x_{n_{k+1}}, g(x_{n_k})\big) \to 0 \ as \ k \to \infty$, for large enough k, we have

$$d\big(g(x_{n_{k+1}}, g(x_{n_k})\big) < \epsilon \ and \ d(gx_{m_{k+1}}, gx_{m_k}) < \epsilon.$$

Thus we obtain

$$\begin{aligned} d(gx_{n_{k+1}}, gx_{m_k}) \geq d(gx_{n_k}, gx_{m_k}) - d(gx_{n_{k+1}}, gx_{n_k}) > \epsilon, \\ d(gx_{n_{k+1}}, gx_{m_{k-1}}) \geq d(gx_n, gx_{m_k}) - d(gx_{m_{k-1}}, gx_{m_k}) - d(gx_{n_{k+1}}, gx_{n_k}) > \epsilon, \end{aligned} \tag{3}$$

we may assume that n_k are even and m_k are odd and that $d(gx_{n_k}, gx_{m_k}) > \epsilon$ for all k.

$$Put \quad r_k = \min\{m_k : d(gx_{n_k}, gx_{m_k}) > \epsilon\}.$$

Now,

$$\epsilon < d(gx_{n_k}, gx_{r_k}) \leq d(gx_{n_k}, gx_{r_{k-2}}) + d(gx_{r_{k-2}}, gx_{r_{k-1}}) + d(gx_{r_{k-1}}, gx_{r_k}),$$

implies that $d(gx_{n_k}, gx_{r_k}) \to \epsilon$ as $k \to \infty$. Furthermore

$$\begin{aligned} d(gx_{n_k}, gx_{r_k}) &- d(gx_{n_k}, gx_{n_{k+1}}) - d(gx_{r_k}, gx_{r_{k+1}}) \\ &\leq d(gx_{n_{k+1}}, gx_{r_{k+1}}) \leq d(gx_{n_k}, gx_{r_k}) + d(gx_{n_k}, gx_{n_{k+1}}) + d(gx_{r_k}, gx_{r_{k+1}}), \end{aligned}$$

gives $d(gx_{n_{k+1}}, gx_{r_{k+1}}) \to \epsilon$ as $k \to \infty$. Therefore

$$\begin{aligned} \psi\big(d(gx_{n_{k+1}}, gx_{r_{k+1}})\big) &= \psi\big(d(fx_{n_k}, fx_{r_k})\big) \\ &\leq \psi\big(d(gx_{n_k}, gx_{r_k})\big) - \varphi\big(d(gx_{n_k}, gx_{r_k})\big). \end{aligned}$$

Taking limit as $k \to \infty$ yields

$$\psi(\epsilon) \leq \psi(\epsilon) - \varphi(\epsilon),$$

which is a contradiction. Hence $\{g(x_n)\}$ is cauchy sequence. Since A is closed subset of the complete metric space X, there exists a point $x \in A$ such that $\lim_{n\to\infty} gx_n \to x$. By (1) and (2) because of the fact f and g are commute proximally, $fx_{n-1} = gx_n$. Therefore, the continuity of f and g and $n \to \infty$ ascertain that $fx = gx$. Since $g(A_0) \subseteq B_0$ and $f(A_0) \subseteq B_0$, there exists $x^* \in A_0$ such that $x_n \to x^*$

$$d(x^*, gx) = dist(A, B),$$
$$d(x^*, fx) = dist(A, B),$$

f and g are commute proximally, $fx^* = gx^*$. Since $g(A_0) \subseteq B_0$ and $f(A_0) \subseteq B_0$, there exists $z \in A_0$ such that

$$d(z, gx^*) = dist(A, B),$$
$$d(z, fx^*) = dist(A, B).$$

Because the pair (A, B) has the P-property

$$\begin{aligned}
\psi\big(d(x^*, z)\big) &= \psi\big(d(gx, gx^*)\big) \\
&= \psi\big(d(fx, fx^*)\big) \\
&\leq \psi\big(d(gx, gx^*)\big) - \varphi\big(d(gx, gx^*)\big) \\
&< \psi\big(d(gx, gx^*)\big) = \psi\big(d(x^*, z)\big),
\end{aligned}$$

which is a contradiction, which implies that $x^* = z$, thus

$$d(x^*, gx^*) = dist(A, B),$$
$$d(x^*, fx^*) = dist(A, B).$$

Then x^* is a common best proximity point of the mappings f and g. Suppose that y is another common best proximity point of the mappings f and g, so that

$$d(y, gy) = dist(A, B),$$
$$d(y, fy) = dist(A, B).$$

As the pair (A, B) has P-property

$$\psi\big(d(x^*, y) \leq \psi\big(d(x^*, y),$$

which is contradiction, therefore $x^* = y$ this gives, x^* is unique common best proximity point of the mappings f and g. □

Corollary 2.2. *Let f and g be two non-self mappings of a metric space (X, d) satisfying*

$$\int_0^{\psi\left(d(fx,fy)\right)} \phi(t)dt \leq \int_0^{\psi\left(d(gx,gy)\right)} \phi(t)dt - \int_0^{\varphi\left(d(gx,gy)\right)} \phi(t)dt, \tag{4}$$

for all $x, y \in A$, where $\phi \in F$ and $\psi, \varphi \in G$. If rang of g contains rang of f and g(X) is complete subspace of A, $f, g : A \to B$ be two non-self mappings satisfying all conditions of Theorem 2.1, then f and g have a unique common best proximity point.

Proof. Define $\Phi : R^+ \to R^+$ by $\Phi(x) = \int_0^x \phi(t)dt$, then $\Phi \in G$ and 4 becomes

$$\Phi\Big(\psi\big(d(fx, fy)\big)\Big) \le \Phi\Big(\psi\big(d(gx, gy)\big)\Big) - \Phi\Big(\varphi\big(d(gx, gy)\big)\Big), \tag{5}$$

which further can be written as

$$\psi_1\big(d(fx, fy)\big) \le \psi_1\big(d(gx, gy)\big) - \varphi_1\big(d(gx, gy)\big), \tag{6}$$

where $\psi_1 = \Phi \circ \psi \in G$. Clearly $\psi_1, \varphi_1 \in G$. Hence by Theorem 2.1 f and g have unique common best proximity point. □

Example 2.3. *Let* $X = [0, 1]$ *with usual metric and* $d(x, y) = |x - y|$ *if* $(x, y) \in [0, 1]$*. Then (X, d) is a complete metric space. Suppose that*

$$A := \{(0, x) : 0 \le x \le 1\},$$
$$B := \{(1, y) : 0 \le y \le 1\},$$

dist$(A, B) = 1$*,* $A_0 = A$ *and* $B_0 = B$ *and (A, B) has P-property. Let* $f : A \to B$ *and* $g : A \to B$ *and* $\psi, \varphi \in G$ *as given in*

$$f(0, x) = (1, x - \frac{1}{2}x^2) \qquad if\ 0 \le x \le 1,$$
$$g(0, x) = (1, x) \qquad if\ 0 \le x \le 1,$$

and

$$\psi(t) = t \qquad if\ 0 \le t \le 1,$$
$$\varphi(t) = \frac{1}{2}t^2 \qquad if\ 0 \le t \le 1.$$

Then for all $x, y \in X$ *we have*

$$\begin{aligned} \psi\big(d(fx, fy)\big) &= |(x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2)| \\ &= |x - y| - \frac{1}{2}|x^2 - y^2| \\ &= |x - y|\big(1 - \frac{1}{2}|x + y|\big) \\ &\le |x - y| - \frac{1}{2}|x - y|^2 \\ &\le \psi\big(d(gx, gy)\big) - \varphi\big(d(gx, gy)\big). \end{aligned}$$

Now, all the required conditions of Theorem 2.1 are satisfied. Hence (0,0) is unique common best proximity point of f and g.

References

[1] Ya. I. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert spaces*, New Results in Operator Theory and its Applications, Birkhauser, Basel, 98(1997), 722.

[2] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal. TMA, 47(2001), 26832693.

[3] K. Fan, *Extensions of two fixed point theorems*, F.E. Browder, Math.Z., 112(1969), 234-240.

[4] W. A. Kirk, S. Reich and P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., 24(78)(2003), 851862.

[5] V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., 74(2011), 48044808.

[6] P. N. Dutta and B. S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed point theory and applications, 2008(2008), Article ID406368, 8 pages.

[7] P. Lolo, S. M. Vaezpour and J. Esmaily, *Common best proximity points theorem for four mappings in metric-type spaces*, Fixed Point Thoery and App., 2015(2015).

[8] A. Abkar and M. Gabeleh, *Best proximity points of non-self mappings*, TOP, 21(2)(2013), 287-295.

[9] M. Gabeleh, *Proximal weakly contractive and proximal nonexpansive non-self mappings in metric and banach spaces*, Journal of Optimization Theory and Applications, 158(2)(2013), 615-625.

[10] E. Karapinar, V. Pragadeeswarar and M. Marudai, *Best proximity point for generalised proximal weak contractions in complete metric space*, Journal of Appilied Mathematics, 2014(2014), Article ID 150941, 6 pages.

[11] M. Abbas and M. Ali Khan, *Common fixed point theorem of two mappings satisfying a generalised weak contractive condition*, Int. Journal of Mathematics and Mathematical Science, 2009(2009), Article ID131068.

[12] S. Sadiq Basha and P. Veeramani, *Best proximity pair theorems for multi functions with open fibres*, J. Approx. Theory, 103(1)(2000), 119-129.

[13] S. Sadiq Basha, *Common best proximity points: global minimization of multi-objective functions*, J. Glob. Optim., DOI 10.1007/s10898-011-9760-8.