Int. J. Math. And Appl., 6(3)(2018), 77–88
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

Existence and Continuous Dependence of the Solutions of the Benjamin-Bona-Mahony-Peregrine-Burger's Equation on the Circle

R. Gokilam[1,*]

1 Department of Mathematics, Periyar University, Salem, Tamil Nadu, India.

Abstract: In this paper, we show the existence and continuous dependence of the solutions of the Benjamin-Bona-Mahony-Peregrine-Burger's(BBMPB) equation in Sobolev spaces H^s, for $s > \frac{3}{2}$. We employ a Galerkin approximation argument to show the existence of solutions of BBMPB equation.

Keywords: Continuous dependence, Sobolev space, Galerkin approximation.

Accepted on: 20.06.2018

1. Introduction

Consider the initial value problem for the Benjamin-Bona-Mahony-Peregrine-Burgers(BBMPB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x + \beta u_{xxx} = 0$$

$$u(x,0) = u_0(x)$$

where α is a positive constant, θ and β are nonzero real numbers. The BBMPB equation can be (and is more conveniently) written in the following non-local form

$$u_t + \theta u u_x = \partial_x(1-\partial x^2)^{-1}(-\theta u u_{xx} + \alpha u_x - \gamma u - \beta u_{xx} + \theta u_x^2)$$

The non-local form can be obtained from BBMPB equation as follows.

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x + \beta u_{xxx} = 0$$

adding and subtracting the terms $3\theta u_x u_{xx}$ and $\theta u u_{xxx}$

$$u_t + \theta u u_x - u_{xxt} - \theta u u_{xxx} + \theta u u_{xxx} - 3\theta u_x u_{xx} + 3\theta u_x u_{xx} - \alpha u_{xx} + \gamma u_x + \beta u_{xxx} = 0$$

$$u_t + \theta u u_x - u_{xxt} - \theta u u_{xxx} - 3\theta u_x u_{xx} = -\theta u u_{xxx} - 3\theta u_x u_{xx} + \alpha u_{xx} - \gamma u_x - \beta u_{xxx}$$

* *E-mail: gokilamrangaswamy@gmail.com*

$$(1-\partial_x^2)(u_t+\theta uu_x) = -\theta uu_{xxx} - \theta u_x u_{xx} - 2\theta u_x u_{xx} + \alpha u_{xx} - \gamma u_x - \beta u_{xxx}$$

$$(1-\partial_x^2)(u_t+\theta uu_x) = \partial_x[-\theta uu_{xx} + \alpha u_x - \gamma u - \beta u_{xx} + \theta u_x^2]$$

multiply bothsides by $(1-\partial_x^2)^{-1}$ we get

$$(u_t+\theta uu_x) = (1-\partial_x^2)^{-1}\partial_x[-\theta uu_{xx} + \alpha u_x - \gamma u - \beta u_{xx} + \theta u_x^2]$$

written this way, the BBMPB equation is a special case in the family of nonlinear wave equations of the form

$$u_t + auu_x = L(u).$$

2. Preliminaries

Definition 2.1. *A Schwarz function $j(x)\in\mathcal{S}(\mathbb{R})$ satisfying $0 \le \hat{j}(\xi) \le 1$ for all $\xi\in\mathbb{R}$, with $\hat{j}(\xi) = 1$ for $|\xi| \le 1$ and $\hat{j}(\xi) = 0$ for $|\xi| \ge 2$. We then define $j_\epsilon(x) = \frac{1}{2\pi}\sum_n \hat{j}(\epsilon n)e^{inx\cdot}$ Given $j_\epsilon(x)$, we define Friedrichs mollifier on a test function f by the convolution $j_\epsilon f = j_\epsilon \star f$.*

Definition 2.2. *For any $s\in\mathbb{R}$ the operator $\Lambda^s = (1-\partial_x^2)^{s/2}$ is defined by*

$$\hat{\Lambda^s u}(k) = (1+k^2)^{s/2}\hat{u}(k)$$

where $\hat{u}$ is the fourier transform

$$\hat{u}(k) = \int_T e^{-ikx}u(x)dx$$

The inverse relation is given by

$$u(x) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\hat{u}(k)e^{ikx}$$

Then, for $u\in H^s(T)$ we have

$$\|u\|^2_{H^s(T)} = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}(1+k^2)^s|\hat{u}(k)|^2 = \|\Lambda^s u\|^2_{L^2(T)}\,.$$

where $\Lambda^{-2} = (1-\partial_x^2)^{-1}$.

Theorem 2.3. *For $r < s$ we have*

$$\| I - J_\epsilon \|_{L(H^s;H^r)} = o(\epsilon^{s-r})$$

Also, for any test function f, we have for all $s > 0$, $J_\epsilon f \longrightarrow f \in H^s$. We similarly have the growth estimate when $r > s$.

Theorem 2.4. *Let $r \ge s$, then for any test function f*

$$\| J_\epsilon f \|_{H^r} \le \epsilon^{s-r} \| f \|_{H^s}$$

Let $\Lambda = (1-\partial_x{}^2)$ so that for any test function f, we have $\mathcal{F}(\Lambda^s f) = (1+k^2)^s\hat{f}(k)$. Then we have the following basic estimates.

Lemma 2.5. *Let f be any test function, and $\sigma \in \mathbb{R}$, then $\|\Lambda^\sigma f\|_{L^2} = \|f\|_{H^\sigma}$, $\left\|(1-\partial_x{}^2)^{-1}f\right\|_{H^\sigma} = \|f\|_{H^{\sigma-2}}$, $\|\partial_x f\|_{H^\sigma} \le \|f\|_{H^{\sigma+1}}$. We define the commutator $[\Lambda^s, f] = \Lambda^s f - f\Lambda^s$, in which a test function f is regarded as a multiplication operator. We will use the following negative Sobolev space estimate.*

Proposition 2.6. *If $s > \frac{3}{2}, r+1 \geq 0$ and $r \leq s-1$, then*

$$\|[\Lambda^r \partial_x, f]\, g\|_{L^2} \leq c_{s,r} \|f\|_{H^s} \|g\|_{H^r}$$

Also, we will using the Kato-Ponce commutator estimate.

Proposition 2.7. *If $s \geq 0$ then*

$$\|[\Lambda^s, f]\, g\|_{L^2} \leq c_s (\|\partial_x f\|_{L^\infty} \left\|\Lambda^{s-1} g\right\|_{L^2} + \|\Lambda^s f\|_{L^2} \|g\|_{L^\infty})$$

Finally, replacing Λ with the J_ϵ operator, we have the commutator estimate.

Proposition 2.8. *Let J_ϵ be the mollifier defined above, and f,g be two test functions, then*

$$\|[J_\epsilon, f] g\|_{L^2} \leq C \|f\|_{Lip} \|g\|_{H^{-1}}.$$

Lemma 2.9 (Algebra Property). *Let $s > \frac{1}{2}$ and $f, g \in H^s$, we have*

$$\|fg\|_{H^s} \leq c_s \|f\|_{H^s} \|g\|_{H^s}.$$

Lemma 2.10 (Sobolev Interpolation Lemma). *Let $s_0 < s < s_1$ be real numbers, then*

$$\|f\|_{H^s} \leq \|f\|_{H^{s_0}}^{\frac{s_1 - s}{s_1 - s_0}} \|f\|_{H^{s_1}}^{\frac{s - s_0}{s_1 - s_0}}.$$

Lemma 2.11. *Let $s > 0$ and J_ϵ be defined as in $J_\epsilon f(x) = j_\epsilon f(x)$. Then for any $f \in H^s$, we have $J_\epsilon f \to f$ in H^s.*

Lemma 2.12. *Let w be such that $\|\partial_x w\|_{L^\infty}$. Then there is a constant $c > 0$ such that for any $f \in L^2$, we have*

$$\|[J_\epsilon, w] \partial_x f\|_{L^2} \leq c \|f\|_{L^2} \|\partial_x w\|_{L^2}.$$

Proposition 2.13. *Given $\sigma = \frac{n}{p} + 1$ and $1 < s < \sigma$, there exists $\theta \in (0,1)$ such that $\|f\|_{H^{s,\frac{p}{\theta}}} \leq c \|f\|_{H^{\sigma,p}}$ and $\|u\|_{L^{\frac{p}{1-\theta}}} \leq c \|u\|_{H^{s-1,p}}$.*

Lemma 2.14. *If $s > k + \frac{n}{2}$, where k is a nonnegative integer then $H^s(\mathbb{R}^n) \subset C^k(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where the inclusion is continuous. In fact,*

$$\sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^\infty} \leq C_s \|u\|_{H^s},$$

where C_s is independent of u.

Lemma 2.15. *Let $\sigma \in \left(\frac{1}{2}, 1\right)$, then*

$$\|fg\|_{H^{\sigma-1}} \leq \|f\|_{H^{\sigma-1}} \|g\|_{H^\sigma}.$$

Lemma 2.16. *Given $q \geq 0$, let $u = u(x) \in H^q$ be any function such that $\|u_x\|_{L^\infty} < \infty$. Then the there is a constant c_q depending only on q such that the following inequalities hold*

$$\left|\int_{\mathbb{R}} \Lambda^q u \Lambda(u u_x) dx\right| \leq c_q \|u_x\|_{L^\infty} \left\|u^2\right\|_{H^q}$$

$$\left|\int_{\mathbb{R}} \Lambda^q u \Lambda(u^2) dx\right| \leq c_q \|u\|_{L^\infty} \|u\|_{H^q}^2$$

On the other hand, one may estimate the following integral using integration by parts

$$\left|\int_{\mathbb{R}} f \Lambda^q u \Lambda^q u_x dx\right| = \frac{1}{2} \left|\int_{\mathbb{R}} f_x (\Lambda^q u)^2 dx\right| \leq \frac{1}{2} \|f_x\|_{L^\infty} \|u\|_{H^q}^2.$$

3. Local Well-posedness

To prove well-posedness, we employ a Galerkin approximation argument. The strategy will be to mollify the nonlinear terms in the BBMPB equation to construct a family of ODEs. Then, we will extract a sequence of solutions to the ODEs, which converges to the solution of the BBMPB equation in an appropriate space. We apply the mollifier J_ε to the BBMPB equation to construct family of ODEs in H^s.

$$\partial_t u_\epsilon + \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon) = \partial_x(1-\partial x^2)^{-1}[-\theta(u_\epsilon \partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon - \beta\partial_x^2 u_\epsilon + \theta(\partial_x u_\epsilon)^2]$$

$$u_\epsilon(x,0) = u_0(x)$$

Using the fact that

$$\lambda^{-2} = (1-\partial_x^2)^{-1}$$

The non local form can be written as

$$\partial_t u_\epsilon + \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon) = \partial_x \lambda^{-2}[-\theta(u_\epsilon \partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon - \beta\partial_x^2 u_\epsilon + \theta(\partial_x u_\epsilon)^2]$$

Our strategy is now to demonstrate that the Cauchy problem satisfies the hypotheses of the Fundamental ODE theorem. We will therefore obtain a unique solution $u_\epsilon(.,t) \in H^s$, $|t| < T_\epsilon$, for some $T_\epsilon > 0$.

Energy estimate and lifespan of solution u_ϵ

For each ϵ, there is a solution u_ϵ to the mollified BBMPB equation. The lifespan of each of these solutions has a lower bound T_ϵ. In this subsection, we shall demonstrate that there is actually a lower bound $T > 0$ that does not depend upon ϵ. To show the existence of T, we shall derive an energy estimate for the u_ϵ. Applying the operator λ^s to both sides of i.v.p, multiplying by $\lambda^s u_\epsilon$, and integrating over the torus yields the H^s-energy of u_ϵ.

$$\int \lambda^s \partial_t u_\epsilon \lambda^s u_\epsilon dx + \int \lambda^s \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon)\lambda^s u_\epsilon dx = \int \lambda^s \partial_x \lambda^{-2}[-\theta(u_\epsilon \partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon - \beta\partial_x^2 u_\epsilon + \theta(\partial_x u_\epsilon)^2]\lambda^s u_\epsilon dx$$

Consider the first term of the left hand side

$$\int \lambda^s \partial_t u_\epsilon \lambda^s u_\epsilon dx = \frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{H^s}^2$$

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{H^s}^2 + \int \lambda^s \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon)\lambda^s u_\epsilon dx = \int \lambda^s \partial_x(1-\partial x^2)^{-1}[-\theta(u_\epsilon \partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon - \beta\partial_x^2 u_\epsilon + \theta(\partial_x u_\epsilon)^2]\lambda^s u_\epsilon dx$$

using the fact that

$$\lambda^{-2} = (1-\partial_x^2)^{-1}$$

$$\frac{1}{2}\frac{d}{dt}\|u_\epsilon\|_{H^s}^2 = -\int \lambda^s \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon)\lambda^s u_\epsilon dx + \int \lambda^s \partial_x \lambda^{-2}\alpha\partial_x u_\epsilon \lambda^s u_\epsilon dx - \gamma\int \lambda^s \partial_x \lambda^{-2} u_\epsilon \lambda^s u_\epsilon dx - \beta\int \lambda^s \partial_x \lambda^{-2}\partial_x^2 u_\epsilon \lambda^s u_\epsilon dx \theta \int \lambda^s \partial_x \lambda^{-2}(\partial_x u_\epsilon)^2 \lambda^s u_\epsilon dx$$

To bound the energy, we will need the following Kato-Ponce commutator estimate. We now rewrite the first term by first commuting the exterior J_ϵ and then commuting the operator λ^s with $(J_\epsilon u_\epsilon)$ arriving at

$$\theta\int \lambda^s J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon)\lambda^s u_\epsilon dx = \theta\int \lambda^s [J_\epsilon u_\epsilon \partial_x J_\epsilon u_\epsilon]\lambda^s J_\epsilon u_\epsilon dx$$

adding and subtracting the term on the right hand side $\theta\int(J_\epsilon u_\epsilon)\lambda^s\partial_x J_\epsilon u_\epsilon\lambda^s J_\epsilon u_\epsilon dx$

$$\begin{aligned}\theta\int\lambda^s J_\epsilon(J_\epsilon u_\epsilon J_\epsilon\partial_x u_\epsilon)\lambda^s u_\epsilon dx &= \theta\int\lambda^s[J_\epsilon u_\epsilon\partial_x J_\epsilon u_\epsilon]\lambda^s J_\epsilon u_\epsilon dx - \theta\int(J_\epsilon u_\epsilon)\lambda^s\partial_x J_\epsilon u_\epsilon\lambda^s J_\epsilon u_\epsilon dx\\ &\quad + \theta\int(J_\epsilon u_\epsilon)\lambda^s\partial_x J_\epsilon u_\epsilon\lambda^s J_\epsilon u_\epsilon dx\\ \theta\int\lambda^s J_\epsilon(J_\epsilon u_\epsilon J_\epsilon\partial_x u_\epsilon)\lambda^s u_\epsilon dx &= \theta\int[\lambda^s, J_\epsilon u_\epsilon]\partial_x J_\epsilon u_\epsilon\lambda^s J_\epsilon u_\epsilon dx + \theta\int(J_\epsilon u_\epsilon)\lambda^s\partial_x J_\epsilon u_\epsilon\lambda^s J_\epsilon u_\epsilon dx\end{aligned}$$

Setting $v = J_\epsilon u_\epsilon$, we can bound the first term of right hand side by first using the Cauchy-Schwarz inequality and then applying the lemma (Kato-Ponce) and using the Sobolev theorem, we get

$$\begin{aligned}\theta\int[\lambda^s, v]\partial_x v\lambda^s v dx &\leq \|[\lambda^s, v]\partial_x v\|_{L^2}\|\lambda^s v\|_{L^2}\\ &\leq \left(c_s\left(\|\lambda^s v\|_{L^2}\|\partial_x v\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\left\|\lambda^{s-1}\partial_x v\right\|_{L^2}\right)\right)\|v\|_{H^s}\\ &\leq \left(c_s\left(\|v\|_{H^s}\|\partial_x v\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\|\partial_x v\|_{H^{s-1}}\right)\right)\|v\|_{H^s}\\ &\leq \left(c_s\left(\|v\|_{H^s}\|\partial_x v\|_{L^\infty} + \|\partial_x v\|_{L^\infty}\|v\|_{H^s}\right)\right)\|v\|_{H^s}\\ &\leq \left(c_s\left(\|v\|_{H^s}\|v\|_{H^s} + \|v\|_{H^s}\|v\|_{H^s}\right)\right)\|v\|_{H^s}\\ &= 2c_s\|v\|_{H^s}^3\end{aligned}$$

Next consider the second term of eqn , integrating by parts and using the Sobolev theorem, we have

$$\begin{aligned}\left|\theta\int v\partial_x\lambda^s v\lambda^s v dx\right| &= \frac{1}{2}\left|\int(\lambda^s v)^2\partial_x v dx\right|\\ &\leq \|\partial_x v\|_{L^\infty}\|v\|_{H^s}^2\\ &\leq \|v\|_{H^s}\|v\|_{H^s}^2\\ &= \|v\|_{H^s}^3\end{aligned}$$

Combining, we get

$$\begin{aligned}\theta\int\lambda^s J_\epsilon(J_\epsilon u_\epsilon J_\epsilon\partial_x u_\epsilon)\lambda^s u_\epsilon dx &\leq (2c_s+1)\|v\|_{H^s}^3\\ &\leq (2c_s+1)\|J_\epsilon u_\epsilon\|_{H^s}^3\\ &\leq (2c_s+1)\|u_\epsilon\|_{H^s}^3\end{aligned}$$

Consider the second term of the right hand side is bounded by first applying the Cauchy-Schwarz inequality and then using the estimate and the algebra property of H^s, we get

$$\begin{aligned}\int\lambda^s\partial_x\lambda^{-2}\alpha J_\epsilon\partial_x u_\epsilon\lambda^s u_\epsilon dx &\leq \left\|\lambda^s\partial_x\lambda^{-2}\alpha\partial_x u_\epsilon\right\|_{L^2}\|\lambda^s u_\epsilon\|_{L^2}\\ &\leq \left\|\partial_x\lambda^{-2}\alpha\partial_x u_\epsilon\right\|_{H^s}\|u_\epsilon\|_{H^s}\\ &\leq \|\alpha\partial_x u_\epsilon\|_{H^{s-1}}\|u_\epsilon\|_{H^s}\\ &\leq \alpha\|u_\epsilon\|_{H^s}\|u_\epsilon\|_{H^s}\\ &\leq \|u_\epsilon\|_{H^s}\|u_\epsilon\|_{H^s}\\ &= \|u_\epsilon\|_{H^s}^2\\ \gamma\int\lambda^s\partial_x\lambda^{-2}u_\epsilon\lambda^s u_\epsilon dx &\leq \left\|\lambda^s\partial_x\lambda^{-2}u_\epsilon\right\|_{L^2}\|\lambda^s u_\epsilon\|_{L^2}\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\partial_x \lambda^{-2} u_\epsilon\right\|_{H^s} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^{s-1}} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^s}^2 \\
\beta \int \lambda^s \partial_x \lambda^{-2} \partial_x^2 u_\epsilon \lambda^s u_\epsilon dx &\leq \left\|\lambda^s \partial_x \lambda^{-2} \partial_x^2 u_\epsilon\right\|_{L^2} \|\lambda^s u_\epsilon\|_{L^2} \\
&\leq \left\|\partial_x \lambda^{-2} \partial_x^2 u_\epsilon\right\|_{H^s} \|u_\epsilon\|_{H^s} \\
&\leq \left\|\partial_x^2 u_\epsilon\right\|_{H^s} \|u_\epsilon\|_{H^s} \\
&\leq \|\partial_x u_\epsilon\|_{H^{s+1}} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^{s+2}} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^s} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^s} \|u_\epsilon\|_{H^s} \\
&= \|u_\epsilon\|_{H^s}^2 \\
\theta \int \lambda^s \partial_x \lambda^{-2} (\partial_x u_\epsilon)^2 \lambda^s u_\epsilon dx &\leq \left\|\lambda^s \partial_x \lambda^{-2} (\partial_x u_\epsilon)^2\right\|_{L^2} \|\lambda^s u_\epsilon\|_{L^2} \\
&\leq \left\|\partial_x \lambda^{-2} \partial_x u_\epsilon^2\right\|_{H^s} \|u_\epsilon\|_{H^s} \\
&\leq \left\|(\partial_x u_\epsilon)^2\right\|_{H^{s-1}} \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^s}^2 \|u_\epsilon\|_{H^s} \\
&\leq \|u_\epsilon\|_{H^s}^3 \\
\frac{1}{2} \frac{d}{dt} \|u_\epsilon\|_{H^s}^2 &\leq (2c_s + 3) \|u_\epsilon\|_{H^s}^3 + 3 \|u_\epsilon\|_{H^s}^2 \\
\frac{1}{2} \frac{d}{dt} \|u_\epsilon\|_{H^s}^2 &\leq (2c_s + 3) \|u_\epsilon\|_{H^s}^3 + 3 \|u_\epsilon\|_{H^s}^3 \\
&= (2c_s + 6) \|u_\epsilon\|_{H^s}^3
\end{aligned}
$$

Solving this inequality, gives

$$
\|u_\epsilon(t)\|_{H^s}^2 \leq \left(\frac{\|u_0\|_{H^s}}{1 - (2c_s + 6)t \|u_0\|_{H^s}} \right)^2
$$

which yields the minimum lifespan, T and energy estimate

$$
T < \frac{1}{2(2c_s + 6) \|u_0\|_{H^s}}
$$

and

$$
\|u_\epsilon(t)\|_{H^s} \leq 2 \|u_0\|_{H^s}
$$

for $|t| < T$.

Refinement 1

Claim : To show that there exists a subsequence $\{u_{\epsilon j}\}$ of $\{u_\epsilon\}$ which converges in $L^\infty([-T, T]; H^s)$.

The family $\{u_\epsilon\}$ is bounded in $L^\infty([-T, T]; H^s)$, since the family $\{u_\epsilon\}$ is bounded (by the previous energy estimate) in $C([-T, T]; H^s)$. Since $L^\infty([-T, T]; H^s)$ is the dual of $L^1([-T, T]; H^s)$, we may apply Alaoglu's theorem. By Alaoglu's theorem there exists a subsequence $\{u_{\epsilon j}\}$ of $\{u_\epsilon\}$ which converges to an element $u \in L^1([-T, T]; H^s)$ in the weak* topology. Moreover, the limit point u, satisfies the same size estimation bound and minimum lifespan estimate as the u_ϵ solutions.

Refinement 2

Claim : To show that there is a further subsequence of our sequence $\{u_\epsilon\}$ which converges to u in $C([-T,T];H^{s-1})$.
To prove this we will employ Ascoli's theorem. First to prove equicontinuity, let t_1 and $t_2 \in [-T,T]$. By the mean value theorem

$$\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-1}} \leq \sup_{t\in[-T,T]} \|\partial_t u_\epsilon\|_{H^{s-1}} |t_1-t_2| \tag{1}$$

Now consider the mollified equation

$$\partial_t u_\epsilon + \theta J_\epsilon(J_\epsilon u_\epsilon J_\epsilon \partial_x u_\epsilon) = \partial_x \lambda^{-2}[-\theta(u_\epsilon \partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon - \beta\partial_x^2 u_\epsilon + \theta\partial_x u_\epsilon^2]$$

Applying norm on both sides and using the triangle inequality and lemma, we have

$$\begin{aligned}
\|\partial_t u_\epsilon\|_{H^{s-1}} &= \left\|\partial_x\lambda^{-2}[-\theta(u_\epsilon\partial_x^2 u_\epsilon) + \alpha\partial_x u_\epsilon - \gamma u_\epsilon \ -\beta\partial_x^2 u_\epsilon + \theta\partial_x u_\epsilon^2]\right\|_{H^{s-1}} \\
&\leq \left\|\partial_x\lambda^{-2}\theta u_\epsilon\partial_x^2 u_\epsilon\right\|_{H^{s-1}} + \left\|\partial_x\lambda^{-2}\alpha\partial_x u_\epsilon\right\|_{H^{s-1}} + \|-\gamma u_\epsilon\|_{H^{s-1}} - \left\|\beta\partial_x^2 u_\epsilon\right\|_{H^{s-1}} + \left\|\theta\partial_x u_\epsilon^2]\right\|_{H^{s-1}} \\
&\leq a\|u_0\|_{H^s}^3 + b\|u_0\|_{H^s}^2 + \|u_0\|_{H^s}
\end{aligned}$$

Substituting in inequality (1), we get

$$\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-1}} \leq \left(a\|u_0\|_{H^s}^3 + b\|u_0\|_{H^s}^2 + c\|u_0\|_{H^s}\right)|t_1-t_2|$$

which implies $\{u_\epsilon(t)\}$ is equicontinuous. Next, we observe that for each $t\in[0,T]$ the set $U(t)=\{u_\epsilon\}_{\epsilon\in(0,1]}$ is bounded in H^s. Since T is a compact manifold, the inclusion mapping $i: H^s \longrightarrow H^{s-1}$ is a compact operator, and therefore we may deduce that $U(t)$ is a precompact set in H^{s-1}. As the two hypotheses of Ascoli's theorem have been satisfied, we have a subsequence $\{u_{\epsilon_v}\}$ that converges in $\left([-T,T];H^{s-1}\right)$. By uniqueness of limits, this subsequence must converge to u.

Refinement 3

Claim : To refine the subsequence we show that the limit u is in the space $C\left([-T,T];H^{s-\sigma}\right)$ for all $\sigma\in(0,1]$.
As in the previous case, we will prove that the family u_ϵ satisfies the hypotheses of Ascoli's theorem, and to do so we will show that the sequence u_ϵ is equicontinuous in $H^{s-\sigma}$ and uniformly bounded. In fact, we will show that the following modulus of continuity

$$\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}} \leq \left(\|u_0\|_{H^s}^3 + \|u_0\|_{H^s}^2 + \|u_0\|_{H^s}\right)|t_1-t_2|^\sigma \tag{2}$$

To prove the above inequality, we begin by estimating

$$\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}} \leq \|u_\epsilon\|_{C^\sigma\left([-T,T];H^{s-\sigma}\right)}|t_1-t_2|^\sigma. \tag{3}$$

By definition of the Holder norm

$$\|u_\epsilon\|_{C^\sigma\left([-T,T];H^{s-\sigma}\right)} = \sup_{t\in[-T,T]}\|u_\epsilon(t)\|_{H^{s-\sigma}} + \sup_{t_1\neq t_2}\frac{\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}}}{|t_1-t_2|^\sigma} \tag{4}$$

The first term of the right hand side of (4) is bounded by $2\|u_0\|_{H^s}$ using the Sobolev embedding theorem followed by estimate.

$$\sup_{t\in[-T,T]}\|u_\epsilon(t)\|_{H^{s-\sigma}} \leq \sup_{t\in[-T,T]}\|u_\epsilon(t)\|_{H^s} \leq 2\|u_0\|_{H^s}.$$

For the second term is more difficult, and we will open the norm to analyze it. We have

$$\frac{\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}}}{|t_1-t_2|^\sigma} = |t_1-t_2|^{-\sigma}\left(\sum_k (1+k^2)^{s-\sigma}|\widehat{u_\epsilon}(k,t_1)-\widehat{u_\epsilon}(k,t_2)|^2\right)^{1/2}$$

First, as $\sigma \in (0,1)$, we have

$$\frac{1}{(1+k^2)^\sigma|t_1-t_2|^{2\sigma}} \le \left(1+\frac{1}{(1+k^2)|t_1-t_2|^2}\right)^\sigma \le 1+\frac{1}{(1+k^2)|t_1-t_2|^2}.$$

Using this inequality

$$\begin{aligned}
\frac{(1+k^2)^s}{(1+k^2)^\sigma|t_1-t_2|^{2\sigma}} &\le (1+k^2)^s+\frac{(1+k^2)^s}{|t_1-t_2|^2(1+k^2)}\\
\frac{\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}}}{|t_1-t_2|^\sigma} &= \left(\sum_k (1+k^2)^s\,|\widehat{u_\epsilon}(k,t_1)-\widehat{u_\epsilon}(k,t_2)|^2+\sum_k \frac{(1+k^2)^s}{|t_1-t_2|^2(1+k^2)}\,|\widehat{u_\epsilon}(k,t_1)-\widehat{u_\epsilon}(k,t_2)|\right)^{1/2}\\
\frac{\|u_\epsilon(t_1)-u_\epsilon(t_2)\|_{H^{s-\sigma}}}{|t_1-t_2|^\sigma} &\le 2\sup_{t\in[-T,T]}\|u_\epsilon\|_{H^s}+\|u_\epsilon\|_{C^1\left([-T,T];H^{s-1}\right)}.
\end{aligned}$$

Using the solution size estimate and the estimate found in the previous refinement, we obtain

$$\|u_\epsilon\|_{C^\sigma\left([-T,T];H^{s-\sigma}\right)} \le (4+c)\,\|u_0\|_{H^s}+a\,\|u_0\|_{H^s}^3+b\,\|u_0\|_{H^s}^2$$

Substituting into inequality (3) we establish a uniform modulus of continuity, and we conclude that the family $\{u_\epsilon\}$ is equicontinuous in the variable t. The precompactness condition is established in exactly the same fashion as the previous case as the inclusion mapping of H^s into $H^{s-\sigma}$ is a compact operator. As the two hypotheses of Ascoli have been satisfied, we may extract a subsequence that converges to u in $C\left([0,T];H^{s-\sigma}\right)$. Similarly, we can refine the sequence $\{u_\epsilon\}$ several times, by finding a sub-sequence of solutions which converges to a solution to BBMPB equation. Hence the proof of existence of a solution to the BBMPB equation.

Continuity of the data-to-solution map

Here we show that the dependence of the solution of the BBMPB equation on initial data is continuous.

Theorem 3.1 (Continuous dependence)**.** *The data-to-solution map $u_0 \longmapsto u(t)$ for the cauchy problem of the BBMPB equation is continuous from $H^s \longrightarrow C(I;H^s)$.*

Proof. Fix $u_0 \in H^s$ and let $\{u_{0,n}\} \subset H^s$ be a sequence with $\lim_{n\longrightarrow\infty} u_{0,n}=u_0$. If u is the solution to the BBMPB equation with initial data u_0 and if u_n is the solution to the BBMPB equation with initial data $u_{0,n}$, we will demonstrate that $\lim_{n\longrightarrow\infty} u_n = u$ in $C(I;H^s)$. Equivalently, let $\eta>0$. We need to show that there exists an $N>0$ such that

$$n>N \Rightarrow \|u-u_n\|_{C(I;H^s)}<\eta$$

As we will be using energy estimates in the H^s norm, to get around the difficulty of estimating the terms, we will use the J_ϵ convolution operator to smoooth out the initial data. Let $\varepsilon\in(0,1]$. We take u^ε be the solution to the Cauchy problem for BBMPB equation with initial data $J_\varepsilon u_0 = j_\varepsilon * u_0$ and u_n^ε be the solution with initial data $J_\varepsilon u_{0,n}$. Applying the triangle inequality, we arrive at

$$\|u-u_n\|_{C(I;H^s)} \le \|u-u^\varepsilon\|_{C(I;H^s)}+\|u^\varepsilon-u_n^\varepsilon\|_{C(I;H^s)}+\|u_n^\varepsilon-u_n\|_{C(I;H^s)}.$$

We will prove that each of these terms can be bounded by $\frac{\eta}{3}$ for suitable choices of ε and N. We note that the ε we have introduced will be independent of N and will only depend on η; whereas, the choice of N will depend on both η and ε.

Estimating $\|u^\varepsilon - u_n^\varepsilon\|_{C(I;H^s)}$: Setting $v = u^\varepsilon - u_n^\varepsilon$

$$\partial_t u^\varepsilon = (-\theta u^\varepsilon \partial_x u^\varepsilon) + \lambda^{-2}\partial_x\left(-\theta u^\varepsilon \partial_x u^\varepsilon + \alpha\partial_x u^\varepsilon - \gamma u^\varepsilon - \beta\partial_x^2 u^\varepsilon + \theta(\partial_x u^\varepsilon)^2\right)$$

$$\partial_t u_n^\varepsilon = (-\theta u_n^\varepsilon \partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x\left(-\theta u_n^\varepsilon \partial_x u_n^\varepsilon + \alpha\partial_x u_n^\varepsilon - \gamma u_n^\varepsilon - \beta\partial_x^2 u_n^\varepsilon + \theta(\partial_x u_n^\varepsilon)^2\right)$$

Subtracting

$$\begin{aligned}\partial_t(u_\varepsilon - u_n^\varepsilon) &= (-\theta u^\varepsilon\partial_x u^\varepsilon + \theta u_n^\varepsilon \partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x(-\theta u^\varepsilon \partial_x u^\varepsilon + \theta u_n^\varepsilon\partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x\alpha\partial_x(u^\varepsilon - u_n^\varepsilon)\\ &\quad -\gamma(u^\varepsilon - u_n^\varepsilon) - \beta\partial_x^2(u^\varepsilon - u_n^\varepsilon) + \theta\left((\partial_x u^\varepsilon)^2 - (\partial_x u_n^\varepsilon)^2\right)\end{aligned}$$

Let

$$F(u^\varepsilon) = (-\theta u^\varepsilon\partial_x u^\varepsilon) + \lambda^{-2}\partial_x\left(-\theta u^\varepsilon\partial_x u^\varepsilon\right) + \lambda^{-2}\partial_x\theta(\partial_x u^\varepsilon)^2$$

Let

$$\begin{aligned}F(u_n^\varepsilon) &= (-\theta u_n^\varepsilon\partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x\left(-\theta u_n^\varepsilon\partial_x u_n^\varepsilon\right) + \lambda^{-2}\partial_x\theta(\partial_x u_n^\varepsilon)^2\\ \partial_t(v) &= (-\theta u^\varepsilon\partial_x u^\varepsilon + \theta u_n^\varepsilon\partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x(-\theta u^\varepsilon\partial_x u^\varepsilon + \theta u_n^\varepsilon\partial_x u_n^\varepsilon) + \lambda^{-2}\partial_x\alpha\partial_x(v)\\ &\quad - \lambda^{-2}\partial_x\gamma(v) - \lambda^{-2}\partial_x\beta\partial_x^2(v) + \lambda^{-2}\partial_x\theta\left((\partial_x u^\varepsilon)^2 - (\partial_x u_n^\varepsilon)^2\right)\\ \partial_t(v) &= (F(u^\varepsilon) - F(u_n^\varepsilon)) + \lambda^{-2}\partial_x\left\{\alpha\partial_x(v) - \gamma(v) - \beta\partial_x^2(v)\right\}\end{aligned}$$

We calculate the H^s energy of v.

$$\int \lambda^s\partial_t v\lambda^s v dx = \int \lambda^s\left(F(u^\varepsilon) - F(u_n^\varepsilon)\right)\lambda^s v dx + \int \lambda^s\lambda^{-2}\partial_x\left\{\alpha\partial_x(v) - \gamma(v) - \beta\partial_x^2(v)\right\}\lambda^s v dx$$

Applying Cauchy-Schwarz inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 &\leq \|\lambda^s(F(u^\varepsilon) - F(u_n^\varepsilon))\|_{L^2}\|\lambda^s v\|_{L^2} + \left\|\lambda^s\lambda^{-2}\partial_x\left\{\alpha\partial_x(v) - \gamma(v) - \beta\partial_x^2(v)\right\}\right\|_{L^2}\|\lambda^s v\|_{L^2}\\ &\leq \|F(u^\varepsilon) - F(u_n^\varepsilon)\|_{H^s}\|v\|_{H^s} + \left\|\lambda^{-2}\partial_x\left\{\alpha\partial_x(v) - \gamma(v) - \beta\partial_x^2(v)\right\}\right\|_{H^s}\|v\|_{H^s}\end{aligned}$$

Consider the first term of inequality

$$\|F(u^\varepsilon) - F(u_n^\varepsilon)\|_{H^s}\|v\|_{H^s} \leq \|v\|_{H^s}^2$$

Consider the second term of inequality and using the triangle inequality

$$\begin{aligned}\left\|\lambda^{-2}\partial_x\left\{\alpha\partial_x(v) - \gamma(v) - \beta\partial_x^2(v)\right\}\right\|_{H^s}\|v\|_{H^s} &\leq \left\|\lambda^{-2}\partial_x\alpha\partial_x(v)\right\|_{H^s} + \gamma\left\|\lambda^{-2}\partial_x v\right\|_{H^s} + \beta\left\|\lambda^{-2}\partial_x\partial_x^2(v)\right\|_{H^s}\\ &\leq \alpha\|v\|_{H^s} + \gamma\|v\|_{H^{s-1}} + \|v\|_{H^{s+1}}\\ &\leq \frac{M}{\varepsilon}\|v\|_{H^s}^2\end{aligned}$$

Where M is positive constant. Combining the above estimates, we obtain the differential inequality

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^s}^2 \leq \left(1 + \frac{M}{\varepsilon}\right)\|v\|_{H^s}^2$$

Let

$$\frac{c_s}{\varepsilon} = (1 + \frac{M}{\varepsilon})$$
$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{H^s} \le \frac{c_s}{\varepsilon}\|v\|^2_{H^s} \tag{5}$$

for some constant c_s, Solving (5) gives for all $t \in I$

$$\begin{aligned}\|v(t)\|_{H^s} &\le e^{\frac{c_s T}{\varepsilon}}\|v(0)\|_{H^s} \\ &\le e^{\frac{c_s T}{T}}\|u_0 - u_{0,n}\|_{H^s}\end{aligned} \tag{6}$$

We observe that (6) does not place any constraints on ε; however, handling the first and third terms of (6) will require ε to be small. After ε is chosen, we take N sufficiently large so that

$$\|u_0 - u_{0,n}\|_{H^s} < \left(\frac{\eta}{3}\right)e^{\frac{c_s T}{\varepsilon}}$$

there by yielding

$$\|u^\varepsilon - u^\varepsilon_n\|_{C(I;H^s)} < \frac{\eta}{3}.$$

Estimation of $\|u - u^\varepsilon\|_{C(I;H^s)}$ and $\|u^\varepsilon - u^\varepsilon_n\|_{C(I;H^s)}$: Let

$$\partial_t u = -(\theta u\partial_x u) + \lambda^{-2}\partial_x[-\theta u\partial_x^2 u + \alpha\partial_x u - \gamma u - \beta\partial_x^2 u + \theta(\partial_x u)^2]$$
$$\partial_t u^\varepsilon = (-\theta u^\varepsilon\partial_x u^\varepsilon) + \lambda^{-2}\partial_x\left(-\theta u^\varepsilon\partial_x u^\varepsilon + \alpha\partial_x u^\varepsilon - \gamma u^\varepsilon - \beta\partial_x^2 u^\varepsilon + \theta(\partial_x u^\varepsilon)^2\right)$$

As the differences $u^\varepsilon - u$ and $u^\varepsilon_n - u_n$ satisfy the same inequalities, we will use the unified notation $v = u^\varepsilon - u$ and $v = u^\varepsilon_n - u_n$ and omit all n subscripts in formulae until we reach the point where different analysis for each case is needed. In constructing this Cauchy problem for v, we note that as we are taking energy estimates in H^s, we will want to avoid having any u coefficients for the Burgers-type part of the equation as this may give rise to an expression of the form $\|u\|_{H^{s+1}}$ which is undefined. There is no such problem for the nonlocal part of the equation so we will have $F(u^\varepsilon - F(u))$ as this can be estimated.

$$\partial_t(u - u_\varepsilon) = \lambda^{-2}\partial_x[\alpha\partial_x(u - u_\varepsilon) - \gamma(u - u_\varepsilon) - \beta\partial_x^2(u - u_\varepsilon)] - [F(u) - F(u^\varepsilon)]$$
$$\partial_t v = \lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v] - [F(u) - F(u^\varepsilon)]$$
$$v(x,0) = J_\varepsilon u_0 - u_0$$

Now let us obtain the H^s energy of v,

$$\int \lambda^s\partial_t v\lambda^s v dx = \int \lambda^s\lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v]\lambda^s v dx - \int \lambda^s\left(F(u) - F(u^\varepsilon)\right)\lambda^s v dx$$

Applying the Cauchy-Schwarz inequality

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{H^s} \le \left\|\lambda^s\lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v]\right\|_{L^2}\|\lambda^s v\|_{L^2} + \|\lambda^s(F(u) - F(u^\varepsilon))\|_{L^2}\|\lambda^s v\|_{L^2}$$

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 \leq \left\|\lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v]\right\|_{H^s}\|v\|_{H^s} + \|(F(u) - F(u^\varepsilon))\|_{H^s}\|v\|_{H^s}$$

Consider the second term of the inequality

$$\|F(u) - F(u^\varepsilon)\|_{H^s}\|v\|_{H^s} \leq \|v\|_{H^s}^2$$

Consider the first term of the inequality and using the triangle inequality

$$\begin{aligned}\left\|\lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v]\right\|_{H^s}\|v\|_{H^s} &\leq \left\|\lambda^{-2}\partial_x\alpha\partial_x v\right\|_{H^s} + \left\|\lambda^{-2}\partial_x\gamma v\right\|_{H^s} + \left\|\lambda^{-2}\partial_x\beta\partial_x^2 v\right\|_{H^s}\\ &\leq \alpha\|v\|_{H^s} + \gamma\|v\|_{H^{s-1}} + \beta\|v\|_{H^{s+1}}\end{aligned}$$

Choosing M a positive constant

$$\left\|\lambda^{-2}\partial_x[\alpha\partial_x v - \gamma v - \beta\partial_x^2 v]\right\|_{H^s}\|v\|_{H^s} \leq \frac{M}{\varepsilon}\|v\|_{H^s}^2$$

Combining the above estimates, we obtain the following differential inequality

$$\frac{1}{2}\frac{d}{dt}\|v\|_{H^s}^2 \leq \left(1 + \frac{M}{\varepsilon}\right)\|v\|_{H^s}^2$$

As in the previous estimation we will get

$$\|u - u^\varepsilon\|_{C(I;H^s)} < \frac{\eta}{3}$$

Similarly we can obtain

$$\|u^\varepsilon - u_n^\varepsilon\|_{C(I;H^s)} < \frac{\eta}{3}$$

By combining all the inequalities ,we get

$$\|u - u_n\|_{C(I;H^s)} < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3}$$

Hence

$$\|u - u_n\|_{C(I;H^s)} < \eta$$

Hence the data-to-solution map is continuous. □

References

[1] R. Camassa and D. Holm, *An integrable shallow water equation with peaked solitons*, Physics Review Letter, 71(1993), 1661-1664.

[2] A. Degasperis, D.D. Holm and A.N.W. Hone, *A new integrable equation with peakon solutions*, Theoretical and Mathematical Physics, 133(2002), 1463-1474.

[3] G. Fornberg and G.B. Whitham, *A numerical and theoretical study of certain nonlinear wave phenomena*, Philosophical Transactions of the Royal Society London, 289(1978), 373-404.

[4] A. Himonas and C. Holliman, *The Cauchy problem for the Novikov equation*, Discrete Continuous Dynamical Systems, 31(2011), 469-488.

[5] A. Himonas and C. Holliman, *On well-posedness of the Degasperis-Procesi equation*, Nonlinearity, 25(2012), 449-479

[6] A. Himonas and C. Holliman, *The Cauchy problem for a generalized Camassa-Holm equation*, Advanced Differential Equations, 19(2014), 161-200.

[7] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-stokes equations*, Communications on Pure and Applied Mathematics, 41(1988), 891-907.

[8] Y. Li and P. Olver, *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*, Journal of Differential Equations, 162(2000), 27-63.

[9] M. Li and Z. Yin, *Blow-up phenomena and local well-posedness for a generalized Camassa-Holm equation with cubic nonlinearity*, Nonlinear Analysis: Theory, Methods and Applications, 151(2017), 208-226.

[10] M. Taylor, *Pseudodifferential Operators and Nonlinear PDE*, Birkhauser, Boston, (1991).

[11] M. Taylor, *Commutator estimates*, Proceedings of the American Mathematical Society, 131(2003), 1501-1507.

[12] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the Novikov equation*, Journal of Nonlinear Differential Equations and Applications, 20(2013), 1157-1169.

[13] A. Yin, *Well-posedness and blow-up phenomena for a class of nonlinear third-order partial differential equations*, Houston Journal of Mathematics, 31(2005), 961-972.

[14] J. Yin, L. Tian and X. Fan, *Classification of traveling waves in the Fornberg-Whitham equation*, Journal o Mathematical Analysis and Applications, 368(2010), 133-143.