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Generalized Ulam-Hyers Stability of $n-$Dimensional AQCQ Functional Equation in Fuzzy Banach Spaces in Two Different Methods

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Abstract: In this paper, the authors achieve the generalized Hyers-Ulam-stability of $n-$dimensional additive-quadratic-cubic-quartic functional equation

$$f\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + f\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) = 4f\left(\sum_{k=1}^{n} x_k\right) + 4f\left(\sum_{k=1}^{n-1} x_k - x_n\right) - 6f\left(\sum_{k=1}^{n-1} x_k\right) + f(2x_n) + f(-2x_n) - 4f(x_n) - 4f(-x_n)$$

in fuzzy Banach spaces using direct and fixed point methods.

MSC: 39B52, 39B82, 26E50, 46S50.

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1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [43] concerning the stability of group homomorphisms. D.H. Hyers [16] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' theorem was generalized by T. Aoki [2] for additive mappings and by Th.M. Rassias [37] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [37] has provided a lot of influence in the development of what we call generalized Hyers-Ulam stability of functional equations. A generalization of the Th.M. Rassias theorem was obtained by P. Gavruta [15] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassias approach. In 1982, J.M. Rassias [33] followed the innovative approach of the Th.M. Rassias theorem [37] in which he replaced the factor $||x||^p + ||y||^p$ by $||x||^p||y||^q$ for $p, q \in R$ with $p + q \neq 1$.

In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained by Ravi etal., [39] by considering the summation of both the sum and the product of two $p-$ norms in the sprit of Rassias approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many

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interesting results concerning this problem (see [1, 13, 17, 21]).

A.K. Katsaras [23] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view [14, 25, 44]. In particular, T. Bag and S.K. Samanta [9], following S.C. Cheng and J.N. Mordeson [11], gave an idea of fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [24]. They established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy normed spaces [10].

We use the definition of fuzzy normed spaces given in [9] and [28–31].

Definition 1.1. *Let X be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*

$(FBS1)$ $N(x, c) = 0$ *for* $c \leq 0$;

$(FBS2)$ $x = 0$ *if and only if* $N(x, c) = 1$ *for all* $c > 0$;

$(FBS3)$ $N(cx, t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$;

$(FBS4)$ $N(x + y, s + t) \geq min\{N(x, s), N(y, t)\}$;

$(FBS5)$ $N(x, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{t\to\infty} N(x, t) = 1$;

$(FBS6)$ *for* $x \neq 0, N(x, \cdot)$ *is (upper semi) continuous on* $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(X, t)$ as the truth-value of the statement the norm of x is less than or equal to the real number t'.

Example 1.2. *Let $(X, ||\cdot||)$ be a normed linear space. Then*

$$N(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \quad x \in X, \\ 0, & t \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. *Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - \lim_{n\to\infty} x_n = x$.*

Definition 1.4. *A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N(x_{n+p} - x_n, t) > 1 - \epsilon$.*

Definition 1.5. *Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.*

The stability of various functional equations in fuzzy normed spaces was investigated in [3, 5–7, 18, 27–31, 40].

The general solution and the generalized Ulam-Hyers stability of a $n-$dimensional additive-quadratic-cubic-quartic functional equation of the form

$$f\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + f\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) = 4f\left(\sum_{k=1}^{n} x_k\right) + 4f\left(\sum_{k=1}^{n-1} x_k - x_n\right) - 6f\left(\sum_{k=1}^{n-1} x_k\right) + f(2x_n) + f(-2x_n) - 4f(x_n) - 4f(-x_n) \quad (1)$$

in Banach spaces using direct and fixed point methods was introduced and investigated by Sandra Pinals et.al [41].

In this paper, the authors estabilish the generalized Ulam-Hyers stability of $n-$dimensional AQCQ functional equation (1) in fuzzy Banach spaces using direct and fixed point methods. The stability result are proved via two different substitutions by considering $n-$ is a odd and even positive integer respectively. Throughout this article, assume that $X, (Z, N')$ and (Y, N') are linear space, fuzzy normed space and fuzzy Banach space, respectively. Now use the following notation for a given mapping $f : X \to Y$

$$\begin{aligned} D\ f(x_1, x_2 \cdots x_{n-1}, x_n) = &f\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + f\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) - 4f\left(\sum_{k=1}^{n} x_k\right) \\ &- 4f\left(\sum_{k=1}^{n-1} x_k - x_n\right) + 6f\left(\sum_{k=1}^{n-1} x_k\right) - f(2x_n) - f(-2x_n) + 4f(x_n) + 4f(-x_n) \end{aligned}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$.

2. Fuzzy Stability Results: n is an Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of AQCQ functional equation (1) by considering n is an Even Positive Integer.

2.1. f is an odd function

Theorem 2.1. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2}\right)^{\rho} < 1$ and*

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{2}$$

for all $x \in X$ and all $r > 0$ and

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 2^{\rho k}r\right) = 1 \tag{3}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{4}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$A(x) = N - \lim_{k\to\infty} \frac{1}{2^{\rho k}}\left(f(2^{(k+1)\rho}x) - 8f(2^{k\rho}x)\right) \tag{5}$$

exists for all $x \in X$ and the mapping $A : X \to Y$ is a unique additive mapping such that

$$N\left(f(2x) - 8f(x) - A(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2-d|r\right) \tag{6}$$

where

$$\begin{aligned} N'\left(\Omega\left(x, x, \cdots, x\right), |2-d|r\right) = min\Bigg\{ &N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2-d|r}{8}\right), \\ &N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2-d|r}{2}\right)\Bigg\} \end{aligned} \tag{7}$$

for all $x \in X$ and all $r > 0$.

Proof. First assume $\rho = 1$. Replacing $(x_1, x_2, \cdots, x_{n-1}, x_n)$ by $\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right)$ and using oddness of f in (4), we get

$$N\left(f(3x) - 4f(2x) + 5f(x), r\right) \geq N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), r\right) \tag{8}$$

for all $x \in X$ and all $r > 0$. Setting $(x_1, x_2, \cdots, x_{n-1}, x_n)$ by $\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right)$ in (4), we obtain

$$N\left(f(4x) - 4f(3x) + 6f(2x) - 4f(x), r\right) \geq N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), r\right) \tag{9}$$

for all $x \in X$ and all $r > 0$. Now, from (8) and (9), we have

$$\begin{aligned} N\left(f(4x) - 10f(2x) + 16f(x), r\right) &\geq min\left\{N\left(4\left(f(3x) - 4f(2x) + 5f(x)\right), \frac{r}{2}\right), N\left(f(4x) - 4f(3x) + 6f(2x) - 4f(x), \frac{r}{2}\right)\right\} \\ &\geq min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{8}\right),\right. \\ &\qquad \left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{2}\right)\right\} \end{aligned} \tag{10}$$

for all $x \in X$ and all $r > 0$. Define

$$\begin{aligned} N'\left(\Omega\left(x, x, \cdots, x\right), r\right) = min&\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{8}\right),\right. \\ &\qquad \left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{2}\right)\right\} \end{aligned} \tag{11}$$

for all $x \in X$ and all $r > 0$. Using (11) in (10), we arrive

$$N\left([f(4x) - 8f(2x)] - 2[f(2x) - 8f(x)], r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{12}$$

for all $x \in X$ and all $r > 0$. Let $a : X \to Y$ be a mapping defined by $a(x) = f(2x) - 8f(x)$. Then we conclude from (12), one can arrive

$$N\left(a(2x) - 2a(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{13}$$

for all $x \in X$ and all $r > 0$. Using $(FBS3)$ in above inequality, we have

$$N\left(\frac{a(2x)}{2} - a(x), \frac{r}{2}\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{14}$$

for all $x \in X$ and all $r > 0$. Replace x by $2^k x$ in (14), we obtain

$$N\left(\frac{a(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}, \frac{r}{2^k 2}\right) \geq N'\left(\Omega\left(2^k x, 2^k x, \cdots, 2^k x\right), r\right) \tag{15}$$

for all $x \in X$ and all $r > 0$. Using (2), $(FBS3)$ in (15), we arrive

$$N\left(\frac{a(2^{k+1}x)}{2^{k+1}} - \frac{a(2^kx)}{2^k}, \frac{r}{2^k 2}\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{r}{d^k}\right) \tag{16}$$

for all $x \in X$ and all $r > 0$. Replacing r by $d^k r$ in (16), we get

$$N\left(\frac{a(2^{k+1}x)}{2^{k+1}} - \frac{a(2^kx)}{2^k}, \frac{d^k r}{2^k 2}\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{17}$$

for all $x \in X$ and all $r > 0$. It is easy to see that

$$\frac{a(2^kx)}{2^k} - a(x) = \sum_{i=0}^{k-1}\left[\frac{a(2^{i+1}x)}{2^{i+1}} - \frac{a(2^ix)}{2^i}\right] \tag{18}$$

for all $x \in X$. From equations (17) and (18), we have

$$\begin{aligned} N\left(\frac{a(2^kx)}{2^k} - a(x), \sum_{i=0}^{k-1}\frac{d^i r}{2^i 2}\right) &\geq min \bigcup_{i=0}^{k-1}\left\{N\left(\frac{a(2^{i+1}y)}{2^{i+1}} - \frac{a(2^ix)}{2^i}, \sum_{i=0}^{k-1}\frac{d^i r}{2^i 2}\right)\right\} \\ &\geq min \bigcup_{i=0}^{k-1}\left\{N'\left(\Omega\left(x, x, \cdots, x\right), r\right)\right\} \\ &\geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \end{aligned} \tag{19}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^m x$ in (19) and using (2), $(FBSF3)$, we obtain

$$N\left(\frac{a(2^{k+m}x)}{2^{(k+m)}} - \frac{a(2^mx)}{2^m}, \sum_{i=0}^{k-1}\frac{d^i\ r}{2^{i+m}2}\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{r}{d^m}\right) \tag{20}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^m r$ in (20), we get

$$N\left(\frac{a(2^{k+m}x)}{2^{(k+m)}} - \frac{a(2^mx)}{2^m}, \sum_{i=0}^{m+k-1}\frac{d^{i+m}\ r}{2^{i+m}2}\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{21}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Using $(FBS3)$ in (21), we obtain

$$N\left(\frac{a(2^{k+m}x)}{2^{(k+m)}} - \frac{a(2^mx)}{2^m}, r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{r}{\sum_{i=m}^{m+k-1}\frac{d^i}{2^i 2}}\right) \tag{22}$$

for all $x \in X$ and all $r > 0$ and all $m, k \geq 0$. Since $0 < d < 2$ and $\sum_{i=0}^{k}\left(\frac{d}{2}\right)^i < \infty$, the cauchy criterion for convergence and $(FBS5)$ implies that $\{\frac{a(2^kx)}{2^k}\}$ is a Cauchy sequence in (Y, N'). Since (Y, N') is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So one can define the mapping $A : X \to Y$ by $A(x) = N - \lim_{k\to\infty}\frac{a(2^kx)}{2^k}$ for all $x \in X$. Letting $m = 0$ in (22), we get

$$N\left(\frac{a(2^kx)}{2^k} - a(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{r}{\sum_{i=0}^{k-1}\frac{d^i}{2^i 2}}\right) \tag{23}$$

for all $x \in X$ and all $r > 0$. Letting $k \to \infty$ in (23) and using $(FBS6)$, we arrive

$$N\left(a(x) - A(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), (2-d)r\right)$$

for all $x \in X$ and all $r > 0$. To prove A satisfies the (1), replacing $(x_1, x_2, \cdots, x_{n-1}, x_n)$ by $(2^k x_1, 2^k x_2, \cdots, 2^k x_{n-1}, 2^k x_n)$ in (4), respectively, we obtain

$$N\left(\frac{1}{2^k} Df(2^k x_1, 2^k x_2, \cdots, 2^k x_{n-1}, 2^k x_n), r\right) \geq N'\left(\Lambda(2^k x_1, 2^k x_2, \cdots, 2^k x_{n-1}, 2^k x_n), 2^k r\right) \tag{24}$$

and all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ for all $r > 0$. Now,

$$\begin{aligned}
&N\left(f\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + f\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) - 4f\left(\sum_{k=1}^{n} x_k\right) - 4f\left(\sum_{k=1}^{n-1} x_k - x_n\right)\right.\\
&\quad \left. +6f\left(\sum_{k=1}^{n-1} x_k\right) - f(2x_n) - f(-2x_n) + 4f(x_n) + 4f(-x_n), r\right)\\
&\quad \geq min\left\{N\left(A\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) - \frac{1}{2^k} a\left(\sum_{k=1}^{n-1} 2^k(x_k + 2x_n)\right), \frac{r}{10}\right),\right.\\
&\qquad N\left(A\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) - \frac{1}{2^k} a\left(\sum_{k=1}^{n-1} 2^k(x_k - 2x_n)\right), \frac{r}{10}\right),\\
&\quad N\left(-4A\left(\sum_{k=1}^{n} x_k\right) + 4\frac{1}{2^k} a\left(\sum_{k=1}^{n} 2^k x_k\right), \frac{r}{10}\right), N\left(4A\left(\sum_{k=1}^{n-1} x_k - x_n\right) - 4\frac{1}{2^k} a\left(\sum_{k=1}^{n-1} 2^k(x_k - x_n)\right), \frac{r}{10}\right),\\
&\quad N\left(6A\left(\sum_{k=1}^{n-1} x_k\right) - 6\frac{1}{2^k} a\left(\sum_{k=1}^{n-1} 2^k x_k\right), \frac{r}{10}\right), N\left(-A(2x_n) + \frac{1}{2^k} a(2^k 2x_n), \frac{r}{10}\right),\\
&\quad N\left(-A(-2x_n) + \frac{1}{2^k} a(-2^k 2x_n), \frac{r}{10}\right), N\left(4A(x_n) - 4\frac{1}{2^k} a(2^k x_n), \frac{r}{10}\right),\\
&\quad N\left(4A(-x_n) - 4\frac{1}{2^k} a(-2^k x_n), \frac{r}{10}\right), N\left(\frac{1}{2^k} 2^k\left(\sum_{k=1}^{n-1} 2^k(x_k + 2x_n)\right) + \frac{1}{2^k} a\left(\sum_{k=1}^{n-1} 2^k(x_k - 2x_n)\right) - \frac{1}{2^k} 4a\left(\sum_{k=1}^{n} 2^k x_k\right)\right.\\
&\quad \left.\left. +\frac{1}{2^k} 4a\left(\sum_{k=1}^{n-1} 2^k(x_k - x_n)\right) + \frac{1}{2^k} 6a\left(\sum_{k=1}^{n-1} 2^k x_k\right) - \frac{1}{2^k} a(2^k 2x_n) - \frac{1}{2^k} a(-2^k 2x_n) + \frac{1}{2^k} 4a(2^k x_n) + \frac{1}{2^k} 4a(-2^k x_n), \frac{r}{10}\right)\right\}
\end{aligned} \tag{25}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. Using (24) and $(FBS5)$ in (25), we arrive

$$\begin{aligned}
&N\left(A\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + A\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) - 4A\left(\sum_{k=1}^{n} x_k\right) - 4A\left(\sum_{k=1}^{n-1} x_k - x_n\right)\right.\\
&\quad \left. +6A\left(\sum_{k=1}^{n-1} x_k\right) - A(2x_n) - A(-2x_n) + 4A(x_n) + 4A(-x_n), r\right)\\
&\quad \geq min\left\{1, 1, 1, 1, 1, 1, 1, 1, 1, N'\left(\Omega\left(2^k x, 2^k x, \cdots, 2^k x\right), (2-d)2^k r\right)\right\}\\
&\quad \geq N'\left(\Omega\left(2^k x, 2^k x, \cdots, 2^k x\right), (2-d)2^k r\right)
\end{aligned} \tag{26}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. Letting $k \to \infty$ in (26) and using (3), we see that

$$\begin{aligned}
&N\left(A\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + A\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) - 4A\left(\sum_{k=1}^{n} x_k\right) - 4A\left(\sum_{k=1}^{n-1} x_k - x_n\right)\right.\\
&\quad \left. +6A\left(\sum_{k=1}^{n-1} x_k\right) - A(2x_n) - A(-2x_n) + 4A(x_n) + 4A(-x_n), r\right) = 1
\end{aligned} \tag{27}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. Using $(FBS2)$ in the above inequality gives

$$A\left(\sum_{k=1}^{n-1} x_k + 2x_n\right) + A\left(\sum_{k=1}^{n-1} x_k - 2x_n\right) = 4A\left(\sum_{k=1}^{n} x_k\right) + 4A\left(\sum_{k=1}^{n-1} x_k - x_n\right) - 6A\left(\sum_{k=1}^{n-1} x_k\right)$$

$$+ A(2x_n) + A(-2x_n) - 4A(x_n) - 4A(-x_n)$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$. Hence A satisfies the $n-$ dimentional additive functional equation (1). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive functional equation satisfying (1) and (6). Hence,

$$\begin{aligned} N(A(x) - A'(x), r) &\geq min\Big\{ N\left(\frac{A(2^k x)}{2^k} - \frac{f(2^k x)}{2^k}, \frac{r}{2}\right), N\left(\frac{f(2^k x)}{2^k} - \frac{A(2^k x)}{2^k}, \frac{r}{2}\right) \Big\} \\ &\geq min\left\{ N'\left(\Omega\left(2^k x, 2^k x, \cdots, 2^k x\right), \frac{(2-d)2^k r}{2}\right), \right. \\ &\qquad\qquad \left. N'\left(\Omega\left(2^k x, 2^k x, \cdots, 2^k x\right), \frac{(2-d)2^k r}{2}\right)\right\} \\ &\geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{(2-d)2^k r}{2d^k}\right) \end{aligned}$$

for all $x \in X$ and all $r > 0$. Since

$$\lim_{k\to\infty} \frac{2^k(2-d)r}{2d^k} = \infty$$

we obtain

$$\lim_{k\to\infty} N'\left(\Omega\left(x, x, \cdots, x\right), \frac{(2-d)2^k r}{2d^k}\right) = 1$$

for all $x \in X$ and all $r > 0$. Thus

$$N(A(x) - A'(x), r) = 1$$

for all $x \in X$ and all $r > 0$, hence $A(x) = A'(x)$. Therefore $A(x)$ is unique. Thus the theorem holds for $\rho = 1$.

Second assume $\rho = -1$, replace x by $\frac{x}{2}$ in (14), we arrive

$$N\left(a(x) - 2a\left(\frac{x}{2}\right), r\right) \geq N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right) \tag{28}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar lines to that of case $\rho = 1$. This completes the proof of the theorem. □

From Theorem 2.1, we obtain the following corollary concerning the stabilities for the functional equation (1).

Corollary 2.2. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^n ||x_i||^s, r\right), & s \neq 1; \\ N'\left(\epsilon \prod_{i=1}^n ||x_i||^s, r\right), & s \neq \frac{1}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right), r\right), & s \neq \frac{1}{n}; \end{cases} \tag{29}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f(2x) - 8f(x) - A(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{30}$$

for all $x \in X$ and all $r > 0$.

Theorem 2.3. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2^3}\right)^{\rho} < 1$*

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{31}$$

for all $x \in X$ and all $r > 0$, and

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 8^{\rho k}r\right) = 1 \tag{32}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{33}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$C(x) = N - \lim_{k\to\infty} \frac{1}{8^{\rho k}}\left(f(2^{(k+1)\rho}x) - 2f(2^{k\rho}x)\right) \tag{34}$$

exists for all $x \in X$ and $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f(2x) - 2f(x) - C(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2^3 - d|r\right) \tag{35}$$

where

$$\begin{aligned} N'\left(\Omega\left(x, x, \cdots, x\right), |2^3 - d|r\right) = min\Bigg\{ & N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^3 - d|r}{8}\right), \\ & N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^3 - d|r}{2}\right)\Bigg\} \end{aligned} \tag{36}$$

for all $x \in X$ and all $r > 0$.

Proof. It is easy to see from (11) and (10) that

$$N\left([f(4x) - 2f(x)] - 8[f(2x) - 2f(x)], r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{37}$$

for all $x \in X$ and all $r > 0$. Let $h : X \to Y$ be a mapping defined by $h(x) = f(2x) - 2f(x)$. Then, we conclude from (37), one can get

$$N\left(h(2x) - 8h(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{38}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1. □

The following corollary is an immediate consequence of Theorem 2.3 concerning the stabilites for the functional equation (1).

Corollary 2.4. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r) \geq \begin{cases} N'(\epsilon, r), \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 3; \\ N'\left(\epsilon \prod_{i=1}^{n} ||x_i||^s, r\right), & s \neq \frac{3}{n}; \\ N'\left(\epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{3}{n}; \end{cases} \tag{39}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N(f(2x) - 2f(x) - C(x), r) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s, \frac{|2^3 - 2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^3 - 2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{2}\right)\right\} \end{cases} \tag{40}$$

for all $x \in X$ and all $r > 0$.

Theorem 2.5. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with the conditions given (2), (31) and $0 < \left(\frac{d}{2}\right)^\rho < 1$, $0 < \left(\frac{d}{2^3}\right)^\rho < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r) \geq N'(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r) \tag{41}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then there exists a unique additive mapping $A : X \to Y$ and unique cubic mapping $C : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N(f(x) - A(x) - C(x), r) \\ &\geq \min\left\{N'\left(\Omega(x, x, \cdots, x), \frac{|2-d|r}{2}\right), N'\left(\Omega(x, x, \cdots, x), \frac{|2^3 - d|r}{2}\right)\right\} \end{aligned} \tag{42}$$

where

$$N'(\Omega(x, x, \cdots, x), |2-d|r) \qquad and \qquad N'\left(\Omega(x, x, \cdots, x), |2^3 - d|r\right)$$

are defined in (7) and (36) for all $x \in X$ and all $r > 0$.

Proof. By Theorems 2.1 and 2.3, there exists a unique additive function $A_1 : X \to Y$ and a unique cubic function $C_1 : X \to Y$ such that

$$N(f(2x) - 8f(x) - A_1(x), r) \geq N'\left(\Omega(x, x, \cdots, x), \frac{|2-d|r}{2}\right) \tag{43}$$

for all $x \in X$ and all $r > 0$ and

$$N(f(2x) - 2f(x) - C_1(x), r) \geq N'\left(\Omega(x, x, \cdots, x), \frac{|2^3 - d|r}{2}\right) \tag{44}$$

for all $x \in X$ and all $r > 0$. Now from (43) and (44), one can see that

$$N\left(f(x) + \frac{1}{6}A_1(x) - \frac{1}{6}C_1(x), 2r\right)$$

$$
\begin{aligned}
&\geq \min\left\{N\left(\frac{f(2x)}{6}-\frac{8}{6}f(x)-\frac{1}{6}A_1(x),\frac{r}{6}\right),N\left(\frac{f(2x)}{6}-\frac{2}{6}f(x)-\frac{1}{6}C_1(x),\frac{r}{6}\right)\right\}\\
&\geq \min\left\{N\left(f(2x)-8f(x)-A_1(x),r\right),N\left(f(2x)-2f(x)-C_1(x),r\right)\right\}\\
&\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2-d|r}{2}\right),N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^3-d|r}{2}\right)\right\}
\end{aligned}
$$

for all $x \in X$ and all $r > 0$. Thus, we obtain (42) by defining $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$ for all $x \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 2.5 concerning the stabilities for the functional equation (1).

Corollary 2.6. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$
N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 1,3;\\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,r\right), & s\neq \frac{1}{n},\frac{3}{n};\\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{1}{n},\frac{3}{n}; \end{cases} \tag{45}
$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$
\begin{aligned}
&N\left(f(x)-A(x)-C(x),r\right)\\
&\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right),N'\left(\epsilon,\frac{|7|r}{8}\right),N'\left(\epsilon,\frac{|7|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon n||x||^s,\frac{|2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2-2^s|r}{2}\right),\right.\\ \qquad\left.N'\left(\epsilon n||x||^s,\frac{|2^3-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2^3-2^s|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\ \ 2^s||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right.\\ \qquad\left.N'\left(\epsilon||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\ \ 2^s||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right.\\ \qquad\left.N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases}
\end{aligned} \tag{46}
$$

for all $x \in X$ and all $r > 0$.

2.2. f is an even function

Theorem 2.7. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2^2}\right)^\rho < 1$*

$$
N'\left(\Lambda\left(2^{\rho k}x,2^{\rho k}x,\cdots,2^{\rho k}x,2^{\beta k}x\right),r\right) \geq N'\left(d^{\rho k}\Lambda\left(x,x,\cdots,x,x\right),r\right) \tag{47}
$$

for all $x \in X$ and all $r > 0$, and

$$
\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1,2^{\rho k}x_2,\cdots,2^{\rho k}x_{n-1},2^{\rho k}x_n\right),4^{\rho k}r\right) = 1 \tag{48}
$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$
N\left(D\ f(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \tag{49}
$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$Q_2(x) = N - \lim_{k\to\infty} \frac{1}{4^{\rho k}} \left(f(2^{(k+1)\rho} x) - 16 f(2^{k\rho} x) \right) \tag{50}$$

exists for all $x \in X$ and the mapping $Q : X \to Y$ is a unique quadratic mapping such that

$$N\left(f(2x) - 16f(x) - Q_2(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2^2 - d|r\right) \tag{51}$$

where

$$N'\left(\Omega\left(x, x, \cdots, x\right), |2^2 - d|r\right) = min \left\{ N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^2 - d|r}{8}\right), N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^2 - d|r}{2}\right) \right\} \tag{52}$$

for all $x \in X$ and all $r > 0$.

Proof. Replacing $(x_1, x_2, \cdots, x_{n-1}, x_n)$ by $\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right)$ in (49), and using evenness of f, we arrive

$$N\left(f(3x) - 6f(2x) + 15f(x), r\right) \geq N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), r\right) \tag{53}$$

for all $x \in X$ and all $r > 0$. Replacing $(x_1, x_2, \cdots, x_{n-1}, x_n)$ by $\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right)$ in (49), we obtain

$$N\left(f(4x) - 4f(3x) + 4f(2x) + 4f(x), r\right) \geq N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), r\right) \tag{54}$$

for all $x \in X$ and all $r > 0$. It follows from (53) and (54) that

$$\begin{aligned}
&N\left(f(4x) - 20f(2x) + 64f(2x), r\right) \\
&\geq min \left\{ N\left(4\left(f(3x) - 24f(2x) + 60f(x)\right), \frac{r}{2}\right), N\left(f(4x) - 4f(3x) + 4f(2x) + 4f(x), \frac{r}{2}\right) \right\} \\
&\geq min \left\{ N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{8}\right), N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{r}{2}\right) \right\} \\
&= N'\left(\Omega\left(x, x, \cdots, x\right), r\right)
\end{aligned} \tag{55}$$

for all $x \in X$ and all $r > 0$. The above inequality can be rewritten as

$$N\left([f(4x) - 16f(2x)] - 4[f(2x) - 16f(x)], r\right) \geq N'\left(\Omega\left(x, x \cdots, x\right), r\right) \tag{56}$$

for all $x \in X$ and all $r > 0$. Let $q_2 : X \to Y$ be a mapping defined by $q_2(x) = f(2x) - 16f(x)$. Then we conclude from (56) one can obtain

$$N\left(q_2(2x) - 4q_2(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{57}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1.

□

The following corollary is an immediate consequence of Theorem 2.7 concerning the stabilites for the functional equation (1).

Corollary 2.8. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 2; \\ N'\left(\epsilon \prod_{i=1}^{n} ||x_i||^s, r\right), & s \neq \frac{2}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{2}{n}; \end{cases} \tag{58}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$\begin{aligned} & N\left(f(2x) - 16f(x) - Q_2(x), r\right) \\ & \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s, \frac{|2^2 - 2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^2 - 2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^2 - 2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^2 - 2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^2 - 2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^2 - 2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{59}$$

for all $x \in X$ and all $r > 0$.

Theorem 2.9. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2^4}\right)^\rho < 1$*

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{60}$$

for all $x \in X$ and all $r > 0$, and

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 16^{\rho k}r\right) = 1 \tag{61}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{62}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$Q_4(x) = N - \lim_{k\to\infty} \frac{1}{16^{\rho k}} \left(f(2^{(k+1)\rho}x) - 4f(2^{k\rho}x) \right) \tag{63}$$

exists for all $x \in X$ and $Q_4 : X \to Y$ is a unique quartic mapping such that

$$N\left(f(2x) - 4f(x) - Q_4(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2^4 - d|r\right) \tag{64}$$

where

$$\begin{aligned} N'\left(\Omega\left(x, x \cdots x\right), |2^4 - d|r\right) = min &\left\{ N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^4 - d|r}{8}\right),\right. \\ &\left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \frac{|2^4 - d|r}{2}\right)\right\} \end{aligned} \tag{65}$$

for all $x \in X$ and all $r > 0$.

Proof. It is easy to see from (55), that

$$N\left([f(4x) - 4f(2x)] - 16[f(2x) - 4f(x)], r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{66}$$

for all $x \in X$ and all $r > 0$. Let $q_4 : X \to Y$ be a mapping defined by $q_4(x) = f(2x) - 4f(x)$. Then we conclude from (66), one can arrive

$$N\left(q_4(2x) - 16q_4(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), r\right) \tag{67}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 2.1. □

The following corollary is an immediate consequence of Theorem 2.9 concerning the Ulam-Hyers stability of the functional equation (1).

Corollary 2.10. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 4; \\ N'\left(\epsilon \prod_{i=1}^{n} ||x_i||^s, r\right), & s \neq \frac{4}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{4}{n}; \end{cases} \tag{68}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} &N\left(f(2x) - 4f(x) - Q_4(x), r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s, \frac{|2^4 - 2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^4 - 2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^4 - 2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^4 - 2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^4 - 2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^4 - 2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{69}$$

for all $x \in X$ and all $r > 0$.

Theorem 2.11. *Let $\rho \in \{-1,1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with the conditions (47), (60) and $0 < \left(\frac{d}{2^2}\right)^\rho < 1$, $0 < \left(\frac{d}{2^4}\right)^\rho < 1$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{70}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then there exists a quadratic mapping $Q_2 : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(x) - Q_2(x) - Q_4(x), r\right) \\ &\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^2 - d|r}{2}\right), N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^4 - d|r}{2}\right)\right\} \end{aligned} \tag{71}$$

where

$$N'\left(\Omega\left(x, x, \cdots, x\right), |2^2 - d|r\right) \qquad \text{and} \qquad N'\left(\Omega\left(x, x, \cdots, x\right), |2^4 - d|r\right)$$

are defined in (52) and (65) for all $x \in X$ and all $r > 0$.

Proof. By Theorems 2.7 and 2.9, there exists a unique quadratic function $Q_{2_1} : X \to Y$ and a unique quartic function $Q_{4_1} : X \to Y$ such that

$$N\left(f(2x) - 16f(x) - Q_{2_1}(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^2 - d|r}{2}\right) \tag{72}$$

for all $x \in X$ and all $r > 0$ and

$$N\left(f(2x) - 4f(x) - Q_{4_1}(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^4 - d|r}{2}\right) \tag{73}$$

for all $x \in X$ and all $r > 0$. Now from (72) and (73), one can see that

$$\begin{aligned} &N\left(f(x) + \frac{1}{12}Q_{2_1}(x) - \frac{1}{12}Q_{4_1}(x), 2r\right) \\ &\geq \min\left\{N\left(\frac{f(2x)}{12} - \frac{16}{12}f(x) - \frac{1}{12}Q_{2_1}(x), \frac{r}{12}\right), N\left(\frac{f(2x)}{12} - \frac{4}{12}f(x) - \frac{1}{12}Q_{4_1}(x), \frac{r}{12}\right)\right\} \\ &\geq \min\left\{N\left(f(2x) - 16f(x) - Q_{2_1}(x), r\right), N\left(f(2x) - 4f(x) - Q_{4_1}(x), r\right)\right\} \\ &\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^2 - d|r}{2}\right), N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^4 - d|r}{2}\right)\right\} \end{aligned}$$

for all $x \in X$ and all $r > 0$. Thus we obtain (71) by defining $Q_2(x) = \frac{-1}{12}Q_{2_1}(x)$ and $Q_4(x) = \frac{1}{12}Q_{4_1}(x)$ for all $x \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 2.11 concerning the stabilities of the functional equation (1).

Corollary 2.12. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^n ||x_i||^s, r\right), & s \neq 2, 4; \\ N'\left(\epsilon \prod_{i=1}^n ||x_i||^s, r\right), & s \neq \frac{2}{n}, \frac{4}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right), r\right), & s \neq \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{74}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(x) - Q_2(x) - Q_4(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon n||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{75}$$

for all $x \in X$ and all $r > 0$.

2.3. f is an odd-even function

Theorem 2.13. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with the condition given (2), (31), (47), (60) and $0 < \left(\frac{d}{2}\right)^\rho < 1$, $0 < \left(\frac{d}{2^2}\right)^\rho < 1$, $0 < \left(\frac{d}{2^3}\right)^\rho < 1$ and $0 < \left(\frac{d}{2^4}\right)^\rho < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{76}$$

for all$x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\ &\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2-d|r}{2}\right)\right. \\ &\qquad N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^2-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^2-d|r}{2}\right) \\ &\qquad N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^3-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^3-d|rr}{2}\right) \\ &\qquad \left. N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^4-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^4-d|r}{2}\right)\right\} \end{aligned}$$

for all $x \in X$ and all $r > 0$.

Proof. Let $f_{ac}(x) = \frac{f_o(x) - f_o(-x)}{2}$ for all $x \in X$. Then $f_{ac}(0) = 0$ and $f_{ac}(-x) = -f_{ac}(x)$ for all $x \in X$. Hence

$$\begin{aligned} &N\left(Df_{ac}(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \\ &\geq min\left\{N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), \frac{r}{2}\right), N'\left(\Lambda(-x_1, -x_2, \cdots, -x_{n-1}, -x_n), \frac{r}{2}\right)\right\} \end{aligned} \tag{77}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. By Theorem 2.5, there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$N\left(f_{ac}(x) - A(x) - C(x), r\right)$$

$$\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2-d|r}{2}\right),N'\left(\Omega\left(-x,-x,\cdots,-x\right),\frac{|2-d|r}{2}\right)\right.$$
$$\left.N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^3-d|r}{2}\right),N'\left(\Omega\left(-x,-x,\cdots,-x\right),\frac{|2^3-d|r}{2}\right)\right\} \tag{78}$$

for all $x \in X$ and all $r > 0$. Also, let $f_{q_2q_4}(x) = \frac{f_e(x)+f_e(-x)}{2}$ for all $x \in X$. Then $f_{q_2q_4}(0) = 0$ and $f_{q_2q_4}(-x) = f_{q_2q_4}(x)$ for all $x \in X$. Hence

$$N\left(Df_{q_2q_4}(x_1,x_2,\cdots,x_{n-1},x_n),r\right)$$
$$\geq min\left\{N'\left(\Lambda(x_1,x_2,\cdots,x_{n-1},x_n),\frac{r}{2}\right),N'\left(\Lambda(-x_1,-x_2,\cdots,-x_{n-1},-x_n),\frac{r}{2}\right)\right\} \tag{79}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n \in X$ and all $r > 0$. By Theorem 2.11, there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f_{q_2q_4}(x)-Q_2(x)-Q_4(x),r\right)$$
$$\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^2-d|r}{2}\right),N'\left(\Omega\left(-x,-x,\cdots,-x\right),\frac{|2^2-d|r}{2}\right)\right.$$
$$\left.N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^4-d|r}{2}\right),N'\left(\Omega\left(-x,-x,\cdots,-x\right),\frac{|2^4-d|r}{2}\right)\right\} \tag{80}$$

for all $x \in X$ and all $r > 0$. Define a function $f(x)$ by

$$f(x) = f_{ac}(x) + f_{q_2q_4}(x) \tag{81}$$

for all $x \in X$. Combining (81), (78) and (80), we arrive our desired result. □

Corollary 2.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^n ||x_i||^s,r\right), & s \neq 1,3,2,4; \\ N'\left(\epsilon\prod_{i=1}^n ||x_i||^s,r\right), & s \neq \frac{1}{n},\frac{3}{n},\frac{2}{n},\frac{4}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s+\sum_{i=1}^n ||x_i||^{ns}\right),r\right), & s \neq \frac{1}{n},\frac{3}{n},\frac{2}{n},\frac{4}{n}; \end{cases} \tag{82}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique quadratic mapping $Q_2 : X \to Y$ and a unique cubic mapping $C : X \to Y$, and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(x)-A(x)-Q_2(x)-C(x)-Q_4(x),r\right)$$

$$\geq \begin{cases} (i) min \left\{ N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right), \right. \\ \quad N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right), \\ \quad N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), \\ \quad \left. N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\ (ii) min \left\{ N'\left(\epsilon n||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2-2^s|r}{2}\right), \right. \\ \quad N'\left(\epsilon n||x||^s, \frac{|2^3-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2^3-2^s|r}{2}\right), \\ \quad N'\left(\epsilon n||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right), \\ \quad \left. N'\left(\epsilon n||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\ (iii) min \left\{ N'\left(\epsilon||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right), \right. \\ \quad N'\left(\epsilon||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\ \quad N'\left(\epsilon||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\ \quad \left. N'\left(\epsilon||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \\ (iv) min \left\{ N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right), \right. \\ \quad N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\ \quad N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\ \quad \left. N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{83}$$

for all $x \in X$ and all $r > 0$.

3. Fuzzy Stability Results: n is an odd Positive Integer

The proof of following theorms and corollaries are similar tracing to that theorms and corollaries of section 2. Hence the details of the proof are omitted.

In this section, we investigate the generalized Hyers-Ulam stability of AQCQ functional equation (1) when n is an odd positive integer in the fuzzy Banach space.

3.1. f is an odd function

Theorem 3.1. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2}\right)^\beta < 1$*

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{84}$$

for all $x \in X$ and all $r > 0$, and

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 2^{\rho k}r\right) = 1 \tag{85}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an odd function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{86}$$

for all$x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$A(x) = N - \lim_{k\to\infty} \frac{1}{2^{\rho k}}\left(f(2^{(k+1)\rho}x) - 8f(2^{k\rho}x)\right) \tag{87}$$

exists for all $x \in X$ *and the mapping* $A : X \to Y$ *is a unique additive mapping such that*

$$N\left(f(2x) - 8f(x) - A(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2-d|r\right) \tag{88}$$

where

$$N'\left(\Omega\left(x, x, \cdots, x\right), |2-d|r\right) = min\left\{ N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{|2-d|r}{8}\right), N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{|2-d|r}{4}\right)\right\} \tag{89}$$

for all $x \in X$ *and all* $r > 0$.

Corollary 3.2. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 1; \\ N'\left(\epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{1}{n}; \end{cases} \tag{90}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ *and all* $r > 0$*, where* ϵ, s *are constants with* $\epsilon > 0$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\begin{aligned} & N\left(f(2x) - 8f(x) - A(x), r\right) \\ & \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{91}$$

for all $x \in X$ *and all* $r > 0$.

Theorem 3.3. *Let* $\rho \in \{-1, 1\}$ *be fixed and let* $\Omega, \Lambda : X^n \to Z$ *be a mapping such that for some* $d > 0$ *with* $0 < \left(\frac{d}{2^3}\right)^\rho < 1$

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{92}$$

for all $x \in X$ *and all* $r > 0$*, and*

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 2^{\rho k}r\right) = 1 \tag{93}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ *and all* $r > 0$. *Suppose that an odd function* $f : X \to Y$ *satisfies the inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{94}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ *and all* $r > 0$. *Then the limit*

$$C(x) = N - \lim_{k\to\infty} \frac{1}{8^{\rho k}}\left(f(2^{(k+1)\rho}x) - 2f(2^{k\rho}x)\right) \tag{95}$$

exists for all $x \in X$ and the mapping $C : X \to Y$ is a unique cubic mapping such that

$$N\left(f(2x) - 2f(x) - C(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), |2^3 - d|r\right) \tag{96}$$

where

$$N'\left(\Omega\left(x, x, \cdots, x\right), |2^3 - d|r\right) = min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{(2^3-d)r}{8}\right),\right.$$
$$\left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{(2^3-d)r}{2}\right)\right\} \tag{97}$$

for all $x \in X$ and all $r > 0$.

Corollary 3.4. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 3; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{3}{n}; \end{cases} \tag{98}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(2x) - 2f(x) - C(x), r\right)$$
$$\geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2^3-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^3-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{99}$$

for all $x \in X$ and all $r > 0$.

Theorem 3.5. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with the condition given (84) and (92) and $0 < \left(\frac{d}{2}\right)^\rho < 1$, $0 < \left(\frac{d}{2^3}\right)^\rho < 1$. Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{100}$$

for all$x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all$r > 0$. Then there exists a additive mapping $A : X \to Y$ and unique cubic mapping $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(x) - A(x) - C(x), r\right)$$
$$\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2-d|r}{2}\right), N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^3-d|r}{2}\right)\right\} \tag{101}$$

where

$$N'\left(\Omega\left(x, x, \cdots, x\right), |2-d|r\right) \qquad and \qquad N'\left(\Omega\left(x, x, \cdots, x\right), |2^3-d|r\right)$$

are defined in (89) and (97) for all $x \in X$ and all $r > 0$.

Corollary 3.6. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 1,3; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{1}{n},\frac{3}{n}; \end{cases} \tag{102}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n \in X$ and all $r>0$, where ϵ, s are constants with $\epsilon>0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &N\left(f(x)-A(x)-C(x),r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right),N'\left(\epsilon,\frac{|7|r}{8}\right),N'\left(\epsilon,\frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2-2^s|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^s,\frac{|2^3-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^3-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{103}$$

for all $x \in X$ and all $r>0$.

3.2. f is an even function

Theorem 3.7. *Let $\rho\in\{-1,1\}$ be fixed and let $\Omega,\Lambda : X^n \to Z$ be a mapping such that for some $d>0$ with $0<\left(\frac{d}{2^2}\right)^{\rho}<1$*

$$N'\left(\Lambda\left(2^{\rho k}x,2^{\rho k}x,\cdots,2^{\rho k}x,2^{\rho k}x\right),r\right)\geq N'\left(d^{\rho k}\Lambda\left(x,x,\cdots,x,x\right),r\right) \tag{104}$$

for all $x\in X$ and all $r>0$, and

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1,2^{\rho k}x_2,\cdots,2^{\rho k}x_{n-1},2^{\rho k}x_n\right),2^{\rho k}r\right)=1 \tag{105}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n\in X$ and all $r>0$. Suppose that an even function $f : X\to Y$ satisfies the inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_{n-1},x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \tag{106}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n\in X$and all $r>0$. Then the limit

$$A_2(x)=N-\lim_{k\to\infty}\frac{1}{16^{\rho k}}\left(f(2^{(k+1)\rho}x)-16f(2^{k\rho}x)\right) \tag{107}$$

exists for all $x\in X$ and the mapping $Q : X\to Y$ is a unique quadratic mapping such that

$$N\left(f(2x)-16f(x)-Q_2(x),r\right)\geq N'\left(\Omega\left(x,x,\cdots,x\right),|2^2-d|r\right) \tag{108}$$

where

$$\begin{aligned} N'\left(\Omega\left(x,x,\cdots,x\right),|2^2-d|r\right)=min\Bigg\{ & N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\frac{|2^2-d|r}{8}\right), \\ & N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\frac{|2^2-d|r}{2}\right)\Bigg\} \end{aligned} \tag{109}$$

for all $x\in X$ and all $r>0$.

Corollary 3.8. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 2; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{2}{n}; \end{cases} \tag{110}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$\begin{aligned} &N\left(f(2x) - 16f(x) - Q_2(x), r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{111}$$

for all $x \in X$ and all $r > 0$.

Theorem 3.9. *Let $\rho \in \{-1, 1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some d with $0 < \left(\frac{d}{2^4}\right)^{\rho} < 1$*

$$N'\left(\Lambda\left(2^{\rho k}x, 2^{\rho k}x, \cdots, 2^{\rho k}x, 2^{\rho k}x\right), r\right) \geq N'\left(d^{\rho k}\Lambda\left(x, x, \cdots, x, x\right), r\right) \tag{112}$$

for all $x \in X$ and all $r > 0$

$$\lim_{k\to\infty} N'\left(\Lambda\left(2^{\rho k}x_1, 2^{\rho k}x_2, \cdots, 2^{\rho k}x_{n-1}, 2^{\rho k}x_n\right), 2^{\rho k}r\right) = 1 \tag{113}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Suppose that an even function $f : X \to Y$ satisfies the inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \tag{114}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then the limit

$$Q_4(x) = N - \lim_{k\to\infty} \frac{1}{2^{\rho k}}\left(f(2^{(k+1)\rho}x) - 16f(2^{k\rho}x)\right) \tag{115}$$

exists for all $x \in X$ and the mapping $Q_4 : X \to Y$ is a unique quartic mapping such that

$$N\left(f(2x) - 4f(x) - Q_4(x), r\right) \geq N'\left(\Omega_{Q_4}\left(x, x, \cdots, x\right), |2^4 - d|r\right) \tag{116}$$

where

$$\begin{aligned} N'\left(\Omega\left(x,x,\cdots,x\right),|2^4-d|r\right) = min\Bigg\{ & N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{|2^4-d|r}{8}\right), \\ & N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \frac{|2^4-d|r}{4}\right)\Bigg\} \end{aligned} \tag{117}$$

for all $x \in X$ and all $r > 0$.

Corollary 3.10. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2\cdots x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 4; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{4}{n}; \end{cases} \tag{118}$$

for all $x_1,x_2\cdots x_{n-1},x_n \in X$ and all $r>0$, where ϵ, s are constants with $\epsilon>0$. Then there exists a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} &N\left(f(2x)-4f(x)-Q_4(x),r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{|15|r}{8}\right),N'\left(\epsilon,\frac{|15|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2^4-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^4-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^4-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{119}$$

for all $x\in X$ and all $r>0$.

Theorem 3.11. *Let $\rho\in\{-1,1\}$ be fixed and let $\Omega,\Lambda : X^n \to Z$ be a mapping such that for some $d>0$ with the condition given (104) and (112) and $0<\left(\frac{d}{2^2}\right)^\rho<1$, $0<\left(\frac{d}{2^4}\right)^\rho<1$. Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x_1,x_2,\cdots,x_{n-1},x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \tag{120}$$

for all $x_1,x_2\cdots x_{n-1},x_n\in X$ and all $r>0$. Then there exists a quadratic mapping $Q_2 : X\to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(x)-Q_2(x)-Q_4(x),r\right) \\ &\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^2-d|r}{2}\right),N'\left(\Omega\left(x,x,\cdots,x\right),\frac{|2^4-d|r}{2}\right)\right\} \end{aligned} \tag{121}$$

where

$$N'\left(\Omega\left(x,x,\cdots,x\right),|2^2-d|r\right) \qquad and \qquad N'\left(\Omega\left(x,x,\cdots,x\right),|2^4-d|r\right)$$

are defined in (109) and (117) for all $x\in X$ and all $r>0$.

Corollary 3.12. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2\cdots x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 2,4; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{2}{n},\frac{4}{n}; \end{cases} \tag{122}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n\in X$ and all $r>0$, where ϵ,s are constants with $\epsilon>0$. Then there exists a unique quadratic mapping $Q_2 : X\to Y$ and a unique quartic mapping $Q_4 : X\to Y$ such that

$$\begin{aligned} &N\left(f(x)-Q_2(x)-Q_4(x),r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{|3|r}{8}\right),N'\left(\epsilon,\frac{|3|r}{2}\right),N'\left(\epsilon,\frac{|15|r}{8}\right),N'\left(\epsilon,\frac{|15|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2^2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^2-2^s|r}{2}\right),\right. \\ \qquad\left.N'\left(\epsilon(n-1)||x||^s,\frac{|2^4-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^4-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^2-2^{ns}|r}{2}\right),\right. \\ \qquad\left.N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^4-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{123}$$

for all $x\in X$ and all $r>0$.

3.3. f is an odd-even function

Theorem 3.13. *Let $\rho \in \{-1,1\}$ be fixed and let $\Omega, \Lambda : X^n \to Z$ be a mapping such that for some $d > 0$ with the condition given (84), (92), (104), (112) and $0 < \left(\frac{d}{2}\right)^{\rho} < 1$, $0 < \left(\frac{d}{2^2}\right)^{\rho} < 1$, $0 < \left(\frac{d}{2^3}\right)^{\rho} < 1$ and $0 < \left(\frac{d}{2^4}\right)^{\rho} < 1$. Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \tag{124}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$. Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned}
&N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\
&\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2-d|r}{2}\right)\right. \\
&\quad N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^2-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^2-d|r}{2}\right) \\
&\quad N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^3-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^3-d|r}{2}\right) \\
&\quad \left. N'\left(\Omega\left(x, x, \cdots, x\right), \frac{|2^4-d|r}{2}\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \frac{|2^4-d|r}{2}\right)\right\}
\end{aligned}$$

for all $x \in X$ and all $r > 0$.

Corollary 3.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^n ||x_i||^s, r\right), & s \neq 1, 3, 2, 4; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right), r\right), & s \neq \frac{1}{n}, \frac{3}{n}, \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{125}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned}
&N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\
&\geq \begin{cases}
(i)\min\left\{N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right),\right. \\
\quad N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right), \\
\quad N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), \\
\quad \left. N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\
(ii)\min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2-2^s|r}{2}\right),\right. \\
\quad N'\left(\epsilon(n-1)||x||^s, \frac{|2^3-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^3-2^s|r}{2}\right), \\
\quad N'\left(\epsilon(n-1)||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right), \\
\quad \left. N'\left(\epsilon(n-1)||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\
(iv)\min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right),\right. \\
\quad N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\
\quad N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\
\quad \left. N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\}
\end{cases}
\end{aligned} \tag{126}$$

for all $x \in X$ and all $r > 0$.

4. Stability Results: Fixed Point Method

In this section, the authors presented the generalized Ulam - Hyers stability of the functional equation (1) in fuzzy normed space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ for some (Lipschitz constant) $L < 1$. Then,

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty}T^n x = x^*$, for any starting point $x \in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \leq \frac{1}{1-L} \ d(T^n x, T^{n+1}x), \forall \ \ n \geq 0, \forall \ \ x \in X$;

(A4) $d(x, x^*) \leq \frac{1}{1-L} \ d(x, x^*), \forall \ \ x \in X$.

Theorem 4.2. *[26](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall \ \ n \geq 0,$$

or

$(B2)$ there exists a natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;

(ii)The sequence $(T^n x)$ is convergent to a fixed point y^* of T

(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \leq \frac{1}{1-L} \ d(y, Ty)$ for all $y \in Y$.

In order to prove the stability results, we define the following:

δ_i is a constant such that

$$\delta_i = \begin{cases} 2 & if \quad i = 0, \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Δ is the set such that

$$\Delta = \{g \mid g : X \to Y, g(0) = 0\}.$$

5. Fixed point Fuzzy Stability Results: n is an Even Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of AQCQ functional equation (1) by considering n is an even Positive Integer.

5.1. f is an odd function

Theorem 5.1. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^k r\right) = 1 \ \ \forall \ x_1, x_2, \cdots, x_n \in X, r > 0 \tag{127}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right),\ \forall\ x_1,x_2,\cdots,x_n \in X, r>0. \tag{128}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(x),r\right) = N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x),r\right) = N'\left(\beta(x),r\right),\ \forall\ x \in X, r>0. \tag{129}$$

Then there exists unique additive function $A: X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x)-8f(x)-A(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{130}$$

where

$$N'\left(\Omega\left(x,x\cdots x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) = min\left\{N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left.N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r>0$.

Proof. Let d be a general metric on Δ, such that

$$d(g,h) = inf\left\{K \in (0,\infty) | N\left(g(x)-h(x),r\right) \geq N'\left(\beta(x),Kr\right), x \in X, r>0\right\}.$$

It is easy to see that (Ω,d) is complete. Define $T:\Delta \to \Delta$ by $Tg(x) = \dfrac{1}{\delta_i}g(\delta_i x)$, for all $x \in X$. For $g,h \in \Delta$, we have $d(g,h) \leq K$

$$\begin{aligned}
&\Rightarrow & N\left(g(x)-h(x),r\right) &\geq N'\left(\beta(x),Kr\right)\\
&\Rightarrow & N\left(\frac{g(\delta_i x)}{\delta_i}-\frac{h(\delta_i x)}{\delta_i},r\right) &\geq N'\left(\beta(\delta_i x),K\delta_i r\right)\\
&\Rightarrow & N\left(Tg(x)-Th(x),r\right) &\geq N'\left(\beta(x),LKr\right)\\
&\Rightarrow & d\left(Tg(x),Th(x)\right) &\leq KL\\
&\Rightarrow & d\left(Tg,Th\right) &\leq Ld(g,h)
\end{aligned} \tag{131}$$

for all $g,h \in \Delta$. There fore T is strictly contractive mapping on Δ with Lipschitz constant L. From (13), we arrive

$$N\left(a(2x)-2a(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),r\right) \tag{132}$$

for all $x \in X$ and all $r>0$. Using (FBS3) in (132), we arrive

$$N\left(\frac{a(2x)}{2}-a(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),2r\right) \tag{133}$$

for all $x \in X$ and all $r > 0$, with the help of (129) when $i = 0$, it follows from (133), we get

$$N\left(\frac{a(2x)}{2} - a(x), r\right) \geq N'\left(\beta(x), Lr\right)$$
$$\Rightarrow d(Ta, a) \leq L = L^{1-i} \tag{134}$$

for all $x \in X$ and all $r > 0$. Letting $x = \frac{x}{2}$ in (132), we obtain

$$N\left(a(x) - 2a\left(\frac{x}{2}\right), r\right) \geq N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right) \tag{135}$$

for all $x \in X$ and all $r > 0$, with the help of (129) when $i = 1$, it follows from (135), we get

$$N\left(a(x) - 2a\left(\frac{x}{2}\right), r\right) \geq N'\left(\beta(x), r\right)$$
$$\Rightarrow \quad d(a, Ta) \leq 1 = L^0 = L^{1-i}. \tag{136}$$

Then from (134) and (136), we conclude

$$d(a, Ta) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Δ such that

$$A(x) = N - \lim_{k\to\infty} \frac{a(2^k x)}{2^k}, \qquad \forall x \in X. \tag{137}$$

By fixed point alternative, A is unique fixed point of T in the set

$$\Psi = \{a \in \Delta | d(a, A) < \infty\},$$

therefore A is a uniqe function such that

$$N\left(a(x) - A(x), r\right) \geq min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), Kr\right),\right.$$
$$\left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), Kr\right)\right\} \tag{138}$$

for all $x \in X$ and all $r > 0$ and $K > 0$. Again using the fixed point alternative, we obtain

$$d(a, A) \leq \frac{1}{1-L} d(a, Ta)$$
$$\Rightarrow \quad d(a, A) \leq \frac{L^{1-i}}{1-L}$$
$$\Rightarrow \quad N\left(a(x) - A(x), r\right) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), \tag{139}$$

firall $x \in X$ and $r > 0$. This completes the proof of the theorem. □

From Theorem 5.1, we obtain the following corollary concerning the stability for the functional equation (1).

Corollary 5.2. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \quad \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 1; \\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,r\right), & s\neq \frac{1}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{1}{n}; \end{cases} \tag{140}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n \in X$ and all $r>0$, where ϵ, s are constants with $\epsilon>0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$N\left(f(2x)-8f(x)-A(x),r\right)$$
$$\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s,\frac{|2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{141}$$

for all $x \in X$ and all $r>0$.

Proof. Setting

$$\Lambda(x_1,x_2,\cdots,x_n)=\begin{cases} \epsilon, \\ \epsilon\sum_{i=1}^{n}||x_i||^s, \\ \epsilon\prod_{i=1}^{n}||x_i||^s, \\ \epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right). \end{cases}$$

for all $x_1,x_2,\cdots,x_n \in X$. Then,

$$N'\left(\Lambda(\delta_i^k x_1,\delta_i^k x_2,\cdots,\delta_i^k x_n),\delta_i^k r\right)$$
$$=\begin{cases} N'\left(\epsilon,\delta_i^k r\right) \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,\delta_i^{(1-s)k}r\right) \\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,\delta_i^{(1-ns)k}r\right) \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),\delta_i^{(1-ns)k}r\right) \end{cases}$$
$$=\begin{cases} \to 1 \text{ as } k\to\infty, \\ \to 1 \text{ as } k\to\infty, \\ \to 1 \text{ as } k\to\infty, \\ \to 1 \text{ as } k\to\infty. \end{cases}$$

Thus, (127) is holds. But from (129) we have $N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$, has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x),r\right)\geq N'\left(\beta(x),r\right)\ \forall\ x\in X, r>0.$$

Hence

$$N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$
$$=\begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{n\epsilon}{2^s}||x||^s,\frac{r}{8}\right),N'\left(\epsilon\frac{2^s+(n-1)}{2^s}||x||^s,\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{2s}}||x||^{ns},\frac{r}{8}\right),N'\left(\frac{\epsilon}{2^{(n-1)s}}||x||^{ns},\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon(n+1)}{2^{ns}}||x||^{ns},\frac{r}{8}\right),N'\left(\epsilon\left(\frac{2^s+(n-1)}{2^s}+\frac{1}{2^{(n-1)s}}\right)||x||^{ns},\frac{r}{2}\right)\right\} \end{cases}$$

Now,

$$N'\left(\frac{1}{\delta_i}\beta(\delta_i x),r\right)$$
$$=\begin{cases} min\left\{N'\left(\frac{\epsilon}{\delta_i},\frac{r}{8}\right),N'\left(\frac{\epsilon}{\delta_i},\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{n\epsilon}{2^s\delta_i}||\delta_i x||^s,\frac{r}{8}\right),N'\left(\epsilon\frac{2^s+(n-1)s}{2^s\delta_i}||\delta_i x||^s,\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{ns}\delta_i}||\delta_i x||^{ns},\frac{r}{8}\right),N'\left(\frac{\epsilon}{2^{(n-1)s}\delta_i}||\delta_i x||^{ns},\frac{r}{2}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon(n+1)}{2^{ns}\delta_i}||\delta_i x||^{ns},\frac{r}{8}\right),N'\left(\frac{\epsilon}{\delta_i}\left(\frac{2^s+(n-1)}{2^s}+\frac{1}{2^{(n-1)s}}\right)||\delta_i x||^{ns},\frac{r}{2}\right)\right\} \end{cases}$$
$$=\begin{cases} min\left\{N'\left(\epsilon,\frac{\delta_i r}{8}\right),N'\left(\epsilon,\frac{\delta_i r}{2}\right)\right\} \\ min\left\{N'\left(\frac{n\epsilon}{2^s}||x||^s,\frac{\delta_i^{1-s}r}{8}\right),N'\left(\epsilon\frac{2^s+(n-1)}{2^s}||x||^s,\frac{\delta_i^{1-s}r}{2}\right)\right\} \\ min\left\{N'\left(\frac{\epsilon}{2^{ns}}||x||^{ns},\frac{\delta_i^{1-ns}r}{8}\right),N'\left(\frac{\epsilon}{2^{(n-1)s}}||x||^{ns},\frac{\delta_i^{1-ns}r}{2}\right)\right\} \\ min\left\{N'\left(\frac{(n+1)\epsilon}{2^{ns}}||x||^{ns},\frac{\delta_i^{1-ns}r}{8}\right),N'\left(\epsilon\left(\frac{2^{ns}+(n-1)}{2^{ns}}+\frac{1}{2^{(n-1)s}}\right)||x||^{ns},\frac{\delta_i^{1-ns}r}{2}\right)\right\} \end{cases}$$

Now from (130), we have the following cases.

Case:1 $L=2^{-1}$ if $i=0$

$$\begin{aligned} N\left(a(x)-A(x),r\right) &\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right),N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right)\right\} \\ &= min\left\{N'\left(\frac{\left(2^{-1}\right)^1}{1-2^{-1}}8\epsilon,r\right),N'\left(\frac{\left(2^{-1}\right)^1}{1-2^{-1}}2\epsilon,r\right)\right\} \\ &= min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right)\right\} \end{aligned}$$

Case:2 $L=2$ if $i=1$

$$\begin{aligned} N\left(a(x)-A(x),r\right) &\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right),N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right)\right\} \\ &= min\left\{N'\left(\frac{\left(2^{1-1}\right)}{1-2}8\epsilon,r\right),N'\left(\frac{\left(2^{1-1}\right)}{1-2}2\epsilon,r\right)\right\} \\ &= min\left\{N'\left(\epsilon,\frac{r}{|8|}\right),N'\left(\epsilon,\frac{r}{|2|}\right)\right\} \end{aligned}$$

Case:3 $L=2^{s-1}$ if $i=0$

$$\begin{aligned} &N\left(a(x)-A(x),r\right) \\ &\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right),N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right)\right\} \end{aligned}$$

$$= min\left\{N'\left(\frac{\left(2^{s-1}\right)^1}{1-2^{s-1}}\frac{8\epsilon}{2^s}||x||^s, r\right), N'\left(\frac{\left(2^{s-1}\right)^1}{1-2^{s-1}}\epsilon\frac{2^s+(n-1)}{2^s}||x||^s, r\right)\right\}$$
$$= min\left\{N'\left(\epsilon n||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2-2^s|r}{2}\right)\right\}$$

Case:4 $L = 2^{1-s}$ if $i = 1$

$$N\left(a(x)-A(x), r\right)$$
$$\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)\right\}$$
$$= min\left\{N'\left(\frac{\left(2^{1-s}\right)^{1-1}}{1-2^{1-s}}\frac{8\epsilon}{2^s}||x||^s, r\right), N'\left(\frac{\left(2^{1-s}\right)^{1-1}}{1-2^{1-s}}\epsilon\frac{2^s+(n-1)}{2^s}||x||^s, r\right)\right\}$$
$$= min\left\{N'\left(\epsilon n||x||^s, \frac{|2^s-2|r}{8}\right), N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s, \frac{|2^s-2|r}{2}\right)\right\}$$

Case:5 $L = 2^{ns-1}$ if $i = 0$

$$N\left(a(x)-A(x), r\right)$$
$$\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)\right\}$$
$$= min\left\{N'\left(\frac{\left(2^{ns-1}\right)^1}{1-2^{ns-1}}\frac{8\epsilon}{2^{ns}}||x||^{ns}, r\right), N'\left(\frac{\left(2^{ns-1}\right)^1}{1-2^{ns-1}}\frac{\epsilon}{2^{(n-1)s}}||x||^{ns}, r\right)\right\}$$
$$= min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\quad 2^s||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\}$$

Case:6 $L = 2^{1-ns}$ if $i = 1$

$$N\left(a(x)-A(x), r\right)$$
$$\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)\right\}$$
$$= min\left\{N'\left(\frac{\left(2^{1-ns}\right)^{1-1}}{1-2^{1-ns}}\frac{8\epsilon}{2^{ns}}||x||^{ns}, r\right), N'\left(\frac{\left(2^{1-ns}\right)^{1-1}}{1-2^{1-ns}}\frac{\epsilon}{2^{(n-1)s}}||x||^{ns}, r\right)\right\}$$
$$= min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^{ns-2}|r}{8}\right), N'\left(\epsilon\quad 2^s||x||^{ns}, \frac{|2^{ns}-2|r}{2}\right)\right\}$$

Case:7 $L = 2^{ns-1}$ if $i = 0$

$$N\left(a(x)-A(x), r\right)$$
$$\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right), N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)\right\}$$
$$= min\left\{N'\left(\frac{\left(2^{ns-1}\right)^1}{1-2^{ns-1}}\frac{8\epsilon(n+1)}{2^{ns}}||x||^{ns}, r\right), N'\left(\frac{\left(2^{ns-1}\right)^1}{1-2^{ns-1}}\epsilon\left(\frac{2^s+(n-1)}{2^s}+\frac{1}{2^{(n-1)s}}\right)||x||^{ns}, r\right)\right\}$$
$$= min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\}$$

Case:8 $L = 2^{1-ns}$ if $i = 1$

$$N\left(a(x)-A(x), r\right)$$

$$\geq min\left\{N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right),N'\left(\frac{L^{1-i}}{1-L}\beta(x),r\right)\right\}$$
$$= min\left\{N'\left(\frac{\left(2^{1-ns}\right)^{1-1}}{1-2^{1-ns}}\frac{8\epsilon(n+1)}{2^{ns}}||x||^{ns},r\right),N'\left(\frac{\left(2^{1-ns}\right)^{1-1}}{1-2^{1-ns}}\epsilon\left(\frac{2^s+(n-1)}{2^s}+\frac{1}{2^{(n-1)s}}\right)||x||^{ns},r\right)\right\}$$
$$= min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^{ns-1}-2|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right)\right\}$$

Hence the proof is complete. □

Theorem 5.3. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1,\delta_i^k x_2,\cdots,\delta_i^k x_n\right),\delta_i^{3k}r\right)=1\quad\forall\ x_1,x_2,\cdots,x_n\in X, r>0 \tag{142}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \ \forall\ x_1,x_2,\cdots,x_n\in X, r>0. \tag{143}$$

If there exists $L = L(i)$, such that the function

$$x\to N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x),r\right)=N'\left(\beta(x),r\right),\ \forall\ x\in X, r>0. \tag{144}$$

Then there exists unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x)-2f(x)-C(x),r\right)\geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{145}$$

where

$$N'\left(\Omega\left(x,x\cdots x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)=min\left\{N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left.N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r > 0$.

Proof. Let d be a general metric on Δ, such that

$$d(g,h)=inf\left\{K\in(0,\infty)|N\left(g(x)-h(x),r\right)\geq N'\left(\beta(x),Kr\right),x\in X,r>0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Delta \to \Delta$ by $Tg(x) = \frac{1}{\delta_i^3}g(\delta_i x)$, for all $x \in X$. For $g, h \in \Delta$, we have $d(g, h) \leq K$

$$\Rightarrow\qquad N\left(g(x)-h(x),r\right)\geq N'\left(\beta(x),Kr\right)$$
$$\Rightarrow\qquad N\left(\frac{g(\delta_i x)}{\delta_i^3}-\frac{h(\delta_i x)}{\delta_i^3},r\right)\geq N'\left(\beta(\delta_i x),K\delta_i^3 r\right)$$

$$\begin{aligned} \Rightarrow \quad & N\left(Tg(x)-Th(x),r\right) \geq N'\left(\beta(x), LKr\right) \\ \Rightarrow \quad & d\left(Tg(x),Th(x)\right) \leq KL \\ \Rightarrow \quad & d\left(Tg,Th\right) \leq Ld(g,h) \end{aligned} \tag{146}$$

for all $g, h \in \Delta$. There fore T is strictly contractive mapping on Δ with Lipschitz constant L. From (38), we arrive

$$N\left(h(2x)-8h(x),r\right) \geq N'\left(\Omega\left(x,x\cdots x\right),r\right) \tag{147}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 5.1. □

The following corollary is an immediate consequence of Theorem 5.3 concerning the Ulam-Hyers stability of the functional equation (1).

Corollary 5.4. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \quad \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s \neq 3; \\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,r\right), & s \neq \frac{3}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s \neq \frac{3}{n}; \end{cases} \tag{148}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} & N\left(f(2x)-2f(x)-C(x),r\right) \\ & \geq \begin{cases} min\left\{N'\left(\epsilon,\frac{|7|r}{8}\right),N'\left(\epsilon,\frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s,\frac{|2^3-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2^3-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{149}$$

for all $x \in X$ and all $r > 0$.

Theorem 5.5. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the conditions (127) and (142), satisfying the functional inequality*

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \forall\ x_1,x_2,\cdots,x_n \in X, r > 0. \tag{150}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(x),r\right) = N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x),r\right) = N'\left(\beta(x),r\right),\ N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x),r\right) = N'\left(\beta(x),r\right),\ \forall\ x \in X, r > 0. \tag{151}$$

Then there exists unique additive function $A : X \to Y$ and unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(x)-A(x)-C(x),r\right) \\ &\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right),N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)\right\} \end{aligned} \tag{152}$$

for all $x \in X$ and all $r > 0$.

Proof. By Theorems 5.1 and 5.3, there exists a unique additive function $A_1 : X \to Y$ and a unique cubic function $C_1 : X \to Y$ such that

$$N\left(f(2x)-8f(x)-A_1(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{153}$$

for all $x \in X$ and all $r > 0$ and

$$N\left(f(2x)-2f(x)-C_1(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{154}$$

for all $x \in X$ and all $r > 0$. Now from (153) and (154), one can see that

$$\begin{aligned} &N\left(f(x)+\frac{1}{6}A_1(x)-\frac{1}{6}C_1(x),2r\right) \\ &= N\left(-\frac{f(2x)}{6}+\frac{8}{6}f(x)+\frac{1}{6}A_1(x)+\frac{f(2x)}{6}-\frac{2}{6}f(x)-\frac{1}{6}C_1(x),2r\right) \\ &\geq \min\left\{N\left(\frac{f(2x)}{6}-\frac{8}{6}f(x)-\frac{1}{6}A_1(x),r\right),N\left(\frac{f(2x)}{6}-\frac{2}{6}f(x)-\frac{1}{6}C_1(x),r\right)\right\} \\ &\geq \min\left\{N\left(f(2x)-8f(x)-A_1(x),r\right),N\left(f(2x)-2f(x)-C_1(x),r\right)\right\} \\ &\geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right),N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)\right\} \end{aligned}$$

for all $x \in X$ and all $r > 0$. Thus we obtain (152) by defining $A(x) = \frac{-1}{6}A_1(x)$ and $C(x) = \frac{1}{6}C_1(x)$ for all $x \in X$ and all $r > 0$. □

Corollary 5.6. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 1,3; \\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,r\right), & s\neq \frac{1}{n},\frac{3}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{1}{n},\frac{3}{n}; \end{cases} \tag{155}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned} &N\left(f(x)-A(x)-C(x),r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right),N'\left(\epsilon,\frac{|7|r}{8}\right),N'\left(\epsilon,\frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s,\frac{|2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2-2^s|r}{2}\right),\right. \\ \qquad\left.N'\left(\epsilon n||x||^s,\frac{|2^3-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2^3-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right. \\ \qquad\left.N'\left(\epsilon||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right. \\ \qquad\left.N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{156}$$

for all $x \in X$ and all $r > 0$.

5.2. f is an even function

Theorem 5.7. *Let $f : X \to Y$ be an even mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^{2k} r\right) = 1 \quad \forall\ x_1, x_2, \cdots, x_n \in X, r > 0 \tag{157}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right)\ \ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{158}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ \ \forall\ x \in X, r > 0. \tag{159}$$

Then there exists unique quadratic function $Q_2 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x) - 16f(x) - Q_2(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) \tag{160}$$

where

$$N'\left(\Omega\left(x, x\cdots x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) = min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left. N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-2}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-2}{2}\ times}, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r > 0$.

Proof. Let d be a general metric on Δ, such that

$$d(g, h) = inf\left\{K \in (0, \infty) | N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right), x \in X, r > 0\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Delta \to \Delta$ by $Tg(x) = \dfrac{1}{\delta_i^2} g(\delta_i x)$, for all $x \in X$. For $g, h \in \Delta$, we have $d(g, h) \leq K$

$$\begin{aligned}
\Rightarrow &\quad N\left(g(x) - h(x), r\right) \geq N'\left(\beta(x), Kr\right) \\
\Rightarrow &\quad N\left(\frac{g(\delta_i x)}{\delta_i^2} - \frac{h(\delta_i x)}{\delta_i^2}, r\right) \geq N'\left(\beta(\delta_i x), K\delta_i^2 r\right) \\
\Rightarrow &\quad N\left(Tg(x) - Th(x), r\right) \geq N'\left(\beta(x), LKr\right) \\
\Rightarrow &\quad d\left(Tg(x), Th(x)\right) \leq KL \\
\Rightarrow &\quad d\left(Tg, Th\right) \leq Ld(g, h)
\end{aligned} \tag{161}$$

for all $g, h \in \Delta$. There fore T is strictly contractive mapping on Δ with Lipschitz constant L. From (57), we arrive

$$N\left(q_2(2x) - 4q_2(x), r\right) \geq N'\left(\Omega\left(x, x\cdots x\right), r\right) \tag{162}$$

for all $x \in X$ and all $r > 0$. The rest of the proof is similar to that of Theorem 5.1.

□

The following corollary is an immediate consequence of Theorem 5.7 concerning the Ulam-Hyers stability of the functional equation (1).

Corollary 5.8. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 2; \\ N'\left(\epsilon\prod_{i=1}^{n}||x_i||^s,r\right), & s\neq \frac{2}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{2}{n}; \end{cases} \tag{163}$$

for all $x_1,x_2,\cdots,x_{n-1},x_n \in X$ and all $r>0$, where ϵ, s are constants with $\epsilon>0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$N\left(f(2x)-16f(x)-Q_2(x),r\right) \geq \begin{cases} min\left\{N'\left(\epsilon,\frac{|3|r}{8}\right),N'\left(\epsilon,\frac{|3|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon n||x||^s,\frac{|2^2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2^2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon||x||^{ns},\frac{|2^2-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2^2-2^{ns}|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2^2-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{164}$$

for all $x \in X$ and all $r>0$.

Theorem 5.9. *Let $f : X \to Y$ be an even mapping for which there exist a function $\Omega,\Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1,\delta_i^k x_2,\cdots,\delta_i^k x_n\right),\delta_i^{4k}r\right)=1 \quad \forall\ x_1,x_2,\cdots,x_n\in X, r>0 \tag{165}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \forall\ x_1,x_2,\cdots,x_n\in X, r>0. \tag{166}$$

a If there exists $L=L(i)$, such that the function

$$x\to N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x),r\right)=N'\left(\beta(x),r\right),\ \forall\ x\in X, r>0. \tag{167}$$

Then there exists unique quartic function $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x)-4f(x)-Q_4(x),r\right)\geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{168}$$

where

$$N'\left(\Omega\left(x,x\cdots x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)=min\left\{N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left.N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-2}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-2}{2}\ times},x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x\in X$ and all $r>0$.

Proof. Let d be a general metric on Δ, such that

$$d(g,h) = inf\left\{K \in (0,\infty) | N\left(g(x)-h(x),r\right) \geq N'\left(\beta(x),Kr\right), x \in X, r>0\right\}.$$

It is easy to see that (Ω,d) is complete. Define $T:\Delta\to\Delta$ by $Tg(x) = \frac{1}{\delta_i^4}g(\delta_i x)$, for all $x \in X$. For $g,h\in\Delta$, we have $d(g,h)\leq K$

$$\begin{aligned}
\Rightarrow & \quad N\left(g(x)-h(x),r\right) \geq N'\left(\beta(x),Kr\right)\\
\Rightarrow & \quad N\left(\frac{g(\delta_i x)}{\delta_i^4}-\frac{h(\delta_i x)}{\delta_i^4},r\right) \geq N'\left(\beta(\delta_i x),K\delta_i^4 r\right)\\
\Rightarrow & \quad N\left(Tg(x)-Th(x),r\right) \geq N'\left(\beta(x),LKr\right)\\
\Rightarrow & \quad d\left(Tg(x),Th(x)\right) \leq KL\\
\Rightarrow & \quad d\left(Tg,Th\right) \leq Ld(g,h) \qquad (169)
\end{aligned}$$

for all $g,h\in\Delta$. There fore T is strictly contractive mapping on Δ with Lipschitz constant L. From (67), we arrive

$$N\left(q_4(2x)-16q_4(x),r\right) \geq N'\left(\Omega\left(x,x\cdots x\right),r\right) \qquad (170)$$

for all $x\in X$ and all $r>0$. The rest of the proof is similar to that of Theorem 5.1. □

The following corollary is an immediate consequence of Theorem 5.9 concerning the Ulam-Hyers stability of the functional equation (1).

Corollary 5.10. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2\cdots x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^n ||x_i||^s,r\right), & s\neq 4;\\ N'\left(\epsilon\prod_{i=1}^n ||x_i||^s,r\right), & s\neq \frac{4}{n};\\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s+\sum_{i=1}^n ||x_i||^{ns}\right),r\right), & s\neq\frac{4}{n}; \end{cases} \qquad (171)$$

for all $x_1,x_2\cdots x_{n-1},x_n\in X$ and all $r>0$, where ϵ, s are constants with $\epsilon>0$. Then there exists a unique quartic mapping $Q_4 : X\to Y$ such that

$$\begin{aligned}
&N\left(f(2x)-4f(x)-Q_4(x),r\right)\\
&\geq \begin{cases}
min\left\{N'\left(\epsilon,\frac{|15|r}{8}\right),N'\left(\epsilon,\frac{|15|r}{2}\right)\right\}\\
min\left\{N'\left(\epsilon n||x||^s,\frac{|2^4-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-1)\right)||x||^s,\frac{|2^4-2^s|r}{2}\right)\right\}\\
min\left\{N'\left(\epsilon||x||^{ns},\frac{|2^4-2^{ns}|r}{8}\right),N'\left(\epsilon\quad 2^s||x||^{ns},\frac{|2^4-2^{ns}|r}{2}\right)\right\}\\
min\left\{N'\left(\epsilon(n+1)||x||^{ns},\frac{|2^4-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^s+2^{ns}+(n-1)\right)||x||^{ns},\frac{|2^4-2^{ns}|r}{2}\right)\right\}
\end{cases} \qquad (172)
\end{aligned}$$

for all $x\in X$ and all $r>0$.

Theorem 5.11. *Let $f : X\to Y$ be an even mapping for which there exist a function $\Omega,\Lambda : X^n\to Z$ with the conditions (157) and (165), satisfying the functional inequality*

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \forall\ x_1,x_2,\cdots,x_n\in X, r>0. \qquad (173)$$

If there exists $L = L(i)$ such that the function

$$x \to N'(\beta(x), r) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'(\beta(x), r), \ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x), r\right) = N'(\beta(x), r), \ \forall\, x \in X, r > 0. \tag{174}$$

Then there exists unique quadratic function $Q_2 : X \to Y$ and unique quartic function $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} & N(f(x) - Q_2(x) - Q_4(x), r) \\ & \geq \min\left\{N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right\} \end{aligned} \tag{175}$$

for all $x \in X$ and all $r > 0$.

Proof. By Theorems 5.7 and 5.9, there exists a unique quadratic function $Q_2 : X \to Y$ and a quartic function $Q_4 : X \to Y$ such that

$$N(f(2x) - 16f(x) - Q_{2_1}(x), r) \geq N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right) \tag{176}$$

for all $x \in X$ and all $r > 0$ and

$$N(f(2x) - 4f(x) - Q_{4_1}(x), r) \geq N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right) \tag{177}$$

for all $x \in X$ and all $r > 0$. Now from (176) and (177), one can see that

$$\begin{aligned} & N\left(f(x) + \frac{1}{12}Q_{2_1}(x) - \frac{1}{12}Q_{4_1}(x), 2r\right) \\ & \geq \min\left\{N\left(\frac{f(2x)}{12} - \frac{16}{12}f(x) - \frac{1}{12}Q_{2_1}(x), \frac{r}{12}\right), N\left(\frac{f(2x)}{12} - \frac{4}{12}f(x) - \frac{1}{12}Q_{4_1}(x), \frac{r}{12}\right)\right\} \\ & \geq \min\left\{N(f(2x) - 16f(x) - Q_{2_1}(x), r), N(f(2x) - 4f(x) - Q_{4_1}(x), r)\right\} \\ & \geq \min\left\{N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega(x, x, \cdots, x), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right\} \end{aligned}$$

for all $x \in X$ and all $r > 0$. Thus we obtain (175) by defining $Q_2(x) = \frac{-1}{12}Q_{2_1}(x)$ and $Q_4(x) = \frac{1}{12}Q_{4_1}(x)$ for all $x \in X$ and all $r > 0$. □

The following corollary is an immediate consequence of Theorem 5.11 concerning the Ulam-Hyers stability of the functional equation(1).

Corollary 5.12. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N(Df(x_1, x_2 \cdots x_{n-1}, x_n), r) \geq \begin{cases} N'(\epsilon, r), \\ N'\left(\epsilon \sum_{i=1}^n ||x_i||^s, r\right), & s \neq 2, 4; \\ N'\left(\epsilon \prod_{i=1}^n ||x_i||^s, r\right), & s \neq \frac{2}{n}, \frac{4}{n}; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right), r\right), & s \neq \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{178}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$
\begin{aligned}
&N\left(f(x) - Q_2(x) - Q_4(x), r\right) \\
&\geq \begin{cases}
min\left\{N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\
min\left\{N'\left(\epsilon n||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right),\right. \\
\qquad \left. N'\left(\epsilon n||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\
min\left\{N'\left(\epsilon||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon \;\; 2^s||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right),\right. \\
\qquad \left. N'\left(\epsilon||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon \;\; 2^s||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \\
min\left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right),\right. \\
\qquad \left. N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\}
\end{cases}
\end{aligned}
\tag{179}
$$

for all $x \in X$ and all $r > 0$.

5.3. f is an odd-even function

Theorem 5.13. *Let $f : X \to Y$ be a mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the conditions (127), (142),(157) and (165), satisfying the functional inequality*

$$
N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right)\ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{180}
$$

If there exists $L = L(i)$ such that the function

$$
x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right),
$$

has the property

$$
\begin{aligned}
&N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ \ N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \\
&N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ \ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ \forall\ x \in X, r > 0.
\end{aligned}
\tag{181}
$$

Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$
\begin{aligned}
&N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\
&\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right)\right. \\
&\qquad N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right) \\
&\qquad N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right) \\
&\qquad \left. N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right)\right\}
\end{aligned}
\tag{182}
$$

for all $x \in X$ and all $r > 0$.

Proof. Let $f_{ac}(x) = \frac{f_o(x)-f_o(-x)}{2}$ for all $x \in X$. Then $f_{ac}(0) = 0$ and $f_{ac}(-x) = -f_{ac}(x)$ for all $x \in X$. Hence

$$N\left(Df_{ac}(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq min\left\{N'\left(\Lambda(x_1, x_2 \cdots x_{n-1}, x_n), \frac{r}{2}\right), N'\left(\Lambda(-x_1, -x_2, \cdots, -x_{n-1}, -x_n), \frac{r}{2}\right)\right\} \tag{183}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. By Theorem 5.5, there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$\begin{aligned}
&N\left(f_{ac}(x) - A(x) - C(x), r\right) \\
&\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right. \\
&\qquad \left. N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right\}
\end{aligned} \tag{184}$$

for all $x \in X$ and all $r > 0$. Also, let $f_{qq}(x) = \frac{f_e(x)+f_e(-x)}{2}$ for all $x \in X$. Then $f_{qq}(0) = 0$ and $f_{qq}(-x) = f_{qq}(x)$ for all $x \in X$. Hence

$$N\left(Df_{qq}(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq min\left\{N'\left(\Lambda(x_1, x_2 \cdots x_{n-1}, x_n), \frac{r}{2}\right), N'\left(\Lambda(-x_1, -x_2, \cdots, -x_{n-1}, -x_n), \frac{r}{2}\right)\right\} \tag{185}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$. By Theorem 5.11, there exists a unique quadratic mapping $Q_2 : X \to Y$, and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned}
&N\left(f_{qq}(x) - Q_2(x) - Q_4(x), r\right) \\
&\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right. \\
&\qquad \left. N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right)\right\}
\end{aligned} \tag{186}$$

for all $x \in X$ and all $r > 0$. Define a function $f(x)$ by

$$f(x) = f_{ac}(x) + f_{qq}(x) \tag{187}$$

for all $x \in X$. Combining (187), (184) and (186), we arrive our result. □

The following corollary is an immediate consequence of Theorem 5.13 concerning the Ulam-Hyers stability of the functional equation (1).

Corollary 5.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), & \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 1, 3, 2, 4; \\ N'\left(\epsilon \prod_{i=1}^{n} ||x_i||^s, r\right), & s \neq \frac{1}{n}, \frac{3}{n}, \frac{2}{n}, \frac{4}{n}; \\ N'\left(\epsilon \left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{1}{n}, \frac{3}{n}, \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{188}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right)$$
$$\geq \begin{cases} (i) min \left\{N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right), \right. \\ \quad N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right), \\ \quad N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), \\ \quad \left. N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\ (ii) min \left\{N'\left(\epsilon n ||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2-2^s|r}{2}\right), \right. \\ \quad N'\left(\epsilon n ||x||^s, \frac{|2^3-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^3-2^s|r}{2}\right), \\ \quad N'\left(\epsilon n ||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right), \\ \quad \left. N'\left(\epsilon n ||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-1)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\ (iii) min \left\{N'\left(\epsilon ||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s ||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right), \right. \\ \quad N'\left(\epsilon ||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s ||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\ \quad N'\left(\epsilon ||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s ||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\ \quad \left. N'\left(\epsilon ||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon \quad 2^s ||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \\ (iv) min \left\{N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right), \right. \\ \quad N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\ \quad N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\ \quad \left. N'\left(\epsilon(n+1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^s + 2^{ns} + (n-1)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{189}$$

for all $x \in X$ and all $r > 0$.

6. Fixed point Fuzzy Stability Results: n is an odd Positive Integer

In this section, we investigate the generalized Ulam-Hyers stability of AQCQ functional equation (1) by considering n is an odd positive Integer.

6.1. f is an odd function

Theorem 6.1. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^k r\right) = 1 \quad \forall\ x_1, x_2, \cdots, x_n \in X, r > 0 \tag{190}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right) \ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{191}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall\ x \in X, r > 0. \tag{192}$$

Then there exists unique additive function $A : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x) - 8f(x) - A(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) \tag{193}$$

where

$$N'\left(\Omega\left(x, x \cdots x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) = min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r > 0$.

Corollary 6.2. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 1; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{1}{n}; \end{cases} \tag{194}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ such that

$$\begin{aligned} &N\left(f(2x) - 8f(x) - A(x), r\right) \\ &\geq \begin{cases} min\left\{N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2-2^{ns}|r}{2}\right)\right\} \end{cases} \end{aligned} \tag{195}$$

for all $x \in X$ and all $r > 0$.

Theorem 6.3. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1, \delta_i^k x_2, \cdots, \delta_i^k x_n\right), \delta_i^{3k} r\right) = 1 \ \ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0 \tag{196}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right)\ \ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{197}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall \ x \in X, r > 0. \tag{198}$$

Then there exists unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x) - 2f(x) - C(x), r\right) \geq N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) \tag{199}$$

where

$$N'\left(\Omega\left(x, x \cdots x\right), \left(\frac{L^{1-i}}{1-L}\right) r\right) = min\left\{N'\left(\Lambda\left(x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right), N'\left(\Lambda\left(2x, \underbrace{x, x, \cdots, x}_{\frac{n-3}{2}\ times}, \underbrace{-x, -x, \cdots, -x}_{\frac{n-3}{2}\ times}, 0, x\right), \left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r > 0$.

Corollary 6.4. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2, \cdots, x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon \sum_{i=1}^n ||x_i||^s, r\right), & s \neq 3; \\ N'\left(\epsilon\left(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}\right), r\right), & s \neq \frac{3}{n}; \end{cases} \tag{200}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(2x) - 2f(x) - C(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2^3 - 2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^3 - 2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^3 - 2^{ns}|r}{2}\right)\right\} \end{cases} \tag{201}$$

for all $x \in X$ and all $r > 0$.

Theorem 6.5. *Let $f : X \to Y$ be an odd mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the conditions (190) and (196), satisfying the functional inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right) \ \forall \ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{202}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \ \forall \ x \in X, r > 0. \tag{203}$$

Then there exists unique additive function $A : X \to Y$ and unique cubic function $C : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(x)-A(x)-C(x),r\right) \geq \min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right),N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)\right\} \tag{204}$$

for all $x \in X$ and all $r > 0$.

Corollary 6.6. *Suppose that an odd function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 1,3; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{1}{n},\frac{3}{n}; \end{cases} \tag{205}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$ such that

$$N\left(f(x)-A(x)-C(x),r\right) \geq \begin{cases} min\left\{N'\left(\epsilon,\frac{r}{8}\right),N'\left(\epsilon,\frac{r}{2}\right),N'\left(\epsilon,\frac{|7|r}{8}\right),N'\left(\epsilon,\frac{|7|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2-2^s|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^s,\frac{|2^3-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^3-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2-2^{ns}|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^3-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^3-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{206}$$

for all $x \in X$ and all $r > 0$.

6.2. f is an even function

Theorem 6.7. *Let $f : X \to Y$ be an even mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1,\delta_i^k x_2,\cdots,\delta_i^k x_n\right),\delta_i^{2k} r\right) = 1 \quad \forall\ x_1,x_2,\cdots,x_n \in X, r>0 \tag{207}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right) \geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \ \forall\ x_1,x_2,\cdots,x_n \in X, r>0. \tag{208}$$

If there exists $L = L(i)$, such that the function

$$x \to N'\left(\beta(x),r\right) = N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x),r\right) = N'\left(\beta(x),r\right),\ \ \forall\ x \in X, r>0. \tag{209}$$

Then there exists unique quadratic function $Q_2 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x)-16f(x)-Q_2(x),r\right) \geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{210}$$

where

$$N'\left(\Omega\left(x,x\cdots x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)=min\left\{N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left.N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x \in X$ and all $r > 0$.

Corollary 6.8. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2,\cdots,x_{n-1},x_n),r\right) \geq \begin{cases} N'\left(\epsilon,r\right), & \\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 2; \\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq \frac{2}{n}; \end{cases} \tag{211}$$

for all $x_1, x_2, \cdots, x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a such that

$$\begin{aligned} &N\left(f(2x)-16f(x)-Q_2(x),r\right)\\ &\geq \begin{cases} min\left\{N'\left(\epsilon,\frac{|3|r}{8}\right),N'\left(\epsilon,\frac{|3|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2^2-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^2-2^s|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^2-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^2-2^{ns}|r}{2}\right)\right\}\end{cases}\end{aligned} \tag{212}$$

for all $x \in X$ and all $r > 0$.

Theorem 6.9. *Let $f : X \to Y$ be an even mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the condition*

$$\lim_{k\to\infty} N'\left(\Lambda\left(\delta_i^k x_1,\delta_i^k x_2,\cdots,\delta_i^k x_n\right),\delta_i^{4k}r\right)=1 \quad \forall\ x_1,x_2,\cdots,x_n\in X, r>0 \tag{213}$$

and satisfying the functional inequality

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \ \forall\ x_1,x_2,\cdots,x_n\in X, r>0. \tag{214}$$

If there exists $L = L(i)$, such that the function

$$x\to N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right)$$

has the property

$$N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x),r\right)=N'\left(\beta(x),r\right),\ \forall\ x\in X, r>0. \tag{215}$$

Then there exists unique quartic function $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$N\left(f(2x)-4f(x)-Q_4(x),r\right)\geq N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right) \tag{216}$$

where

$$N'\left(\Omega\left(x,x\cdots x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)=min\left\{N'\left(\Lambda\left(x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{8}\right),\right.$$
$$\left.N'\left(\Lambda\left(2x,\underbrace{x,x,\cdots,x}_{\frac{n-3}{2}\ times},\underbrace{-x,-x,\cdots,-x}_{\frac{n-3}{2}\ times},0,x\right),\left(\frac{L^{1-i}}{1-L}\right)\frac{r}{2}\right)\right\}$$

for all $x\in X$ and all $r>0$.

Corollary 6.10. *Suppose that an even function $f:X\to Y$ satisfies the inequality*

$$N\left(Df(x_1,x_2\cdots x_{n-1},x_n),r\right)\geq\begin{cases}N'\left(\epsilon,r\right),\\ N'\left(\epsilon\sum_{i=1}^{n}||x_i||^s,r\right), & s\neq 4;\\ N'\left(\epsilon\left(\prod_{i=1}^{n}||x_i||^s+\sum_{i=1}^{n}||x_i||^{ns}\right),r\right), & s\neq\frac{4}{n};\end{cases}\tag{217}$$

for all $x_1,x_2\cdots x_{n-1},x_n\in X$ and all $r>0$, where ϵ,s are constants with $\epsilon>0$. Then there exists a unique quartic mapping $Q_4:X\to Y$ such that

$$\begin{aligned}&N\left(f(2x)-4f(x)-Q_4(x),r\right)\\&\geq\begin{cases}min\left\{N'\left(\epsilon,\frac{|15|r}{8}\right),N'\left(\epsilon,\frac{|15|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon(n-1)||x||^s,\frac{|2^4-2^s|r}{8}\right),N'\left(\epsilon\left(2^s+(n-2)\right)||x||^s,\frac{|2^4-2^s|r}{2}\right)\right\}\\ min\left\{N'\left(\epsilon(n-1)||x||^{ns},\frac{|2^4-2^{ns}|r}{8}\right),N'\left(\epsilon\left(2^{ns}+(n-2)\right)||x||^{ns},\frac{|2^4-2^{ns}|r}{2}\right)\right\}\end{cases}\end{aligned}\tag{218}$$

for all $x\in X$ and all $r>0$.

Theorem 6.11. *Let $f:X\to Y$ be an even mapping for which there exist a function $\Omega,\Lambda:X^n\to Z$ with the conditions (207) and (213), satisfying the functional inequality*

$$N\left(D\ f(x_1,x_2,\cdots,x_n),r\right)\geq N'\left(\Lambda(x_1,x_2,\cdots,x_n),r\right)\ \forall\ x_1,x_2,\cdots,x_n\in X,r>0.\tag{219}$$

If there exists $L=L(i)$ such that the function

$$x\to N'\left(\beta(x),r\right)=N'\left(\Omega\left(\frac{x}{2},\frac{x}{2},\cdots,\frac{x}{2}\right),r\right),$$

has the property

$$N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x),r\right)=N'\left(\beta(x),r\right),\ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x),r\right)=N'\left(\beta(x),r\right),\ \forall\ x\in X,r>0.\tag{220}$$

Then there exists unique quadratic function $Q_2:X\to Y$ and unique quartic function $Q_4:X\to Y$ satisfying the functional equation (1) and

$$N\left(f(y)-Q_2(y)-Q_4(y),r\right)\geq\min\left\{N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right),N'\left(\Omega\left(x,x,\cdots,x\right),\left(\frac{L^{1-i}}{1-L}\right)r\right)\right\}\tag{221}$$

for all $x\in X$ and all $r>0$.

Corollary 6.12. *Suppose that an even function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon \sum_{i=1}^{n} ||x_i||^s, r\right), & s \neq 2, 4; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} ||x_i||^s + \sum_{i=1}^{n} ||x_i||^{ns}\right), r\right), & s \neq \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{222}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$N\left(f(x) - Q_2(x) - Q_4(x), r\right) \geq \begin{cases} min\left\{N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^2-2^s|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)||x||^s, \frac{|2^4-2^s|r}{2}\right)\right\} \\ min\left\{N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^2-2^{ns}|r}{2}\right),\right. \\ \qquad \left. N'\left(\epsilon(n-1)||x||^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)||x||^{ns}, \frac{|2^4-2^{ns}|r}{2}\right)\right\} \end{cases} \tag{223}$$

for all $x \in X$ and all $r > 0$.

6.3. f is an odd-even function

Theorem 6.13. *Let $f : X \to Y$ be a mapping for which there exist a function $\Omega, \Lambda : X^n \to Z$ with the conditions (190), (197),(206) and (213), satisfying the functional inequality*

$$N\left(D\ f(x_1, x_2, \cdots, x_n), r\right) \geq N'\left(\Lambda(x_1, x_2, \cdots, x_n), r\right)\ \forall\ x_1, x_2, \cdots, x_n \in X, r > 0. \tag{224}$$

If there exists $L = L(i)$ such that the function

$$x \to N'\left(\beta(x), r\right) = N'\left(\Omega\left(\frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{2}\right), r\right),$$

has the property

$$\begin{aligned} &N'\left(L\frac{1}{\delta_i}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ N'\left(L\frac{1}{\delta_i^2}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right), \\ &N'\left(L\frac{1}{\delta_i^3}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ N'\left(L\frac{1}{\delta_i^4}\beta(\delta_i x), r\right) = N'\left(\beta(x), r\right),\ \forall\ x \in X, r > 0. \end{aligned} \tag{225}$$

Then there exists a unique additive mapping $A : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$, a unique cubic cubic mapping $C : X \to Y$ and unique quartic mapping $Q_4 : X \to Y$ satisfying the functional equation (1) and

$$\begin{aligned} &N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\ &\geq \min\left\{N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right)\right. \\ &\quad N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right) \\ &\quad N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right) \\ &\quad \left.N'\left(\Omega\left(x, x, \cdots, x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right), N'\left(\Omega\left(-x, -x, \cdots, -x\right), \left(\frac{L^{1-i}}{1-L}\right)r\right)\right\} \end{aligned} \tag{226}$$

for all $x \in X$ and all $r > 0$.

Corollary 6.14. *Suppose that a function $f : X \to Y$ satisfies the inequality*

$$N\left(Df(x_1, x_2 \cdots x_{n-1}, x_n), r\right) \geq \begin{cases} N'\left(\epsilon, r\right), \\ N'\left(\epsilon \sum_{i=1}^{n} \|x_i\|^s, r\right), & s \neq 1, 3, 2, 4; \\ N'\left(\epsilon\left(\prod_{i=1}^{n} \|x_i\|^s + \sum_{i=1}^{n} \|x_i\|^{ns}\right), r\right), & s \neq \frac{1}{n}, \frac{3}{n}, \frac{2}{n}, \frac{4}{n}; \end{cases} \tag{227}$$

for all $x_1, x_2 \cdots x_{n-1}, x_n \in X$ and all $r > 0$, where ϵ, s are constants with $\epsilon > 0$. Then there exists a unique additive mapping $A : X \to Y$ and a unique cubic mapping $C : X \to Y$, a unique quadratic mapping $Q_2 : X \to Y$ and a unique quartic mapping $Q_4 : X \to Y$ such that

$$\begin{aligned} & N\left(f(x) - A(x) - Q_2(x) - C(x) - Q_4(x), r\right) \\ & \geq \begin{cases} (i) min \left\{ N'\left(\epsilon, \frac{r}{8}\right), N'\left(\epsilon, \frac{r}{2}\right), \right. \\ \quad N'\left(\epsilon, \frac{|7|r}{8}\right), N'\left(\epsilon, \frac{|7|r}{2}\right), \\ \quad N'\left(\epsilon, \frac{|3|r}{8}\right), N'\left(\epsilon, \frac{|3|r}{2}\right), \\ \quad \left. N'\left(\epsilon, \frac{|15|r}{8}\right), N'\left(\epsilon, \frac{|15|r}{2}\right) \right\} \\ (ii) min \left\{ N'\left(\epsilon(n-1)\|x\|^s, \frac{|2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)\|x\|^s, \frac{|2-2^s|r}{2}\right), \right. \\ \quad N'\left(\epsilon(n-1)\|x\|^s, \frac{|2^3-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)\|x\|^s, \frac{|2^3-2^s|r}{2}\right), \\ \quad N'\left(\epsilon(n-1)\|x\|^s, \frac{|2^2-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)\|x\|^s, \frac{|2^2-2^s|r}{2}\right), \\ \quad \left. N'\left(\epsilon(n-1)\|x\|^s, \frac{|2^4-2^s|r}{8}\right), N'\left(\epsilon\left(2^s + (n-2)\right)\|x\|^s, \frac{|2^4-2^s|r}{2}\right) \right\} \\ (iv) min \left\{ N'\left(\epsilon(n-1)\|x\|^{ns}, \frac{|2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)\|x\|^{ns}, \frac{|2-2^{ns}|r}{2}\right), \right. \\ \quad N'\left(\epsilon(n-1)\|x\|^{ns}, \frac{|2^3-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)\|x\|^{ns}, \frac{|2^3-2^{ns}|r}{2}\right), \\ \quad N'\left(\epsilon(n-1)\|x\|^{ns}, \frac{|2^2-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)\|x\|^{ns}, \frac{|2^2-2^{ns}|r}{2}\right), \\ \quad \left. N'\left(\epsilon(n-1)\|x\|^{ns}, \frac{|2^4-2^{ns}|r}{8}\right), N'\left(\epsilon\left(2^{ns} + (n-2)\right)\|x\|^{ns}, \frac{|2^4-2^{ns}|r}{2}\right) \right\} \end{cases} \end{aligned} \tag{228}$$

for all $x \in X$ and all $r > 0$.

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