Int. J. Math. And Appl., 6(4)(2018), 1–4
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of Mathematics And its Applications

On $\beta^\star$-closed Spaces in Terms of Nets

P. Maragatha Meenakshi[1] and J. Sathya[2,*]

1 Department of Mathematics, Periyar E.V.R College, Trichirapalli, Tamilnadu, India.

2 Department of Mathematics, Thanthai Hans Roever College, Perambalur, Tamilnadu, India.

Abstract: The purpose of this paper is to obtain various characterizations of $\beta^\star$-closed spaces interms nets.

MSC: 54C10, 54C08, 54C05.

Keywords: Topological spaces, $\beta^\star$-open sets, $\beta^\star$-closed spaces.

Accepted on: 23.06.2018

1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real analysis concerns the various modified forms of continuity, separation axioms etc. by utilizing generalized open sets. Recently, Ali M. Mubarki [1] introduced a new class of generalized open sets called $\beta^\star$-open sets into the field of topology. The purpose of this paper is to obtain various characterizations of $\beta^\star$-closed spaces in terms of nets.

For a subset A of a topological space (X, τ), $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ denote the closure of A and the interior of A, respectively.

Definition 1.1 ([2]). *The δ-closure of A, denoted by $\mathrm{Cl}_\delta(A)$, is defined to be the set of all $x \in X$ such that $A \cap \mathrm{Int}(\mathrm{Cl}(U)) \neq \varnothing$ for every open neighbourhood U of X. If $A = \mathrm{Cl}_\delta(A)$, then A is called δ-closed. The complement of a δ-closed set is a called δ-open set. The δ-interior of A is defined by the union of all δ-open sets contained in A and is denoted by $\mathrm{Int}_\delta(A)$.*

Definition 1.2 ([1]). *A subset S of a topological space (X, τ) is said to be $\beta^\star$-open if $S \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(S))) \cup \mathrm{Int}(\mathrm{Cl}_\delta(S))$. The complement of a $\beta^\star$-closed set is called a $\beta^\star$-open set. The family of all $\beta^\star$-open ($\beta^\star$-closed) subsets of (X, τ) is denoted by $\beta^\star O(X)(\beta^\star C(X))$. The family of all $\beta^\star$-open sets of (X, τ) containing a point $x \in X$ is denoted by $\beta^\star O(X, x)$.*

Definition 1.3 ([1]). *The intersection of all $\beta^\star$-closed sets containing $A \subset X$ is called the $\beta^\star$-closure of A and is denoted by $\beta^\star\,\mathrm{Cl}(A)$. The union of all $\beta^\star$-open sets contained in $A \subset X$ is called the $\beta^\star$-interior of A and is denoted by $\beta^\star\,\mathrm{Int}(A)$.*

2. $\beta^\star$-closed Spaces

Definition 2.1. *A topological space X is said to be $\beta^\star$-closed if every cover of X by $\beta^\star$-open sets ($= \beta^\star$-open cover) has a finite subcover whose $\beta^\star$-closures cover X.*

* *E-mail: satvelan@gmail.com*

Lemma 2.2 ([1]). *Let A and B be subsets of a topological space X. If $A \in \beta^\star O(X)$ and B is δ-open in X, then $A \cap B \in \beta^\star O(B)$.*

Theorem 2.3. *Suppose that A and B are subsets of X such that $A \subset B \subset X$ and B is δ-open in X. Then A is $\beta^\star$-closed relative to the subspace B if and only if A is $\beta^\star$-closed relative to X.*

Proof. Let $\{V_\alpha : \alpha \in I\}$ be a $\beta^\star$-open cover of A. Then by Lemma 2.2, $B \cap V_\alpha \in \beta^\star O(B)$. Since A is $\beta^\star$-closed relative to B, there is a finite subfamily I_0 of I such that $A \subset \cup\{\beta^\star \operatorname{Cl}(B \cap V_\alpha) : \alpha \in I_0\}$ Using Lemma 2.2 once again we have $A \subset \cup\{\beta^\star \operatorname{Cl}(B \cap V_\alpha) : \alpha \in I_0\} \subset \cup\{\beta^\star \operatorname{Cl}(V_\alpha) : \alpha \in I_0\}$. This shows that A is $\beta^\star$-closed relative to X. Conversely, suppose that $\{V_\alpha : \alpha \in I\}$ is a cover of A, where $V_\alpha \in \beta^\star O(B)$ for each $\alpha \in I$. Then by Lemma 2.2 we have $V_\alpha \in \beta^\star O(X)$ for each $\alpha \in I$. Since A is $\beta^\star$-closed relative to X, $A \subset \cup\{\beta^\star \operatorname{Cl}(V_\alpha) : \alpha \in I_0\}$ for some finite subfamily I_0 of I. Again, in view of Lemma 2.2, $A \subset \cup\{\beta^\star \operatorname{Cl}(V_\alpha) : \alpha \in I_0\}$; hence A is $\beta^\star$-closed relative to X. □

Corollary 2.4. *A δ-open subset A of a topological space X is $\beta^\star$-closed if and only if it is $\beta^\star$-closed relative to X.*

The following two theorems are easy consequences of the definitions and hence omitted.

Theorem 2.5. *The union of a finite number of sets in a topological space X, each of which is $\beta^\star$-closed relative to X, is $\beta^\star$-closed relative to X.*

Theorem 2.6. *If A is a $\beta^\star$-open as well as $\beta^\star$-closed subset of a topological space X, then it is $\beta^\star$-closed relative to X.*

Definition 2.7. *A filterbase $\mathcal{F}$ on a topological space X is said to be:*

(1). $\beta^\star$-converge to a point $x \in X$, written $\mathcal{F}\underset{\to}{\beta^\star}x$, if for each $\beta^\star$-open set U containing x, there exists $F \in \mathcal{F}$ such that $F \subset \beta^\star \operatorname{Cl}(U)$.

(2). $\beta^\star$-adhere at $x \in X$, written $x \in \beta^\star - ad(\mathcal{F})$, if for each $\beta^\star$-open set U containing x and each $F \in \mathcal{F}$, $F \cap \beta^\star \operatorname{Cl}(U) \neq \emptyset$.

Definition 2.8. *Let A be a subset of a topological space X. Then a net $\{x_\alpha : \alpha \in (D, \geq)\}$ in A said to be:*

(1). $\beta^\star$-adhere at x, written $x \in \beta^\star - ad(x_\alpha)$, if for each $U \in \beta^\star O(X, x)$ and each $\alpha \in D$ there exists $\beta^\star \in D$ with $\beta^\star \geq \alpha$ such that $x \in \beta^\star \operatorname{Cl}(U)$.

(2). $\beta^\star$-converge at $x \in X$, denoted by $x_\alpha \underset{\to}{\beta^\star} x$, if the net is eventually in $\beta^\star \operatorname{Cl}(U)$ for all $U \in \beta^\star O(X, x)$.

Theorem 2.9. *For a nonempty set A of a topological space (X, τ), the following are statements are equivalent:*

(1). A is $\beta^\star$-closed relative to X.

(2). Every maximal filterbase on X which meets A $\beta^\star$-converges to some point of A.

(3). Every maximal filterbase on A $\beta^\star$-converges to some point of A.

(4). Every filterbase on X which meets A $\beta^\star$-converges to some point of A.

(5). For every family $\{U_\alpha : \alpha \in I\}$ of nonempty $\beta^\star$-closed sets with $(\bigcap_{\alpha\in I} U_\alpha) \cap A = \emptyset$, there is a finite subset I_0 of I such that $(\bigcap_{\alpha\in I} \beta^\star \operatorname{Int}(U_\alpha)) \cap A = \emptyset$.

(6). Every filterbase on A $\beta^\star$-adhere at some point of A.

(7). For every family $\{U_\alpha : \alpha \in I\}$ of nonempty $\beta^\star$-closed sets with $(\bigcap_{\alpha\in I} \beta^\star \operatorname{Cl}(B_\alpha)) \cap A = \emptyset$, there is a finite subset I_0 of I such that $(\bigcap_{\alpha\in I} B_\alpha) \cap A = \emptyset$.

(8). Every net in A $\beta^\star$-adheres at some point of A.

(9). Every ultranet in A $\beta^\star$-adheres at some point of A.

(10). Every net in A has a $\beta^\star$-convergent subnet.

Proof. (1) $\Rightarrow$ (2): Suppose that $\mathcal{F}$ is a maximal filterbase on X, which meets A and does not $\beta^\star$-converge to any point of A. Then for each $x \in A$, there exists $V_x \in \beta^\star O(X,x)$ such that $F \cap (X \backslash \beta^\star \operatorname{Cl}(V_x)) \neq \emptyset$ for every $F \in \mathcal{F}$. The maximality of the filterbase $\mathcal{F}$ then implies that there is some $F_x \in \mathcal{F}$ with $F_x \subset X \backslash \beta^\star \operatorname{Cl}(V_x)$ then $F_x \cap \beta^\star \operatorname{Cl}(V_x) = \emptyset$. Since $\mathcal{U} = \{V_x : x \in A\}$ is a $\beta^\star$-open cover of A, $A \subset \bigcap_{i=1}^{n} \beta^\star \operatorname{Cl}(V_{x_i})$ for some finite subcollection $\{V_{x_1}, V_{x_2}, ...V_{x_n}\}$ of $\mathcal{U}$. Let $F \in \mathcal{F}$ such that $F \subset \bigcap_{i=1}^{n} F_{x_i}$. Then $F \cap A \subset \bigcup_{i=1}^{n} \beta^\star \operatorname{Cl}(V_{x_i}) = \emptyset$, which is a contradiction as $\mathcal{F}$ meets A.

(2) $\Leftrightarrow$ (3): It is clear because of the fact that whenever $\mathcal{F}$ is a maximal filterbase on X, which meets A, the filterbase $\mathcal{F}' = \{F \cap A : F \in \mathcal{F}\}$ on A is also maximal.

(2) $\Rightarrow$ (4): Let $\mathcal{F}$ be a given filterbase $\mathcal{F}$ on X, which meets A. Then $\mathcal{F}$ is contained in a maximal filterbase $\mathcal{F}^*$ which meets A. Since $\mathcal{F}\underset{\rightarrow}{\beta^\star} x$ for some $x \in A$, for every $V \in \beta^\star O(X,x)$ there exists $F_0 \in \mathcal{F}^*$ such that $F_0 \subset \beta^\star \operatorname{Cl}(V)$. Since $F \cap F_0 \neq \emptyset$ for each $F \in \mathcal{F}$, we have $\beta^\star \operatorname{Cl}(V) \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. It follows that $x \in \beta^\star - ad(\mathcal{F})$.

(4) $\Rightarrow$ (1): If possible, let there exists a $\beta^\star$-open cover $\mathcal{U}$ of A such that for every finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ $A \backslash \bigcup_{U \in \mathcal{U}_0} (U) \neq \emptyset$. Then $\mathcal{F} = \{A \backslash \bigcup_{U \in \mathcal{U}_0} \beta^\star \operatorname{Cl}(U)$: $\mathcal{U}_0$ is a finite subfamily of $\mathcal{U}\}$ is a filterbase on X, which meets A. By (iv), there is $a \in A$ such that $a \in \beta^\star - ad\mathcal{U}$. Now $\mathcal{U}$ being a cover of A, there is $U_a \in \mathcal{U}$ such that $a \in U_a$. But then $X \backslash \beta^\star \operatorname{Cl}(U_a) \in \mathcal{F}$ containing the fact that $a \in \beta^\star - ad\mathcal{F}$.

(1) $\Rightarrow$ (5): If $\{U_\alpha : \alpha \in I\}$ is a family of nonempty $\beta^\star$-closed sets with $(\bigcap_{\alpha \in I} U_\alpha) \cap A = \emptyset$, then $A \subset X \backslash \bigcap_{\alpha \in I_0} U_\alpha = \bigcup_{\alpha \in I} (X \backslash U_\alpha)$, that is, $\{(X \backslash U_\alpha) : \alpha \in I\}$ is $\beta^\star$-open cover of A. By (i), there is a finite subset I_0 of I such that $A \subset \bigcup_{\alpha \in I_0} \beta^\star \operatorname{Cl}(X \backslash U_\alpha) = \bigcup_{\alpha \in I_0} (X \backslash \beta^\star \operatorname{Int}(U_\alpha)) = X \backslash \bigcap_{\alpha \in I_0} \beta^\star \operatorname{Int}(U_\alpha)$. Hence $A \cap (\bigcap_{\alpha \in I_0} \beta^\star \operatorname{Int}(U_\alpha)) = \emptyset$.

(5) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in I\}$ be any $\beta^\star$-open cover of A. If $U_\alpha = X$ foe some $\alpha \in I$, then we are through. If $U_\alpha \neq X$ for each $\alpha \in I$, then $\{X \backslash U_\alpha : \alpha \in I\}$ is a family of nonempty $\beta^\star$-closed sets such that $(\bigcap_{\alpha \in I} (X \backslash U_\alpha)) \cap A = (X \backslash \bigcup_{\alpha \in I} U_\alpha) \cap A = \emptyset$. By (v), there is a finite subset I_0 of I such that $\emptyset \neq A \cap (\bigcap_{\alpha \in I} (X \backslash \beta^\star \operatorname{Cl}(U_\alpha))) = A \cap (X \backslash \bigcup_{\alpha \in I_0} (\beta^\star \operatorname{Cl}(U_\alpha)))$; hence $A \subset \bigcup_{\alpha \in I_0} (\beta^\star \operatorname{Cl}(U_\alpha)$ proving that A is $\beta^\star$-closed relative to X.

(4) $\Rightarrow$ (6): Obvious.

(6) $\Rightarrow$ (7): Let $\mathcal{B} = \{B_\alpha : \alpha \in I\}$ be a family of nonempty sets in X such that for every finite subset I_0 of I, $(\bigcap_{\alpha \in I_0} B\alpha \cap A \neq \emptyset$. Then $\mathcal{F} = \{(\bigcap_{\alpha \in I_0} B_\alpha) \cap A$: I_0 is a finite subset of $I\}$ is a filterbase on A. By (vi), let $a \in A \cap \beta^\star - ad\mathcal{F}$. then for each $\alpha \in I$ and each $U \in \beta^\star O(X,a)$, $A \cap B_\alpha \cap \beta^\star \operatorname{Cl}(B\alpha) \neq \emptyset$, that is $B_\alpha \cap \beta^\star \operatorname{Cl}(U) \neq \emptyset$. Hence $a \in \beta^\star \operatorname{Cl}(B_\alpha)$ for each $\alpha \in I$ and consequently, $(\bigcap_{\alpha \in I} \beta^\star \operatorname{Cl}(B_\alpha)) \cap A \neq \emptyset$.

(7) $\Rightarrow$ (1): Let $\{U_\alpha : \alpha \in I\}$ be a $\beta^\star$-open cover of A. Then $A \cap (\bigcap_{\alpha \in I} (X \backslash U_\alpha)) = \emptyset$. If foe some $\alpha \in I$, $X \backslash \beta^\star \operatorname{Cl}(U_\alpha) = \emptyset$, then (i) follows. If $X \backslash \beta^\star \operatorname{Cl}(U_\alpha) = B_\alpha$ (say), $\neq \emptyset$, for each $\alpha \in I$, then $\mathcal{B} = \{B_\alpha : \alpha \in I\}$ is a family of nonempty sets such that $(\bigcap_{\alpha \in I} \beta^\star \operatorname{Cl}(B_\alpha)) \cap A \subset A \cap (\bigcap_{\alpha \in I} (X \backslash U_\alpha)) = \emptyset$ $(*)$. In fact, let $x \in \beta^\star \operatorname{Cl}(B_\alpha) = \beta^\star \operatorname{Cl}(X \backslash \beta^\star \operatorname{Cl}(U_\alpha))$. Then for every $V_x \in \beta^\star O(X)$, $(X \backslash \beta^\star \operatorname{Cl}(U_\alpha)) \cap (\beta^\star \operatorname{Cl}(V_\alpha)) \neq \emptyset$. Since $U_\alpha \in \beta^\star O(X)$, if $x \in U_\alpha$, then $(X \backslash \beta^\star \operatorname{Cl}(U_\alpha)) \cap (\beta^\star \operatorname{Cl}(V_\alpha)) \neq \emptyset$, which is not possible. Thus, $x \notin U_\alpha$ so that $x \in U_\alpha$. Hence $\beta^\star \operatorname{Cl}(B_\alpha) \subset X \backslash U_\alpha$ and $(*)$ follows. By (vii), ther is a finite subset I_0 of I such that $(\bigcap_{\alpha \in I} (B_\alpha) \cap A = \emptyset$, that is, $A \subset X \backslash \bigcap_{\alpha \in I_0} (X \backslash \beta^\star \operatorname{Cl}(B_\alpha)) = \bigcap_{\alpha \in I_0} \beta^\star \operatorname{Cl}(U_\alpha)$.

(6) $\Rightarrow$ (8): Let $\{x_n : n \in (D, \geq)\}$ be a net in A. Consider the filterbase $\mathcal{F} = \{T_n : n \in D\}$ generated by the net, where $T_n = \{x_m : m \in D$ and $m \geq n\}$. By (vi), there exists $a \in A \cap \beta^\star - ad\mathcal{F}$. Then for each $U \in \beta^\star O(X,a)$ and each $F \in \mathcal{F}$, $\beta^\star \operatorname{Cl}(U) \cap F \neq \emptyset$, that is, $\beta^\star \operatorname{Cl}(U) \cap T_n \neq \emptyset$ for all $n \in D$. Hence $a \in A \cap \beta^\star - ad(x_n)$.

(8) $\Rightarrow$ (9): Let $\{x_n : n \in (D, \geq)\}$ be an ultranet in A. By $(viii)$, there exists $a \in \beta^\star - ad(x_n) \cap A$. Let $U \in \beta^\star O(X,a)$. Since the given net is an ultranet in A, it is eventually is either $A \cap \beta^\star \operatorname{Cl}(U)$ or $A \backslash (A \cap \beta^\star \operatorname{Cl}(U))$. But since the net is frequently

in $A \cap \beta^\star \operatorname{Cl}(U)$, we conclude that the net is eventually in $\beta^\star \operatorname{Cl}(U)$. Hence $x_n \underset{\rightarrow}{\beta^\star} a$.

(9) $\Leftrightarrow$ (10): Let $\{x_n : n \in (D, \geq)\}$ be a net in A. Since net has a subnet, the subnet of the given net $\beta^\star$-converges to some point of A by (ix), and (x) follows.

(8) $\Leftrightarrow$ (10): Let $T : E \to A$ be a $\beta^\star$-convergent subnet of a given net $S : D \to A$, and suppose $T \underset{\rightarrow}{\beta^\star} a \in A$. Then $T = S \circ N$, where $N : E \to D$ is a function such that for each $n \in D$, there exists $P \in E$ with the property that $N(m) \geq n$ in D whenever $m \in E$ with $m \geq p$. Let $U \in \beta^\star O(X, a)$ and $n \in D$, there is $m_1 \in E$ such that $T(m) \in \beta^\star \operatorname{Cl}(U)$ for all $m \geq m_1$ ($m \in E$). For the given $n \in D$, let $p \in E$ with the above stated property and $m_2 \in E$ such that $m_2 \geq p, m_1$. Then $N(m_2) \geq n$ in D, and we have $T(m_2) = S \circ N(m_2) \in \beta^\star \operatorname{Cl}(U)$ (since $m_2 \geq m_1$). Hence $a \in \beta^\star - ad(S) \cap A$. This completes the proof of the Theorem. □

Putting $A = X$ in the above Theorem, we now obtain the following characterization of a $\beta^\star$-closed space.

Theorem 2.10. *For a nonempty set A of a topological space (X, τ), the following are statements are equivalent:*

(1). X is a $\beta^\star$-closed space.

(2). Every maximal filterbase on X $\beta^\star$-converges.

(3). Every filterbase on X $\beta^\star$-adherent.

(4). For every family $\{U_\alpha : \alpha \in I\}$ of nonempty $\beta^\star$-closed sets in X with $\underset{\alpha \in I_0}{\cap} U_\alpha = \emptyset$, there is a finite subset I_0 of I such that $\underset{\alpha \in I_0}{\cap} \beta^\star \operatorname{Int}(U_\alpha) = \emptyset$.

(5). For every family $\{B_\alpha : \alpha \in I\}$ of nonempty closed sets in X with $\underset{\alpha \in I_0}{\cap} \beta^\star \operatorname{Cl}(B_\alpha) = \emptyset$, there is a finite subset I_0 of I such that $\underset{\alpha \in I_0}{\cap} B_\alpha = \emptyset$.

(6). Every net in X has a $\beta^\star$-adherent point.

(7). Every ultranet in X $\beta^\star$-converges.

(8). Every net in X has a $\beta^\star$-convergent subnet.

Theorem 2.11. *A topological space X is $\beta^\star$-closed if and only if every filterbase on X with atmost one $\beta^\star$-adherent point is $\beta^\star$-convergent.*

Proof. Let X be $\beta^\star$-closed, and a filterbase $\mathcal{F}$ on X with atmost one $\beta^\star$-adherent point by Theorem 2.10. let x_0 be a unique $\beta^\star$-adherent point of $\mathcal{F}$ and if possible, let $\mathcal{F}$ does not $\beta^\star$-converge to x_0. Then for some $U \in \beta^\star O(X, x_0)$ and for each $F \in \mathcal{F}$, $F \cap (X \backslash \beta^\star \operatorname{Cl}(U)) \neq \emptyset$. So $y = \{F \cap (X \backslash \beta^\star \operatorname{Cl}(U)) : F \in \mathcal{F}\}$ is a filterbase on X and hence a $\beta^\star$-adherent point x in X. Since $U \in \beta^\star O(x, x_0)$ and $\beta^\star \operatorname{Cl}(U) \cap G = \emptyset$ for all $G \in \mathcal{F}$, we have $x \neq x_0$. Now for each $V \in \beta^\star O(X, x)$ and each $F \in \mathcal{F}$, $\beta^\star \operatorname{Cl}(U) \supset \beta^\star \operatorname{Cl}(X) \cap (X \backslash \beta^\star \operatorname{Cl}(U)) \neq \emptyset$, that is, $F \cap \beta^\star \operatorname{Cl}(V) \neq \emptyset$. Thus, x is a $\beta^\star$-adherent point of $\mathcal{F}$. The converse is clear in view of Theorem 2.10 and the fact that a point x is necessarily a $\beta^\star$-adherent point of a filterbase $\mathcal{F}$ if $\mathcal{F} \underset{\rightarrow}{\beta^\star} x$. □

References

[1] Ali M. Mubarki, Massed M. Al-Rshudi and Mohammad A. Al-Juhani, *$\beta^\star$-Open sets and $\beta^\star$-continuity in topological spaces*, Journal of Taibah University for Science, 8(2014), 142-148.

[2] N. V. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl., 78(2)(1968), 103-118.