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Differential Inequalities and Comparison Principles for Linearly Perturbed Differential Equations of First Type

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Abstract: In this paper, some results concerning the global existence as well as comparison theorems for an initial value problem of first order hybrid differential equations with a linear perturbation of first type have been proved. The main results rely on the hybrid fixed point technique of Dhage involving the sum of two operators in a Banach space. Our results include several basic results for unperturbed nonlinear differential equations as special cases.

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1. Statement of the Problem

Given a bounded interval $J = [t_0, t_0 + a]$ in $\mathbb{R}$ for some fixed $t_0, a \in \mathbb{R}$ with $a > 0$, consider the initial value problem of nonlinear hybrid differential equation (in short HDE) with a linear perturbation of first type perturbed by a nonlinear term,

$$\left.\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t))) + g(t, x(t), x(\eta(t))), \quad t \in J, \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1}$$

where, $f, g : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\theta, \eta : J \to J$ are continuous functions. By a *solution* of the HDE (1) we mean a function $x \in C(J, \mathbb{R})$ that satisfies the equations in (1), where $C(J, \mathbb{R})$ is the space of continuous real-valued functions defined on J. The importance of the investigations of nonlinear differential and hybrid differential equations lies in the fact that they include several dynamic systems as special cases. See Hartman [14] and references therein. The consideration of hybrid differential equations is implicit in the works of Krasnoselskii [15] and Burton [2] and extensively treated in the several papers on hybrid differential equations with different perturbations. See Dhage and Jadhav [10], Dhage and Lakshmikatham [11] and the references therein. This class of hybrid differential equations includes the perturbations of original differential equations in different ways. A sharp classification of different types of perturbations of differential equations appears in Dhage [3] which can be treated with hybrid fixed point theory (see Dhage [3–6] and the references therein). In this paper, we initiate the basic theory of hybrid differential equations of linear perturbations of first type involving two nonlinearities and prove the basic result such as local existence theorem and existence of maximal and minimal solutions etc. We claim that the results of this paper are basic and important contribution to the theory of nonlinear ordinary differential equations.

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2. Global Existence Result

In this section, we prove an existence result for the HDE (1) on a closed and bounded interval $J = [t_0, t_0 + a]$ under mixed Lipschitz and compactness type conditions on the nonlinearities involved in it. We place the HDE (1) in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. Define a supremum norm $\|\cdot\|$ in $C(J, \mathbb{R})$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly $C(J, \mathbb{R})$ is a Banach space with respect to the above supremum norm. We prove the existence of solution for the HDE (1) via the following hybrid fixed point theorem involving the sum of two nonlinear operators in a Banach space due to Dhage [4–6]. Before stating the hybrid fixed point theorem, we give some preliminaries and definitions that will be used in what follows.

Definition 2.1. *A mapping $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a **dominating function** or, in short, $\mathcal{D}$-**function** if it is upper semi-continuous and nondecreasing function satisfying $\psi(0) = 0$. A mapping $Q : E \to E$ is called* $\mathcal{D}$**-Lipschitz** *if there is a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying*

$$\|Q\phi - Q\xi\| \le \psi(\|\phi - \xi\|) \tag{2}$$

for all $\phi, \xi \in E$. The function ψ is called a $\mathcal{D}$-function of Q on E. If $\psi(r) = k\,r$, $k > 0$, then Q is called **Lipschitz** *with the Lipschitz constant k. In particular, if $k < 1$, then Q is called a* **contraction** *on X with the contraction constant k. Further, if $\psi(r) < r$ for $r > 0$, then Q is called* **nonlinear $\mathcal{D}$-contraction** *and the function ψ is called $\mathcal{D}$-function of Q on X.*

The details of different types of contractions appear in the papers of Dhage [5, 7] and in the monograph of Granas and Dugundji [13]. There do exist $\mathcal{D}$-functions in the literature and the commonly used $\mathcal{D}$-functions are $\psi(r) = k\,r$ and $\psi(r) = \dfrac{r}{1+r}$, etc. These $\mathcal{D}$-functions have been widely used in the theory of nonlinear differential and integral equations for proving the existence results via fixed point methods. Another notion that we need in the sequel is the following definition.

Definition 2.2. *An operator Q on a Banach space E into itself is called compact if $Q(E)$ is a relatively compact subset of E. Q is called totally bounded if for any bounded subset S of E, $Q(S)$ is a relatively compact subset of E. If Q is continuous and totally bounded, then it is called completely continuous on E.*

Theorem 2.3 ([5, 7])**.** *Suppose that S is a non-empty, closed, convex and bounded subset of the Banach space E and let $A : E \to E$ and $B : S \to E$ be two operators such that*

(a). A is nonlinear $\mathcal{D}$-contraction,

(b). B is compact and continuous, and

(c). $x = Ax + By$ for all $y \in S \Longrightarrow x \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

We consider the following hypotheses in what follows.

(A_1) There exists a constant $L > 0$ such that

$$|f(t, x_1, x_2) - f(t, y_1, y_2)| \le \frac{L \max\{|x_1 - y_1|\,, |x_2 - y_2|\}}{K + \max\{|x_1 - y_1|\,, |x_2 - y_2|\}}$$

for all $t \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Moreover, $La \le K$.

(A_2) There exists a constant $M > 0$ such that $|g(t, x, y)| \leq M$ for all $t \in J$ and $x, y \in \mathbb{R}$.

The following useful lemma follows from the fundamental theorem of calculus.

Lemma 2.4. *For any continuous function $h : J \to \mathbb{R}$, the function $x \in C(J, \mathbb{R})$ is a solution of the HDE*

$$\left.\begin{aligned} x'(t) &= h(t), \;\; t \in J, \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{3}$$

if and only if x satisfies the hybrid integral equation (HIE)

$$x(t) = x_0 + \int_{t_0}^{t} h(s)\, ds, \;\; t \in J. \tag{4}$$

Now we are in a position to prove the following existence theorem for the HDE (1) defined on J.

Theorem 2.5. *Assume that the hypotheses (A_1) and (A_2) hold. Then the HDE (1) has a solution defined on J.*

Proof. Set $E = C(J, \mathbb{R})$ and define a subset $\overline{B_r(x_0)}$ of E defined by

$$\overline{B_r(x_0)} = \{x \in E \mid \|x - x_0\| \leq r\} \tag{5}$$

where, $r = (L + M)a + F_0$ and $F_0 = \sup\limits_{t \in J} \int_{t_0}^{t} |f(s, 0, 0)|\, ds$. Clearly $\overline{B_r(x_0)}$ is a closed, convex and bounded subset of the Banach space E. Below we show that every solution of the HDE (1) that exists in E belongs to $\overline{B_r(x_0)}$ and consequently the solutions are of global in nature. Now, using the hypotheses (A_2) it can be shown by an application of Lemma 2.4 that the HDE (1) is equivalent to the nonlinear hybrid integral equation (in short HIE)

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x(\theta(s))\, ds + \int_{t_0}^{t} g(s, x(s), \eta(s))\, ds \tag{6}$$

for $t \in J$. Define two operators $\mathcal{A} : E \to E$ and $\mathcal{B} : \overline{B_r(x_0)} \to E$ by

$$\mathcal{A}x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x(\theta(s))\, ds, \;\; t \in J, \tag{7}$$

and

$$\mathcal{B}x(t) = \int_{t_0}^{t} g(s, x(s), x(\eta(s))\, ds, \;\; t \in J. \tag{8}$$

Then, the HIE (14) is transformed into an operator equation as

$$\mathcal{A}x(t) + \mathcal{B}x(t) = x(t), \;\; t \in J. \tag{9}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.3. First, we show that $\mathcal{A}$ is a nonlinear contraction on E with a $\mathcal{D}$ function ψ. Let $x, y \in E$. Then, by hypothesis (A_1),

$$\begin{aligned} |\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq \int_{t_0}^{t} |f(t, x(s), x(\theta(s)) - f(t, y(s), y(\theta(s))|\, ds \\ &\leq \int_{t_0}^{t} \frac{L \max\{|x(s) - y(s)|\,,\, |x(\theta(s)) - y(\theta(s))|\}}{K + \max\{|x(s) - y(s)|\,,\, |x(\theta(s)) - y(\theta(s))|\}}\, ds \\ &\leq \frac{La\|x - y\|}{K + \|x - y\|} \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|\mathcal{A}x - \mathcal{A}y\| \le \frac{La\|x-y\|}{K+\|x-y\|}$$

for all $x, y \in E$. This shows that $\mathcal{A}$ is a nonlinear contraction on E with the $\mathcal{D}$-function ψ defined by $\psi(r) = \dfrac{La\,r}{K+r}$. Next, we show that $\mathcal{B}$ is a compact and continuous operator on $\overline{B_r(x_0)}$ into E. First we show that $\mathcal{B}$ is continuous on S. Let $\{x_n\}$ be a sequence in $\overline{B_r(x_0)}$ converging to a point $x \in S$. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \int_{t_0}^{t} g(s, x_n(s), x_n(\eta(s)))\, ds \\
&= \int_{t_0}^{t} \Big[\lim_{n\to\infty} g(s, x_n(s), x_n(\eta(s)))\Big]\, ds \\
&= \int_{t_0}^{t} g(s, x(s), x(\eta(s)))\, ds \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{\mathcal{B}x_n\}$ is an equicontinuous sequence of functions in E. Now, following the arguments similar to that given in Granas *et al.* [12], it is proved that B is a a continuous operator on S.

Next, we show that $\mathcal{B}$ is compact operator on $\overline{B_r(x_0)}$. It is enough to show that $\mathcal{B}(\overline{B_r(x_0)})$ is a uniformly bounded and equi-continuous set in E. Let $x \in \overline{B_r(x_0)}$ be arbitrary. Then by hypothesis (A$_2$),

$$|\mathcal{B}x(t)| \le \int_{t_0}^{t} |g(s, x(s), x(\eta(s))| \, ds \le \int_{t_0}^{t_0+\alpha} M\, ds \le Ma$$

for all $t \in J$. Taking supremum over t, $\|\mathcal{B}x\| \le Ma$ for all $x \in \overline{B_r(x_0)}$. This shows that $\mathcal{B}$ is uniformly bounded on $\overline{B_r(x_0)}$. Again, let $t_1, t_2 \in J$. Then for any $x \in \overline{B_r(x_0)}$, one has

$$\begin{aligned}
|\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| &= \left| \int_{t_0}^{t_1} g(s, x(s), \eta(s))\, ds - \int_{t_0}^{t_2} g(s, x(s), x(\eta(s))\, ds \right| \\
&\le \left| \int_{t_2}^{t_1} |g(s, x(s), x(\eta(s))| \, ds \right| \\
&\le M|t_1 - t_2|.
\end{aligned}$$

Hence, for $\epsilon > 0$, there exists a $\delta = \dfrac{\epsilon}{M} > 0$ such that

$$|t_1 - t_2| < \delta \quad \implies \quad |\mathcal{B}x(t_1) - \mathcal{B}x(t_2)| < \epsilon$$

uniformly for all $t_1, t_2 \in J$ and for all $x \in \overline{B_r(x_0)}$. This shows that $\mathcal{B}(\overline{B_r(x_0)})$ is an equi-continuous set in E. Now the set $\mathcal{B}(\overline{B_r(x_0)})$ is uniformly bounded and equicontinuous set in E, so it is compact by Arzelá-Ascoli theorem. As a result, $\mathcal{B}$ is a continuous and compact operator on $\overline{B_r(x_0)}$ into E.

Next, we show that hypothesis (c) of Theorem 2.3 is satisfied by the operators $\mathcal{A}$ and $\mathcal{B}$. Let $x \in E$ be fixed and $y \in S$ be arbitrary such that $x = \mathcal{A}x + \mathcal{B}y$. Then, by assumption (A$_1$), we have

$$\begin{aligned}
|x(t) - x_0| &\le \int_{t_0}^{t} |f(s, x(s), x(\theta(s))| \, ds + \int_{t_0}^{t} |g(s, y(s), y(\eta(s))| \, ds \\
&\le \int_{t_0}^{t} \big[|f(s, x(s), x(\theta(s)) - f(s,0,0)| + |f(s,0,0)|\big]\, ds + \int_{t_0}^{t} |g(s, y(s), y(\eta(s))| \, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq La + F_0 + \int_{t_0}^{t_0+\alpha} M\, ds \\
&\leq (L+M)a + F_0.
\end{aligned}
$$

Taking supremum over t, $\|x - x_0\| \leq r$ and therefore, $x \in \overline{B_r(x_0)}$. Thus, all the conditions of Theorem 2.3 are satisfied and hence the operator equation $Ax + Bx = x$ has a solution in S. As a result, the HDE (1) has a solution defined on J. This completes the proof. □

3. Maximal and Minimal Solutions

In this section, we shall prove the existence of maximal and minimal solutions for the HDE (1) on $J = [t_0, t_0 + a]$. We need the following definition in what follows.

Definition 3.1. *A solution r of the HDE (1) is said to be maximal if for any other solution x to the HDE (1) one has $x(t) \leq r(t)$ for all $t \in J$. Again, a solution ρ of the HDE (1) is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any other solution of the HDE (1) existing on J.*

We begin by stating the basic results dealing with hybrid differential inequalities without proof, because the proofs are similar to the case of unperturbed nonlinear differential equations given in Lakshmikantham and Leela [16]. See also Dhage [3, 5] and the references therein.

Theorem 3.2. *Suppose that there exist $y, z \in C(J, \mathbb{R})$ such that*

$$
y'(t) \leq f(t, y(t), y(\theta(t)) + g(t, y(t), y(\eta(t)), \quad t \in J, \tag{10}
$$

and

$$
z'(t) \geq f(t, z(t), z(\theta(t)) + g(t, z(t), z(\eta(t)), \quad t \in J. \tag{11}
$$

If one of the inequalities (10) and (11) is strict and

$$
y(t_0) < z(t_0), \tag{12}
$$

then,

$$
y(t) < z(t) \tag{13}
$$

for all $t \in J$.

The next result is about the nonstrict inequality for the HDE (1) on J which requires a one-sided Lipschitz condition.

Theorem 3.3. *Assume that the hypotheses of Theorem 2.3 hold. Suppose also that there exists a real number $L > 0$ such that*

$$
f(t, y(t), y(\theta(t)) - f(t, z(t), z(\theta(t)) + g(t, y(t), y(\eta(t)) - g(t, z(t), z(\eta(t)) \leq L\big[y(t) - z(t)\big] \tag{14}
$$

whenever $y(t) \geq z(t)$, $t_0 \leq t < t_0 + a$. Then,

$$
y(t_0) \leq z(t_0) \tag{15}
$$

implies

$$
y(t) \leq z(t) \tag{16}
$$

for all $t \in J$.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the similar arguments with appropriate modifications. Given a arbitrary small real number $\epsilon > 0$, consider the the following initial value problem of HDE,

$$\left.\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t)) + g(t, x(t), x(\eta(t)) + \epsilon, \;\; t \in J, \\ x(t_0) &= x_0 + \epsilon, \end{aligned}\right\} \tag{17}$$

where $f, g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. An existence theorem for the HDE (10) can be stated as follows:

Theorem 3.4. *Assume that the hypotheses (A_1) and (A_2) hold. Then for every arbitrary real number $\epsilon > 0$, the HDE (1) has a solution defined on J.*

Proof. The proof is similar to Theorem 3.2 and we omit the details. □

Our main existence theorem concerning the maximal solution for the HDE (1) is as follows.

Theorem 3.5. *Assume that the hypotheses (A_1) and (A_2) hold. Then the HDE (1) has a maximal solution defined on J.*

Proof. Let $\{\epsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then for any solution u of the HDE (1), by Theorem 2.3, one has

$$u(t) < r(t, \epsilon_n) \tag{18}$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$, where $r(t, \epsilon_n)$ is a solution of the HDE,

$$\left.\begin{aligned} x'(t) &= f(t, x(t), x(\theta(t))) + g(t, x(t), x(\eta(t))) + \epsilon_n, \;\; t \in J, \\ x(t_0) &= x_0 + \epsilon_n \end{aligned}\right\} \tag{19}$$

defined on J. Since, by Theorems 3.2 and 3.3, $\{r(t, \epsilon_n)\}$ is a decreasing sequence of positive real numbers, so it is bounded below by 0 and the limit

$$r(t) = \lim_{n\to\infty} r(t, \epsilon_n) \tag{20}$$

exists for each $t \in J$. We show that the convergence in (20) is uniform on J. To finish, it is enough to prove that the sequence $\{r(t, \epsilon_n)\} = \{r_{\epsilon_n}(t)\}$ is an equi-continuous set in $C(J, \mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$\begin{aligned} \big|r(t_1, \epsilon_n) - r(t_2, \epsilon_n)\big| &\le \left|\int_{t_0}^{t_1} f(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\theta(s)))\,ds - \int_{t_0}^{t_2} f(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\theta(s)))\,ds\right| \\ &\quad + \left|\int_{t_0}^{t_1} g(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\eta(s)))\,ds - \int_{t_0}^{t_2} g(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\eta(s)))\,ds\right| + \left|\int_{t_0}^{t_1} \epsilon_n\,ds - \int_{t_0}^{t_2} \epsilon_n\,ds\right| \\ &= \left|\int_{t_1}^{t_2} |f(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\theta(s)))|\,ds\right| + \left|\int_{t_1}^{t_2} |g(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\eta(s)))|\,ds\right| + \left|\int_{t_1}^{t_2} \epsilon_n\,ds\right| \\ &\le \left|\int_{t_1}^{t_2} |f(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\theta(s))) - f(s,0,0)|\,ds\right| + \left|\int_{t_1}^{t_2} |f(s,0,0)|\,ds\right| + \left|\int_{t_1}^{t_2} M\,ds\right| + \left|\int_{t_1}^{t_2} \epsilon_n\,ds\right| \\ &= (L+M)|t_1 - t_2| + |F(t_1) - F(t_2)| + |t_1 - t_2|\,\epsilon_n, \end{aligned} \tag{21}$$

where $F(t) = \int_{t_0}^{t} |f(s,0,0)|\,ds$. Since F is continuous on compact set J, it is uniformly continuous there. Hence,

$$|F(t_1) - F(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly on J. Therefore, from the above inequality (14), it follows that

$$|r(t_1, \epsilon_n) - r(t_2, \epsilon_n)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t, \epsilon_n) \to r(t) \quad \text{as} \quad n \to \infty$$

uniformly for all $t \in J$. Next, we show that the function $r(t)$ is a solution of the HDE (10) defined on J. Now, since $r(t, \epsilon_n)$ is a solution of the HDE (12), we have

$$r(t, \epsilon_n) = x_0 + \epsilon_n + \int_{t_0}^{t} f(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\theta(s)))\, ds + \int_{t_0}^{t} g(s, r_{\epsilon_n}(s), r_{\epsilon_n}(\eta(s)))\, ds \tag{22}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above equation (15) yields

$$r(t) = x_0 + \int_{t_0}^{t} f(s, r(s), r(\theta(s)))\, ds + \int_{t_0}^{t} g(s, r(s), r(\eta(s)))\, ds$$

for $t \in J$. Thus, the function r is a solution of the HDE (1) on J. Finally, form the inequality (13) it follows that

$$u(t) \le r(t)$$

for all $t \in J$. Hence the HDE (1) has a maximal solution on J. This completes the proof. □

4. Comparison Principle

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the HDE (1). See Beesack [1], Lakshmikatham and Leela [16], Dhage and Dhage [9] and references therein. In this section we prove that the maximal and minimal solutions serve as the bounds for the solutions of the related differential inequality to HDE (1) on $J = [t_0, t_0 + a]$.

Theorem 4.1. *Assume that the hypotheses (A_1) and (A_2) hold. Further, if there exists a function $u \in C(J, \mathbb{R})$ such that*

$$\left.\begin{aligned} & u'(t) \le f(t, u(t), u(\theta(t)) + g(t, u(t), u(\eta(t))), \quad t \in J, \\ & u(t_0) \le x_0. \end{aligned}\right\} \tag{23}$$

Then,

$$u(t) \le r(t) \tag{24}$$

for all $t \in J$, where r is a maximal solution of the HDE (1) on J.

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 4.2, $r(t, \epsilon)$ is a maximal solution of the HDE (23) and that the limit

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \tag{25}$$

is uniform on J and the function r is a maximal solution of the HDE (1) on J. Hence, we obtain

$$\left.\begin{aligned} & r'(t, \epsilon) = g(t, r(t, \epsilon), r(\theta(t), \epsilon)) + g(t, r(t, \epsilon), r(\eta(t), \epsilon)) + \epsilon \\ & r(t_0, \epsilon) = x_0 + \epsilon. \end{aligned}\right\} \tag{26}$$

for all $t \in J$. From above inequality it follows that

$$\left.\begin{aligned} & r'(t, \epsilon) > g(t, r(t, \epsilon), r(\theta(t), \epsilon)) + g(t, r(t, \epsilon), r(\eta(t), \epsilon)) \\ & r(t_0, \epsilon) > x_0. \end{aligned}\right\} \tag{27}$$

for all $t \in J$. Now we apply Theorem 2.3 to the inequalities (23) and (27) and conclude that

$$u(t) < r(t, \epsilon) \tag{28}$$

for all $t \in J$. This further in view of limit (25) implies that inequality (24) holds on J. This completes the proof. □

Theorem 4.2. *Assume that the hypotheses (A_1) and (A_2) hold. Further, if there exists a function $v \in C(J, \mathbb{R})$ such that*

$$\left.\begin{aligned} v'(t) &\geq f(t, v(t), v(\theta(t)) + g(t, v(t), v(\eta(t)), \ \ t \in J, \\ v(t_0) &\geq x_0. \end{aligned}\right\} \tag{29}$$

Then,

$$v(t) \geq \rho(t) \tag{30}$$

for all $t \in J$, where ρ is a minimal solution of the HDE (1) on J.

Note that Theorem 4.1 is useful to prove the boundedness and uniqueness of the solutions for the HDE (1) on J. A result in this direction is as follows.

Theorem 4.3. *Suppose that there exists a function $G : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\left.\begin{aligned} \big|f(t, x_1, x_2) - f(t, y_1, y_2)\big| &\leq F\left(t, \mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid\right) \\ \big|g(t, x_1, x_2) - g(t, y_1, y_2)\big| &\leq G\left(t, \mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid\right) \end{aligned}\right\} \tag{31}$$

for all $t \in J$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. If identically zero function is the only solution of the scalar differential equation

$$\begin{aligned} m'(t) &= F(t, m(t), m(\theta(t)) + G(t, m(t), m(\eta(t)), \\ m(t_0) &= 0 \end{aligned} \tag{32}$$

for allt $\in J$, where $F, G : J \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfy (A_1) and (A_2), then the HDE (1) has a unique solution defined on J.

Proof. By Theorem 3.3, the HDE (1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of the HDE (1) existing on J. Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = |u_1(t) - u_2(t)|. \tag{33}$$

Then, we have

$$m(\theta(t)) = |u_1(\theta(t)) - u_2(\theta(t))|$$

and

$$m(\eta(t)) = |u_1(\eta(t)) - u_2(\eta(t))|$$

for all $t \in J$. As $\big(|m(t)|\big)' \leq |m'(t)|$ for $t \in J$, we have that

$$\begin{aligned} m'(t) &\leq |u_1'(t) - u_2'(t)| \\ &\leq |f(t, u_1(t), u_1(\theta(t))) - f(t, u_2(t), u_1(\theta(t)))| + |g(t, u_1(t), u_1(\eta(t))) - g(t, u_2(t), u_1(\eta(t)))| \\ &\leq F(t, |u_1(t) - u_2(t)|, |u_1(\theta(t)) - u_2(\theta(t))|) + G(t, |u_1(t) - u_2(t)|, |u_1(\eta(t)) - u_2(\eta(t))|) \\ &= F(t, m(t), m(\theta(t))) + G(t, m(t), m(\eta(t))) \end{aligned}$$

for all $t \in J$; and that $m(t_0) = 0$. Now, we apply Theorem 4.1 to get that $m(t) = 0$ for all $t \in J$. This gives $u_1(t) = u_2(t)$ for all $t \in J$. This completes the proof. □

When $f \equiv 0$ and $g(t,x,y) = g(t,x)$ in our results, we obtain the differential inequalities and other related results given in Lakshmikantam and Leela [16] for the IVP of ordinary nonlinear differential equation

$$x'(t) = g(t, x(t)),\ t \in J,\ \ x(t_0) = x_0. \tag{34}$$

Remark 4.4. *The hybrid differential equations is a rich area for variety of nonlinear ordinary as well as partial differential equations. Here, in this paper, we have considered a very simple hybrid differential equation involving two nonlinearities, however, a more complex hybrid differential equation can also be studied on the similar lines with appropriate modifications. Again, the results proved in this paper are very fundamental in nature and therefore, all other problems for the hybrid differential equation in question are still open. In a forthcoming research project we plan to prove some approximation results for the hybrid differential equation considered in this paper.*

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