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On Multiplication Groups of Middle Bol Loop Related to Left Bol Loop

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Abstract: In this paper, we show that an element of Bryant-Schneider group of middle Bol loop $(Q, \circ)$ is an automorphism and pseudo-automorphism. We proved that multiplication groups of middle autotopy nuclei of middle Bol loop $(Q, \circ)$ coincided with the corresponding left Bol loop $(Q, \circ)$ and their multiplication groups were show to be normal subgroups. It was found that the corresponding left Bol loop $(Q, \circ)$ is an indexed two.

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1. Introduction

A non-empty set Q with binary operation $'A$ is called a groupoid (Q, A). Let (Q, A) be a groupoid and a be fixed element in Q then the translation maps L_a and R_a are defined by $xL_a = ax$ and $xR_a = xa$ for all $x \in Q$. A groupoid (Q, A) is called quasigroup $(Q, \cdot)$ if the maps $L(a) : G \to G$ and $R(a) : G \to G$ are bijections for all $a \in Q$ and if the equations $ax = b$ and $ya = b$ have respectively unique solutions $x = a \setminus b$ and $y = b/a$ for all $a, b \in Q$. A quasigroup $(Q, \cdot)$ is called a loop if $a \cdot 1 = a = 1 \cdot a$, for all $a \in Q$. The group generated by these mappings are called multiplication group $Mlp(Q, \cdot)$. We donate the groups generated by left (right and middle translations) of a quasigroup $(Q, \cdot)$ by $LM(Q, \cdot)$, $RM(Q, \cdot)$ and $PM(Q, \cdot)$ respectively [4]. A loop $(Q, \cdot)$ is called a middle Bol loop if every isotope of $(Q, \cdot)$ satisfies the identity $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$.(that is if the anti-automorphic inverse property is universal in $(Q, \cdot)$ [3]. A loop $(Q, \cdot)$ is called a middle Bol loop if is satisfies the identity $x(yz \setminus x) = (x/z)(y \setminus x)$. In [5] Gvaramiya proved that a loop $(Q, \circ)$ is middle Bol if there exist a right Bol loop $(Q, \cdot)$ such that $x \circ y = (y \cdot xy^{-1})y$. This imply that if $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding right Bol loop then $x \circ y = x^{-1} \setminus y$ and $x \cdot y = y//x^{-1}$, where '/'('//') is the right division in $(Q, \circ)$. If $(Q, \circ)$ is a middle Bol loop and $(Q, \cdot)$ is the corresponding left Bol loop then $x \circ y = x/y^{-1}$ and $x \cdot y = x//y^{-1}$ where ('\') '//' is the left right division in $(Q, \circ)$ (respectively, in $(Q, \circ)$). Let $(Q, \cdot)$ be a loop and the set of autotopisms are defined as follow:

- (α, i, γ) of $(Q, \cdot)$ is called left autotopy nucleus (left A-nucleus).
- (i, β, γ) of $(Q, \cdot)$ is called right autotopy nucleus (right A-nucleus).

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- (α, β, i) of $(Q, \cdot)$ is called middle autotopy nucleus (middle A-nucleus).

$$N_\lambda = a \in Q \mid ax \cdot y = a \cdot xy \quad \forall \quad x, y \in Q$$

$$N_\mu = a \in Q \mid xa \cdot y = x \cdot ay \quad \forall \quad x, y \in Q$$

$$N_\rho = a \in Q \mid xy \cdot a = x \cdot ya \quad \forall \quad x, y \in Q$$

$$Z = N \cap C = a \cdot x = x \cdot a, \quad \text{where } i \text{ is an identity mapping.}$$

Let $(Q, \cdot)$ be a loop, a bijection $\alpha \in Q$ is called pseudo-automorphism such that $(\alpha, \alpha R_g, \alpha R_g)$ is an autotopism where g is a companion in Q. Let $(Q, \cdot)$ be a loop. We donate the following translations $RM(Q, \cdot) = xR_a \mid a \in Q = (x \cdot a \mid x \in Q)$.

$$LM(Q, \cdot) = xL_a \mid a \in Q = (a \cdot x \mid x \in Q).$$

$$PM(Q, \cdot) = xP_a s \mid a \in Q = (x \cdot s = a \mid x, s \in Q),$$

where L_a, R_a and P_a are permutations of the set Q. Let $(Q, \cdot)$ be a groupoid (quasigroup, loop) and α, β, and γ be three bijections that map Q onto Q. The triple $\Phi = (\alpha, \beta, \gamma)$ is called an autotopism of $(Q, \cdot)$ if and only if $\alpha x \cdot \beta y = \gamma(x \cdot y)$ for all$x, y \in Q$. If $\alpha = \beta = \gamma$, then Φ is called the automorpism of $(Q, \cdot)$. Let $(Q, \cdot)$ a loop and $BS(Q, \cdot)$ be the set of all bijections α of Q such that $< \alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha >$ is an autotopism of Q for all $f, g \in Q$, then $BS(Q, \cdot)$ is called Bryant-Schneider group of the loop $(Q, \cdot)$.

2. Preliminaries

Lemma 2.1 ([2]). *Let $(Q, \cdot)$ left Bol loop and $(Q, \circ)$ be the corresponding middle Bol loop. Then$(\alpha, \beta, \gamma) \in S_Q^3$ is an autotopism of $(Q, \cdot)$ if and only if $(\gamma, I\beta I, \alpha)$ autotopism $(Q, \circ)$.*

Lemma 2.2 ([6]). *Let $(Q, \cdot)$ be any loop.Then $(Q, \cdot)$ is a middle Bol loop if and only if $(IP_x^{-1}, IP_x, IP_x L_x)$ is an autotopism of Q.*

Lemma 2.3 ([6]). *Let $(Q,)$ be a middle Bol loop. Then $(A, B, C) \in Atp(Q,)$ if and only if $(ICI, IBI, IAI) \in Atp(Q, \cdot)$.*

3. Main Results

Theorem 3.1. *Let $(Q, \cdot)$ be a symmetric entropy middle Bol loop with $\alpha \in BS(Q, \cdot)$ such that $A = \langle \alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha \rangle$ is an autotopism of Q for some $g \in Q$. Then α^{-1} is an automorphism of Q if Q is of exponent two and $g \in Z$ the center of Q.*

Proof. In symmetric entropy of middle Bol loop, $(IP_x^{-1}, IP_x, IP_x L_x) = B = (IL_x, IR_x, IR_x L_x)$ is an autotopism of Q for all $x \in Q$. And $A = \langle \alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha \rangle$ is an autotopism of $BS(Q, \cdot)$ for some $g \in Q$. Consider $AB = \langle \alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha \rangle \langle (IL_x, IR_x, IR_x L_x)) \rangle = \langle \alpha R_{g^{-1}} IL_x, \alpha L_{g^{-1}} IR_x, \alpha IR_x L_x \rangle$ is also an autotopism of Q. Here, $\alpha R_{g^{-1}} IL_x = \alpha^{-1} R_g L_x$. So, $AB = \langle \alpha^{-1} R_g L_x, \alpha^{-1} L_g R_x, \alpha^{-1} R_x L_x, \rangle$ if we set $g = x$, we have $AB = \langle \alpha^{-1} R_{x^2}, \alpha^{-1} R_{x^2}, \alpha^{-1} R_{x^2} \rangle \in Atp(Q, \cdot)$, since $x \in Z$ (center of Q) that is $L_x = R_x$. And $R_{x^2} = 1$ (that is of exponent 2) for all $x \in Q$, this imply that, $AB = \langle \alpha^{-1}, \alpha^{-1}, \alpha^{-1} \rangle \in Atp(Q, \cdot)$, now for all $x \in Q$ is of exponent 2, $R_{x^2} = L_{x^2} = I$ the identity mapping. Therefor, $AB = \langle \alpha^{-1}, \alpha^{-1}, \alpha^{-1} \rangle$ is also an autotopism of Q. This means that α^{-1} is an automorphism. □

Theorem 3.2. *Let $(Q, \cdot)$ be a middle Bol loop and $(Q, \circ)$ be the corresponding right Bol loop, if $\alpha \in BS(Q, \circ)$ and $B = \langle \alpha R_g, \alpha L_{g^{-1}}, \alpha \rangle \in Atp(Q, \circ)$ for some $g \in Q$. Then $IP_x^{-1} \alpha$ is a pseudo-automorphism with companion g.*

Proof. Suppose that $(Q,\cdot)$ is a middle Bol loop and $B=\langle \alpha R_g, \alpha L_{g^{-1}}, \alpha\rangle \in BS(Q,\circ)$ is an autotopism of $BS(Q,\circ)$. From the Lemma 2.2, $A=(IP_x^{-1}, IP_x, IP_xL_x)$ is an autotopism of $(Q,\cdot)$. And $B=\langle \alpha R_g, \alpha L_{g^{-1}}, \alpha\rangle$, from Proposition 1.2 imply that $B=\langle \alpha, I\alpha L_{-g}I, \alpha R_g\rangle$ is an autotopism of $BS(Q,\circ)$. Here, let's consider $AB=\langle IP_x^{-1}\alpha, IP_xI\alpha L_{g^{-1}}I, IP_xL_x\alpha R_g\rangle$ which is also an autotopism of Q. Using AB, we have $(a)IP_x^{-1}\alpha\cdot(b)IP_xI\alpha L_{g^{-1}}I=(ab)IP_xL_x\alpha R_g$ for all $a,b\in Q$, and for some $x\in Q$, this imply that $(a)IP_x^{-1}\alpha\cdot(b^{-1}\setminus x)I\alpha L_g^{-1}I=(ab)IP_xL_x\alpha R_g$. If $b=e$, we have $(a)IP_x^{-1}\alpha\cdot(x\cdot x^{-1})\alpha L_g=(a)IP_xL_x\alpha R_g$, this imply that $(a)IP_x^{-1}\alpha\cdot g=(a)IP_xL_x\alpha R_g$ this follow from last equality $(a)IP_x^{-1}\alpha R_g=(a)IP_xL_x\alpha R_g$, hence, $IP_x^{-1}\alpha R_g=IP_xL_x\alpha R_g$; $AC=\langle IP_x^{-1}\alpha, IP_x^{-1}\alpha R_g, IP_x^{-1}\alpha R_g\rangle$ is an autotopism of Q for all $x\in Q$. Therefor, $IP_x^{-1}\alpha$ is a right pseudo-automorphism with companion g. ☐

Lemma 3.3. *Let (Q,o) be a middle Bol loop and $(Q,\cdot)$ be the corresponding left Bol loop. If (α,β,i) is an autotopism of the middle Bol loop $(Q,\circ)$, then the following equalities hold*

(i). $P_z^{(\circ)}=\beta L_z^{(\cdot)}\alpha^{-1}$.

(ii). $P_z^{(\circ)}=\beta R_{Iz}^{(\cdot)-1}I\alpha^{-1}$.

(iii). $L_z^{(\cdot)}=R_{Iz}^{(\cdot)-1}I$.

(iv). $PM(Q,\circ)\triangleright RM(Q,\cdot$.

(v). $PM(Q,\circ)\triangleright LM(Q,\cdot)$.

Proof. (i). Suppose $(\alpha,\beta,i)\in Atp(Q,\circ)$, then

$$\alpha x\circ\beta y=x\circ y=z \tag{1}$$

for any $x,y\in Q$ and a fixed element $z\in Q$. $x\circ y=z\Rightarrow x/Iy=z$ where $/$ is a right division in the left Bol loop $(Q,\cdot)$ this follow from last equality $x=z\cdot Iy\Rightarrow z\setminus x=Iy\Rightarrow$

$$L_z^{(\cdot)-1}x=Iy \tag{2}$$

From (1), $\alpha x\circ\beta y=z$ for any fixed element $z\in Q$, $\alpha x\setminus^{\circ} z=\beta y\Rightarrow P_z^{(\circ)}\alpha x=\beta y\Rightarrow$

$$y=\beta^{-1}P_z^{(\circ)}\alpha x \tag{3}$$

Using (2) and (3), we have $L_z^{(\cdot)-1}=I\beta^{-1}P_z^{(\circ)}\alpha\Rightarrow\beta^{-1}P_z^{(\circ)}\alpha=IL_z^{(\cdot)-1}=L_z^{(\cdot)}\Rightarrow P_z^{(\circ)}=\beta L_z^{(\cdot)}\alpha^{-1}$.

(ii). Also, recall the equality above $x=z\cdot Iy\Rightarrow Ix=y\cdot Iz\Rightarrow Ix/Iz=y\Rightarrow$

$$R_{Iz}^{(\cdot)-1}Ix=y \tag{4}$$

Using equality (3) and (4), we have $\beta^{-1}P_z^{(\circ)}\alpha=R_{Iz}^{(\cdot)-1}I\Rightarrow P_z^{(\circ)}=\beta R_{Iz}^{(\cdot)-1}I\alpha^{-1}$, $P_z^{(\circ)}=\beta R_{Iz}^{(\cdot)-1}I\alpha^{-1}$.

(iii). From (i) and(ii) we have $L_z^{(\cdot)}=R_z^{(\cdot)-1}I$.

(iv). Using (3) and (4), $P_z^{(\circ)}\alpha x=\beta R_{Iz}^{(\cdot)^{-1}}Ix$. Here, if we set $Ix=x$, we obtain $P_z^{(\circ)}\alpha=\beta R_{Iz}^{(\cdot)^{-1}}\Rightarrow\beta^{-1}P_z^{(\circ)}\alpha=R_{Iz}^{(\cdot)^{-1}}\Rightarrow$

$$R_z^{(\cdot)}=\beta P_{Iz}^{(\circ)-1}\alpha^{-1} \tag{5}$$

Setting $\alpha = \beta$ in the last equality, we obtain

$$R_z^{(\cdot)} = \alpha P_{Iz}^{(\circ)-1}\alpha^{-1} \Rightarrow R_z^{(\cdot)-1} = \alpha^{-1}P_{Iz}^{(\circ)}\alpha \tag{6}$$

Also from equality (1), $x \circ y = z$ this imply that $x \setminus^{\circ} z = y$ that is

$$P_z^{(\circ)}x = y \tag{7}$$

Using also (3) and (7) gives $P_z^{(\circ)}\alpha x = \alpha P_z^{(\circ)}x \Rightarrow P_z^{(\circ)} = \alpha P_z^{(\circ)}\alpha^{-1}$,

$$P_z^{(\circ)} = \alpha P_z^{(\circ)}\alpha^{-1} \Rightarrow P_z^{(\circ)-1} = \alpha^{-1}P_z^{(\circ)-1}\alpha \tag{8}$$

Let $e \in Q$ be the unit element of the middle Bol loop $(Q, \circ)$ and any fixed element $z \in Q$, then $z = z \circ e = \alpha(z) \circ \beta e \Rightarrow z = \alpha(z)/I\beta(e) \Rightarrow \alpha(z) = z \cdot I\beta(e) \Rightarrow \alpha(z)/I\beta(e) = z \Rightarrow R_{I\alpha(e)}^{(\cdot)-1}\alpha(z) = z$ thus

$$\alpha = R_{I\alpha(e)}^{(\cdot)} \tag{9}$$

Since we set $\alpha = \beta$. Here, equality (6) become $R_z^{(\cdot)} = R_{I\alpha(e)}^{(\cdot)}P_{Iz}^{(\circ)-1}R_{I\alpha(e)}^{(\cdot)-1}$. Now, for any fixed element $z \in Q$, using (6) and (8), we want to show that for every $R_z^{(\cdot)} \in RM(Q, \cdot)$ and $P_z^{(\circ)} \in PM(Q, \circ)$, we have $R_z^{(\cdot)}P_z^{(\circ)}R_z^{(\cdot)-1} \in PM(Q, \circ)$:

$$R_z^{(\cdot)}P_z^{(\circ)}R_z^{(\cdot)-1} = \alpha P_{Iz}^{(\circ)-1}\alpha^{-1}P_z^{(\circ)}\alpha^{-1}P_{Iz}^{(\circ)}\alpha = R_{I\alpha(e)}^{(\cdot)}P_{Iz}^{(\circ)-1}R_{I\alpha(e)}^{(\cdot)-1}P_z^{(\circ)}R_{I\alpha(e)}^{(\cdot)-1}P_{Iz}^{(\circ)}R_{I\alpha(e)}^{(\cdot)} = P_{Iz}^{(\circ)-1}P_z^{(\circ)}P_{Iz}^{(\circ)} \in PM(Q, \circ)$$

and using (6) and (8), we also show that for every $R_z^{(\cdot)} \in RM(Q, \cdot)$ and $P_z^{(\circ)} \in PM(Q, \circ)$ we have $R_z^{(\cdot)-1}P_z^{(\circ)}R_z^{(\cdot)} \in PM(Q, \circ)$:

$$R_z^{(\cdot)-1}P_z^{(\circ)}R_z^{(\cdot)} = \alpha^{-1}P_{Iz}^{(\circ)}\alpha P_z^{(\circ)}\alpha P_{Iz}^{(\circ)-1}\alpha^{-1} = R_{I\alpha(e)}^{(\cdot)-1}P_{Iz}^{(\circ)}R_{I\alpha(e)}^{(\cdot)}P_z^{(\circ)}R_{I\alpha(e)}^{(\cdot)}P_{Iz}^{(\circ)-1}R_{I\alpha(e)}^{(\cdot)-1} = P_{Iz}^{(\circ)}P_z^{(\circ)}P_{Iz}^{(\circ)-1} \in PM(Q, \circ)$$

Also as

$$R_z^{(\cdot)}P_z^{(\circ)-1}R_z^{(\cdot)-1} = (R_z^{(\cdot)-1}P_z^{(\circ)}R_z^{(\cdot)})^{-1} = (P_{Iz}^{(\circ)}P_z^{(\circ)}P_{Iz}^{(\circ)-1})^{-1} \in PM(Q, \circ)$$

and

$$R_z^{(\cdot)-1}P_z^{(\circ)-1}R_z^{(\cdot)} = (R_z^{(\cdot)}P_z^{(\circ)}R_z^{(\cdot)-1})^{-1} = (P_{Iz}^{(\circ)-1}P_z^{(\circ)}P_{Iz}^{(\circ)})^{-1} \in PM(Q, \circ).$$

Here, we obtained that $\phi P_z^{(\circ)}\phi^{-1}, \phi^{-1}P_z^{(\circ)}\phi, \phi P_z^{(\circ)-1}\phi^{-1}, \phi^{-1}P_z^{(\circ)-1}\phi \in PM(Q, \circ)$ for each $\phi \in RM(Q, \circ)$ we have show that $RM(Q, \cdot) \triangleleft PM(Q, \circ)$.

(v). Here, using equalities (2) and (3) with setting $Iy = y$ we obtain $L_z^{(\cdot)-1} = \beta^{-1}P_z^{(\circ)}\alpha \Rightarrow L_z^{(\cdot)-1} = \beta^{-1}P_z^{(\circ)}\alpha$ since $\alpha = \beta$, gives

$$L_z^{(\cdot)} = \alpha P_z^{(\circ)-1}\alpha^{-1} \tag{10}$$

Let $e \in Q$ be the unit element of the middle Bol loop $(Q, \circ)$ and any fixed element $z \in Q$, then $z = e \circ z = \alpha(e) \circ \beta z \Rightarrow z = \alpha(e)/I\beta(z) \Rightarrow \alpha(e) = z \cdot I\beta(z) \Rightarrow \alpha(e) \setminus z = I\beta(z) \Rightarrow L_{\alpha(e)}^{(\cdot)-1}z = I\beta(z)$ thus

$$\alpha = L_{\alpha(e)}^{(\cdot)} \tag{11}$$

Since we set $\alpha = \beta$. Here, equality (10) become $L_z^{(\cdot)} = L_{\alpha(e)}^{(\cdot)} P_z^{(\circ)-1} L_{\alpha(e)}^{(\cdot)-1}$. Now, for any fixed element $z \in Q$, using (8) and (10), we want to show that for every $L_z^{(\cdot)} \in LM(Q, \cdot)$ and $P_z^{(\circ)} \in PM(Q, \circ)$,we have $L_z^{(\cdot)} P_z^{(\circ)} L_z^{(\cdot)-1} \in PM(Q, \circ)$:

$$L_z^{(\cdot)} P_z^{(\circ)} L_z^{(\cdot)-1} = \alpha P_z^{(\circ)-1} \alpha^{-1} P_z^{(\circ)} \alpha^{-1} P_z^{(\circ)} \alpha = L_{\alpha(e)}^{(\cdot)} P_z^{(\circ)-1} L_{\alpha(e)}^{(\cdot)-1} P_z^{(\circ)} L_{\alpha(e)}^{(\cdot)-1} P_z^{(\circ)} L_{\alpha(e)}^{(\cdot)} = P_z^{(\circ)-1} P_z^{(\circ)} P_z^{(\circ)} \in PM(Q, \circ).$$

Also, for any fixed element $z \in Q$, using (8) and (10), we want to show that for every $L_z^{(\cdot)} \in LM(Q, \cdot)$ and $P_z^{(\circ)} \in PM(Q, \circ)$, we have $L_z^{(\cdot)-1} P_z^{(\circ)} L_z^{(\cdot)} \in PM(Q, \circ)$:

$$L_z^{(\cdot)-1} P_z^{(\circ)} L_z^{(\cdot)} = \alpha^{-1} P_z^{(\circ)} \alpha P_z^{(\circ)} \alpha P_z^{(\circ)-1} \alpha^{-1} = L_{\alpha(e)}^{(\cdot)-1} P_z^{(\circ)} L_{\alpha(e)}^{(\cdot)} P_z^{(\circ)} L_{\alpha(e)}^{(\cdot)} P_z^{(\circ)-1} L_{\alpha(e)}^{(\cdot)-1} = P_z^{(\circ)} P_z^{(\circ)} P_z^{(\circ)-1} \in PM(Q, \circ).$$

Also as

$$L_z^{(\cdot)} P_z^{(\circ)-1} L_z^{(\cdot)-1} = (L_z^{(\cdot)-1} P_z^{(\circ)} L_z^{(\cdot)})^{-1} = (P_z^{(\circ)} P_z^{(\circ)} P_z^{(\circ)-1})^{-1} \in PM(Q, \circ)$$

and

$$L_z^{(\cdot)-1} P_z^{(\circ)-1} L_z^{(\cdot)} = (L_z^{(\cdot)} P_z^{(\circ)} L_z^{(\cdot)-1})^{-1} = (P_z^{(\circ)-1} P_z^{(\circ)} P_z^{(\circ)})^{-1} \in PM(Q, \circ).$$

Here, we obtained that $\Phi P_z^{(\circ)} \Phi^{-1}, \Phi^{-1} P_z^{(\circ)} \Phi, \Phi P_z^{(\circ)-1} \Phi^{-1}, \Phi^{-1} P_z^{(\circ)-1} \Phi \in PM(Q, \circ)$ for each $\phi \in LM(Q, \circ)$ we have show that $LM(Q, \cdot) \lhd PM(Q, \circ)$. □

Corollary 3.4. *Let (Q, o) be a middle Bol loop and $(Q, .)$ be the corresponding left Bol loop. If (α, β, i) is an autotopism of the middle Bol loop (Q, o), then $PM(Q, \circ) = RM(Q, .)$.*

Proof. Suppose that $(\alpha, \beta, i) \in Aut(Q, \circ)$. recalling the equality (5) and (6) in Proposition 1, $R_z^{(\cdot)} = \alpha P_{Iz}^{(\circ)-1} \alpha^{-1} \in RM(Q, \cdot)$, this imply that $PM(Q, \circ) \subseteq RM(Q, \cdot)$. Also, using (5) and (6), we have $P_z^{(\circ)} = \alpha R_{Iz}^{(\cdot)^{-1}} \alpha^{-1} \in PM(Q, \circ)$ this imply that $RM(Q, \cdot) \subseteq PM(Q, \circ)$, so $PM(Q, \cdot) = PM(Q, \circ)$. □

Corollary 3.5. *Let (Q, o) be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If (α, β, i) is an autotopism of the middle Bol loop (Q, o), then $PM(Q, \circ) = LM(Q, .)$.*

Proof. Suppose that $(\alpha, \beta, i) \in Aut(Q, \circ)$. Using this equality from Proposition 1, $P_z^{(\circ)} = \beta L_z^{(\cdot)} \alpha^{-1} \in PM(Q, \circ)$ and using (10) with Corollary 3.1, gives the desire result. □

Corollary 3.6. *Let (Q, o) be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If (α, β, i) is an autotopism of the middle Bol loop $(Q, \circ)$, then $(Q, \cdot)$ is an indexed two.*

Proof. Using Lemma 3.1, $L_z^{(\cdot)} = R_{Iz}^{(\cdot)-1} I \Rightarrow L_z^{(\cdot)} a = R_{Iz}^{(\cdot)-1} Ia$ for any $a \in Q$ this follow the last equality $z \cdot a = Ia/Iz \Rightarrow Ia = (z \cdot a) \cdot Iz$ setting $a = y \cdot x$ gives $I(y \cdot x) = (z \cdot yx) \cdot Iz \Leftrightarrow I(y \cdot x) = (z \cdot (y \cdot x)) \cdot Iz$. setting $z = 1$ gives the desire result. □

Proposition 3.7. *Let $(Q, \circ)$ be a middle Bol loop and $(Q, .)$ be the corresponding left Bol loop. If Φ is an automorphism of the middle Bol loop $(Q, \circ)$, then, the following equalities hold:*

(i). $P_z^{(\circ)} = \Phi L_{\Phi^{-1} z}^{(\cdot)} \Phi^{-1}$.

(ii). $P_z^{(\circ)} = \Phi R_{\Phi z}^{(\cdot)-1} I \Phi^{-1}$.

(iii). $L_z^{(\cdot)} = R_{\Phi z}^{(\cdot)-1} I$.

(iv). $PM(Q, \circ) \rhd RM(Q, \cdot$.

(v). $PM(Q,\circ) \triangleright LM(Q,\cdot)$.

Proof. (i). Suppose Φ is an automorphism of the middle Bol loop, then $\Phi x \circ \Phi y = \Phi(x \circ y)$ for all $x, y \in Q$. Let $\Phi x \circ \Phi y = \Phi(x \circ y) = z$ for any fixed element $z \in Q$. This follow from last equality $\Phi x \circ \Phi y = z \Rightarrow \Phi x \setminus z = \Phi y$ this imply that

$$P_z^{(\circ)} \Phi x = \Phi y \tag{12}$$

. Consider the equality above $\Phi(x \circ y) = z$ for any fixed element $z \in Q$, we have $x \circ y = \Phi^{-1} z \Rightarrow x/Iy = \Phi^{-1} z \Rightarrow x = \Phi^{-1} z \cdot Iy$, where $/$ is a right division in $(Q,\cdot)$, this follow $\Phi^{-1} z \setminus x = Iy \Rightarrow L_{\Phi^{-1}z}^{(\cdot)-1} x = Iy \Rightarrow y = L_{\Phi^{-1}z}^{(\cdot)}$ and using (12) we have $P_z^{(\circ)} \Phi x = \Phi L_{\Phi^{-1}z}^{(\cdot)} x \Leftrightarrow P_z^{(\circ)} = \Phi L_{\Phi^{-1}z}^{(\cdot)} \Phi^{-1}$.

(ii). Consider the equality above $\Phi(x \circ y) = z$ for any fixed element $z \in Q$, we have $x \circ y = \Phi^{-1} z \Rightarrow x/Iy = \Phi^{-1} z \Rightarrow x = \Phi^{-1} z \cdot Iy \Leftrightarrow Ix = y \cdot \Phi z$, where $/$ is a right division in $(Q,\cdot)$, this last equality imply $Ix/\Phi z = y \Rightarrow R_{\Phi z}^{(\cdot)-1} Ix = y$, using (12) with the last equality gives $\Phi R_{\Phi z}^{(\cdot)-1} Ix = P_z^{(\circ)} \Phi x$, hence $P_z^{(\circ)} = \Phi R_{\Phi z}^{(\cdot)-1} I \Phi^{-1}$.

(iii). Using (i) and (ii) the proof is obvious.

(iv). Consider the equality in (ii), $\Phi R_{\Phi z}^{(\cdot)-1} Ix = P_z^{(\circ)} \Phi x$, setting $x = Ix$ will give us

$$\Phi R_{\Phi z}^{(\cdot)-1} = P_z^{(\circ)} \Phi \Leftrightarrow R_{\Phi z}^{(\cdot)-1} = \Phi^{-1} P_z^{(\circ)} \Phi \Leftrightarrow R_{\Phi z}^{(\cdot)} = \Phi P_z^{(\circ)-1} \Phi^{-1} \tag{13}$$

Setting $z = \Phi^{-1} z$, (13) becomes $R_z^{(\cdot)} = \Phi P_{\Phi^{-1}z}^{(\circ)-1} \Phi^{-1}$. Also $\Phi(x \circ y) = z$ for any fixed $z \in Q$ this follow the last equality $x \circ y = \Phi^{-1} z \Leftrightarrow x \setminus \Phi^{-1} z = y$,

$$P_{\Phi^{-1}z}^{(\circ)} = y \tag{14}$$

Using (12) and (14), $P_z^{(\circ)} \Phi(x) = \Phi P_{\Phi^{-1}z}^{(\circ)}(x) \Leftrightarrow P_z^{(\circ)} = \Phi P_{\Phi^{-1}z}^{(\circ)} \Phi^{-1} \Leftrightarrow$

$$P_z^{(\circ)-1} = \Phi^{-1} P_{\Phi^{-1}z}^{(\circ)-1} \Phi \tag{15}$$

Now, for any fixed element $z \in Q$, using (13) and (15), we want to show that for every $R_z^{(\cdot)} \in RM(Q,\cdot)$ and $P_z^{(\circ)} \in PM(Q,\circ)$, we have $R_z^{(\cdot)} P_z^{(\circ)} R_z^{(\cdot)-1} \in PM(Q,\circ)$:

$$R_z^{(\cdot)} P_z^{(\circ)} R_z^{(\cdot)-1} = \Phi P_{\Phi^{-1}(z)}^{(\circ)^{-1}} \Phi^{-1} P_z^{(\circ)} \Phi^{-1} P_{\Phi^{-1}(z)}^{(\circ)} \Phi = P_z^{(\circ)-1} P_z^{(\circ)} P_z^{(\circ)} \in PM(Q,\circ)$$

and

$$R_z^{(\cdot)^{-1}} P_z^{(\circ)} R_z^{(\cdot)} = \Phi^{-1} P_{\Phi^{-1}(z)}^{(\circ)} \Phi P_z^{(\circ)} \Phi P_{\Phi^{-1}(z)}^{(\circ)-1} \Phi^{-1} = P_z^{(\circ)} P_z^{(\circ)} P_z^{(\circ)-1} \in PM(Q,\circ)$$

So as

$$R_z^{(\cdot)} P_z^{(\circ)^{-1}} R_z^{(\cdot)^{-1}} = (R_{\Phi^{-1}z}^{(\cdot)} P_z^{(\circ)} R_{\Phi^{-1}z}^{(\cdot)-1})^{-1} = (P_z^{(\circ)^{-1}} P_z^{(\circ)} P_z^{(\circ)})^{-1} \in PM(Q,\circ)$$

and

$$R_z^{(\cdot)-1} P_z^{(\circ)^{-1}} R_z^{(\cdot)} = (R_{\Phi^{-1}z}^{(\cdot)-1} P_z^{(\circ)} R_{\Phi^{-1}z}^{(\cdot)})^{-1} = (P_z^{(\circ)} P_z^{(\circ)} P_z^{(\circ)-1})^{-1} \in PM(Q,\circ)$$

Now, we have obtained $\phi P_z^{(\circ)} \phi^{-1}, \phi^{-1} P_z^{(\circ)} \phi, \Phi P_z^{(\circ)-1} \phi^{-1}, \phi^{-1} P_z^{(\circ)-1} \phi \in PM(Q,\circ)$ for each $\phi \in RM(Q,\cdot)$ we have show that $RM(Q,\cdot) \triangleleft PM(Q,\circ)$.

(v). Consider the equality above $L^{(\cdot)-1}_{\Phi^{-1}z}x = Iy$, setting $Iy = y$ and using (12), gives

$$P^{(\circ)}_z = \Phi L^{(\cdot)-1}_{\Phi^{-1}z}\Phi^{-1} \Leftrightarrow \Phi^{-1}P^{(\circ)-1}_{\Phi z}\Phi = L^{(\cdot)}_z \Leftrightarrow L^{(\cdot)-1}_z = \Phi^{-1}P^{(\circ)}_{\Phi z}\Phi \tag{16}$$

Now, for any fixed element $z \in Q$, using (15) and (16), we want to show that for every $L^{(\cdot)}_z \in LM(Q,\cdot)$ and $P^{(\circ)}_z \in PM(Q,\circ)$, we have $L^{(\cdot)}_z P^{(\circ)}_z L^{(\cdot)-1}_z \in PM(Q,\circ)$:

$$L^{(\cdot)}_z P^{(\circ)}_z L^{(\cdot)-1}_z = \Phi^{-1}P^{(\circ)-1}_{\Phi z}\Phi P^{(\circ)}_z \Phi^{-1}P^{(\circ)}_{\Phi z}\Phi = P^{(\circ)-1}_z P^{(\circ)}_z P^{(\circ)}_z \in PM(Q,\circ)$$

and for every $L^{(\cdot)}_z \in LM(Q,\cdot)$ and $P^{(\circ)}_z \in PM(Q,\circ)$, we have $L^{(\cdot)-1}_z P^{(\circ)}_z L^{(\cdot)}_z \in PM(Q,\circ)$:

$$L^{(\cdot)-1}_z P^{(\circ)}_z L^{(\cdot)}_z = \Phi^{-1}P^{(\circ)}_{\Phi z}\Phi P^{(\circ)}_z \Phi^{-1}P^{(\circ)-1}_{\Phi z}\Phi = P^{(\circ)}_z P^{(\circ)}_z P^{(\circ)-1}_z \in PM(Q,\circ).$$

So as

$$L^{(\cdot)-1}_z P^{(\circ)-1}_z L^{(\cdot)}_z = (L^{(\cdot)}_z P^{(\circ)}_z L^{(\cdot)-1}_z)^{-1} = (P^{(\circ)-1}_z P^{(\circ)}_z P^{(\circ)}_z)^{-1} \in PM(Q,\circ)$$

and

$$L^{(\cdot)}_z P^{(\circ)-1}_z L^{(\cdot)-1}_z = (L^{(\cdot)-1}_z P^{(\circ)}_z L^{(\cdot)}_z)^{-1} = (P^{(\circ)}_z P^{(\circ)}_z P^{(\circ)-1}_z)^{-1} \in PM(Q,\circ)$$

Here, we have obtained $\phi P^{(\circ)}_z\phi^{-1}, \phi^{-1}P^{(\circ)}_z\phi, \Phi P^{(\circ)-1}_z\phi^{-1}, \phi^{-1}P^{(\circ)-1}_z\phi \in PM(Q,\circ)$ for each $\phi \in LM(Q,\cdot)$, we have show that $LM(Q,\cdot) \lhd PM(Q,\circ)$. □

Corollary 3.8. *Let $(Q,\circ)$ be a middle Bol loop and $(Q,\cdot)$ be the corresponding left Bol loop. If Φ is an automorphism of the middle Bol loop $(Q,\circ)$, then,$PM(Q,\circ) = RM(Q,\cdot)$.*

Proof. Using equality in Proposition 3.7, $P^{(\circ)}_z = \Phi R^{(\cdot)-1}_{\Phi z} I\Phi^{-1} \subseteq PM(Q,\circ)$ and using the equality (13), $R^{(\cdot)}_z = \Phi P^{(\circ)-1}_{\Phi^{-1}z}\Phi^{-1} \subseteq RM(Q,\cdot)$. The two equalities imply that $PM(Q,\circ) = RM(Q,\cdot)$. □

Corollary 3.9. *Let $(Q,\circ)$ be a middle Bol loop and $(Q,\cdot)$ be the corresponding left Bol loop. If Φ is an automorphism of the middle Bol loop $(Q,\circ)$, then, $PM(Q,\circ) = LM(Q,\cdot)$.*

Proof. Using equality (16) in Proposition 3.7, the result is obvious. □

Corollary 3.10. *Let $(Q,\circ)$ be a middle Bol loop and $(Q,\cdot)$ be the corresponding left Bol loop. If Φ is an automorphism of the middle Bol loop $(Q,\circ)$, then $(Q,\cdot)$ is an indexed two.*

Proof. Using Proposition 3.7, with this equality $L^{(\cdot)}_z = R^{(\cdot)-1}_{\Phi z} I$, the proof is simple. □

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