On Multiplication Groups of Middle Bol Loop Related to Left Bol Loop

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Abstract: In this paper, we show that an element of Bryant-Schneider group of middle Bol loop \((Q, \circ)\) is an automorphism and pseudo-automorphism. We proved that multiplication groups of middle autotopy nuclei of middle Bol loop \((Q, \circ)\) coincided with the corresponding left Bol loop \((Q, \circ)\) and their multiplication groups were show to be normal subgroups. It was found that the corresponding left Bol loop \((Q, \circ)\) is an indexed two.

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1. Introduction

A non-empty set \(Q\) with binary operation \(\cdot\) is called a groupoid \((Q, \cdot)\). Let \((Q, \cdot)\) be a groupoid and \(a\) be fixed element in \(Q\) then the translation maps \(L_a\) and \(R_a\) are defined by \(xL_a = ax\) and \(xR_a = xa\) for all \(x \in Q\). A groupoid \((Q, \cdot)\) is called quasigroup \((Q, \cdot)\) if the maps \(L(a) : G \to G\) and \(R(a) : G \to G\) are bijections for all \(a \in Q\) and if the equations \(ax = b\) and \(ya = b\) have respectively unique solutions \(x = a\backslash b\) and \(y = b/a\) for all \(a, b \in Q\). A quasigroup \((Q, \cdot)\) is called a loop if \(a \cdot 1 = a = 1 \cdot a\), for all \(a \in Q\). The group generated by these mappings are called multiplication group \(M\! Lp(Q, \cdot)\). We donate the groups generated by left (right and middle translations) of a quasigroup \((Q, \cdot)\) by \(LM(Q, \cdot), RM(Q, \cdot)\) and \(PM(Q, \cdot)\) respectively [4]. A loop \((Q, \cdot)\) is called a middle Bol loop if every isotope of \((Q, \cdot)\) satisfies the identity \((x \cdot y)^{-1} = y^{-1} \cdot x^{-1}\), (that is if the anti-automorphic inverse property is universal in \((Q, \cdot)\) [3]. A loop \((Q, \cdot)\) is called a middle Bol loop if it satisfies the identity \(x(yz \backslash x) = (x/z)(y \backslash x)\). In [5] Gvaramiya proved that a loop \((Q, \circ)\) is middle Bol if there exist a right Bol loop \((Q, \cdot)\) such that \(x \circ y = (y \cdot xy^{-1})y\). This imply that if \((Q, \circ)\) is a middle Bol loop and \((Q, \cdot)\) is the corresponding right Bol loop then \(x \circ y = x^{-1} \backslash y\) and \(x \cdot y = y / x^{-1}\), where \('\backslash'\) and \('/'\) are the right division in \((Q, \cdot)\). If \((Q, \circ)\) is a middle Bol loop and \((Q, \cdot)\) is the corresponding left Bol loop then \(x \circ y = x/y^{-1}\) and \(x \cdot y = x/y^{-1}\), where \('\backslash'\) and \('/'\) is the left right division in \((Q, \circ)\) (respectively, in \((Q, \cdot)\)). Let \((Q, \cdot)\) be a loop and the set of autotopisms are defined as follow:

- \((\alpha, i, \gamma)\) of \((Q, \cdot)\) is called left autotopy nucleus (left A-nucleus).
- \((i, \beta, \gamma)\) of \((Q, \cdot)\) is called right autotopy nucleus (right A-nucleus).

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• \((\alpha, \beta, i)\) of \((Q, \cdot)\) is called middle autotopy nucleus (middle A-nucleus).

\[
N_\lambda = a \in Q \mid ax \cdot y = a \cdot xy \ \forall \ x, y \in Q
\]

\[
N_\mu = a \in Q \mid x_a \cdot y = x \cdot ay \ \forall \ x, y \in Q
\]

\[
N_\sigma = a \in Q \mid xy \cdot a = x \cdot ya \ \forall \ x, y \in Q
\]

\[
Z = N \cap C = a \cdot x = x \cdot a, \ \text{where} \ i \ \text{is an identity mapping.}
\]

Let \((Q, \cdot)\) be a loop, a bijection \(\alpha \in Q\) is called pseudo-automorphism such that \((\alpha, \alpha R_g, \alpha R_g)\) is an autotopism where \(g\) is a companion in \(Q\). Let \((Q, \cdot)\) be a loop. We donate the following translations \(RM(Q, \cdot) = x R_a \mid a \in Q = (x \cdot a \mid x \in Q)\).

\[
LM(Q, \cdot) = x L_a \mid a \in Q = (a \cdot x \mid x \in Q)
\]

\[
PM(Q, \cdot) = x P_a s \mid a \in Q = (x \cdot s = a \mid x, s \in Q),
\]

where \(L_a, R_a\) and \(P_a\) are permutations of the set \(Q\). Let \((Q, \cdot)\) be a groupoid (quasigroup, loop) and \(\alpha, \beta, \) and \(\gamma\) be three bijections that map \(Q\) onto \(Q\). The triple \(\Phi = (\alpha, \beta, \gamma)\) is called an autotopism of \((Q, \cdot)\) if and only if \(\alpha x \cdot \beta y = \gamma(x \cdot y)\) for all \(x, y \in Q\). If \(\alpha = \beta = \gamma\), then \(\Phi\) is called the automorphism of \((Q, \cdot)\). Let \((Q, \cdot)\) a loop and \(BS(Q, \cdot)\) be the set of all bijections \(\alpha\) of \(Q\) such that \(< \alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha >\) is an autotopism of \(Q\) for all \(f, g \in Q\), then \(BS(Q, \cdot)\) is called Bryant-Schneider group of the loop \((Q, \cdot)\).

2. Preliminaries

Lemma 2.1 ([2]). Let \((Q, \cdot)\) left Bol loop and \((Q, \circ)\) be the corresponding middle Bol loop. Then \((\alpha, \beta, \gamma) \in S^3_Q\) is an autotopism of \((Q, \cdot)\) if and only if \((\gamma, I \beta I, I \alpha I)\) autotopism \((Q, \circ)\).

Lemma 2.2 ([6]). Let \((Q, \cdot)\) be any loop. Then \((Q, \cdot)\) is a middle Bol loop if and only if \((IP_x^{-1}, IP_x, IP_x L_x)\) is an autotopism of \(Q\).

Lemma 2.3 ([6]). Let \((Q, \cdot)\) be a middle Bol loop. Then \((A, B, C) \in Atp(Q, \cdot)\) if and only if \((I CI, I BI, I AI) \in Atp(Q, \cdot)\).

3. Main Results

Theorem 3.1. Let \((Q, \cdot)\) be a symmetric entropy middle Bol loop with \(\alpha \in BS(Q, \cdot)\) such that \(A = (\alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha)\) is an autotopism of \(Q\) for some \(g \in Q\). Then \(\alpha^{-1}\) is an automorphism of \(Q\) if \(Q\) is of exponent two and \(g \in Z\) the center of \(Q\).

Proof. In symmetric entropy of middle Bol loop, \((IP_x^{-1}, IP_x, IP_x L_x) = B = (IL_x, IR_x, IR_x L_x)\) is an autotopism of \(Q\) for all \(x \in Q\). And \(A = (\alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha)\) is an autotopism of \(BS(Q, \cdot)\) for some \(g \in Q\). Consider \(AB = (\alpha R_{g^{-1}}, \alpha L_{g^{-1}}, \alpha)((IL_x, IR_x, IR_x L_x)) = (\alpha R_{g^{-1}} IL_x, \alpha L_{g^{-1}} IR_x, \alpha IR_x L_x)\) is also an autotopism of \(Q\). Here, \(\alpha R_{g^{-1}} IL_x = \alpha^{-1} R_g L_x\). So, \(AB = (\alpha^{-1} R_g L_x, \alpha^{-1} L_q R_x, \alpha^{-1} R_z L_x)\) if we set \(g = x\), we have \(AB = (\alpha^{-1} R_{z_2}, \alpha^{-1} R_{z_2}, \alpha^{-1} R_{z_2}) \in Atp(Q, \cdot)\), since \(x \in Z\) (center of \(Q\)) that is \(L_x = R_x\). And \(R_{z_2} = 1\) (that is of exponent 2) for all \(x \in Q\), this imply that, \(AB = (\alpha^{-1}, \alpha^{-1}, \alpha^{-1}) \in Atp(Q, \cdot)\), now for all \(x \in Q\) is of exponent 2, \(R_{z_2} = L_{z_2} = I\) the identity mapping. Therefore, \(AB = (\alpha^{-1}, \alpha^{-1}, \alpha^{-1})\) is also an autotopism of \(Q\). This means that \(\alpha^{-1}\) is an automorphism.

\[\square\]

Theorem 3.2. Let \((Q, \cdot)\) be a middle Bol loop and \((Q, \circ)\) be the corresponding right Bol loop, if \(\alpha \in BS(Q, \circ)\) and \(B = (\alpha R_g, \alpha L_{g^{-1}}, \alpha) \in Atp(Q, \circ)\) for some \(g \in Q\). Then \(IP_x^{-1} \alpha\) is a pseudo-automorphism with companion \(g\).
Proof. Suppose that \((Q, \cdot)\) is a middle Bol loop and \(B = \langle \alpha R_g, \alpha L_g^{-1}, \alpha \rangle \in BS(Q, \circ)\) is an autotopism of \(BS(Q, \circ)\). From the Lemma 2.2, \(A = (IP_x^{-1}, IP_x, IP_x L_x)\) is an autotopism of \((Q, \cdot)\). And \(B = \langle \alpha R_g, \alpha L_g^{-1}, \alpha \rangle\), from Proposition 1.2 imply that \(B = \langle \alpha, IaL_{g^{-1}}a, \alpha R_g \rangle\) is an autotopism of \(BS(Q, \circ)\). Here, let’s consider \(AB = (IP_x^{-1} \alpha, IP_x IaL_{g^{-1}}a, IP_x L_x \alpha R_g)\) which is also an autotopism of \(Q\). Using \(AB\), we have \((a)IP_x^{-1} \alpha \cdot \langle b \rangle IP_x IaL_{g^{-1}}a = \langle ab \rangle IP_x L_x \alpha R_g\) for all \(a, b \in Q\), and for some \(x \in Q\) this imply that \((a)IP_x^{-1} \alpha \cdot (b^{-1} x)IaL_{g^{-1}}a = \langle ab \rangle IP_x L_x \alpha R_g\). If \(b = e\), we have \((a)IP_x^{-1} \alpha \cdot (x^{-1} \alpha L_g = (a)IP_x L_x \alpha R_g\), this imply that \((a)IP_x^{-1} \alpha \cdot g = (a)IP_x L_x \alpha R_g\) this follow from last equality \((a)IP_x^{-1} \alpha R_g = (a)IP_x L_x \alpha R_g\), hence, \(IP_x^{-1} \alpha R_g = IP_x L_x \alpha R_g\); \(AC = (IP_x^{-1} \alpha, IP_x^{-1} \alpha R_g, IP_x^{-1} \alpha R_g)\) is an autotopism of \(Q\) for all \(x \in Q\). Therefor, \(IP_x^{-1} \alpha\) is a right pseudo-automorphism with companion \(g\).

\(\square\)

Lemma 3.3. Let \((Q, \circ)\) be a middle Bol loop and \((Q, \cdot)\) be the corresponding left Bol loop. If \((\alpha, \beta, i)\) is an autotopism of the middle Bol loop \((Q, \circ)\), then the following equalities hold

(i). \(P_x^{(\circ)} = \beta L_x^{(\circ)} \alpha^{-1}\).

(ii). \(P_x^{(\circ)} = \beta R_x^{(\circ)} I_{\circ}^{-1} \alpha^{-1}\).

(iii). \(L_x^{(\circ)} = R_x^{(\circ)} I_{\circ}^{-1} \cdot I_{\circ}^{-1}\).

(iv). \(PM(Q, \circ) \triangleright RM(Q, \cdot)\).

(v). \(PM(Q, \circ) \triangleright LM(Q, \cdot)\).

Proof. (i). Suppose \((\alpha, \beta, i) \in \text{Atp}(Q, \circ)\), then

\[
\alpha x \circ \beta y = x \circ y = z
\]

for any \(x, y \in Q\) and a fixed element \(z \in Q\). \(x \circ y = z \Rightarrow x/ I y = z\) where / is a right division in the left Bol loop \((Q, \cdot)\) this follow from last equality \(x = z \cdot I y \Rightarrow z \cdot I = y \Rightarrow L_x^{(\circ)} = I y\)

Using (2) and (3), we have \(L_x^{(\circ)} = I \beta^{-1} P_x^{(\circ)} \alpha \Rightarrow \beta^{-1} P_x^{(\circ)} \alpha = I L_x^{(\circ)} = L_x^{(\circ)} = P_x^{(\circ)} = \beta L_x^{(\circ)} \alpha^{-1}\).

(ii). Also, recall the equality above \(x = z \cdot I y \Rightarrow Ix = y \cdot Iz \Rightarrow Ix/Iz = y \Rightarrow R_x^{(\circ)} I x = y\)

Using equality (3) and (4), we have \(\beta^{-1} P_x^{(\circ)} \alpha = R_x^{(\circ)} I \Rightarrow P_x^{(\circ)} = \beta R_x^{(\circ)} I \alpha^{-1}, P_x^{(\circ)} = \beta R_x^{(\circ)} I \alpha^{-1}\).

(iii). From (i) and (ii) we have \(L_x^{(\circ)} = R_x^{(\circ)} I_{\circ}^{-1}\).

(iv). Using (3) and (4), \(P_x^{(\circ)} \alpha x = \beta R_x^{(\circ)} I_{\circ}^{-1} I x\). Here, if we set \(I x = x\), we obtain \(P_x^{(\circ)} \alpha = \beta R_x^{(\circ)} I_{\circ}^{-1} \Rightarrow \beta^{-1} P_x^{(\circ)} \alpha = R_x^{(\circ)} I_{\circ}^{-1} \Rightarrow R_x^{(\circ)} = \beta P_x^{(\circ)} \alpha^{-1}\).
Setting $\alpha = \beta$ in the last equality, we obtain

$$R_z^{(1)} = \alpha P_{z \alpha}^{(o)} \alpha^{-1} \Rightarrow R_z^{(1)-1} = \alpha^{-1} P_{z \alpha}^{(o)} \alpha$$  \hspace{1cm} (6)

Also from equality (1), $x \circ y = z$ this imply that $x \setminus^c z = y$ that is

$$P_z^{(o)} x = y$$  \hspace{1cm} (7)

Using also (3) and (7) gives $P_z^{(o)} x = \alpha P_z^{(o)} x \Rightarrow P_z^{(o)} = \alpha P_z^{(o)} \alpha^{-1}$,

$$P_z^{(o)} = \alpha P_z^{(o)} \alpha^{-1} \Rightarrow P_z^{(o)} \alpha^{-1} = \alpha^{-1} P_z^{(o)} \alpha$$  \hspace{1cm} (8)

Let $e \in Q$ be the unit element of the middle Bol loop $(Q, \circ)$ and any fixed element $z \in Q$, then $z = e \circ e = \alpha (z) \circ \beta e \Rightarrow z = \alpha (z) / I \beta (e)$ \Rightarrow $\alpha (z) = z \cdot I \beta (e) \Rightarrow \alpha (z) / I \beta (e) = z \Rightarrow R_{I \beta (e)}^{(1)-1} (\alpha (z) = z$ thus

$$\alpha = R_{I \beta (e)}^{(1)}$$  \hspace{1cm} (9)

Since we set $\alpha = \beta$. Here, equality (6) become $R_z^{(1)} = R_{I \alpha (e)}^{(1)} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1}$. Now, for any fixed element $z \in Q$, using (6) and (8), we want to show that for every $R_z^{(1)} \in RM(Q, \cdot)$ and $P_z^{(o)} \in PM(Q, \circ)$, we have $R_z^{(1)} P_z^{(o)} R_z^{(1)-1} \in PM(Q, \circ)$:

$$R_z^{(1)} P_z^{(o)} R_z^{(1)-1} = \alpha P_{z \alpha}^{(o)} \alpha^{-1} - P_{z \alpha}^{(o)} \alpha^{-1} - P_{z \alpha}^{(o)} \alpha^{-1} = R_{I \alpha (e)}^{(1)} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} = P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} \in PM(Q, \circ)$$

and using (6) and (8), we also show that for every $R_z^{(1)} \in RM(Q, \cdot)$ and $P_z^{(o)} \in PM(Q, \circ)$ we have $R_z^{(1)} P_z^{(o)} R_z^{(1)-1} \in PM(Q, \circ)$:

$$R_z^{(1)} P_z^{(o)} R_z^{(1)-1} = \alpha^{-1} P_{z \alpha}^{(o)} \alpha P_{z \alpha}^{(o)} \alpha^{-1} = R_{I \alpha (e)}^{(1)} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} P_{z \alpha}^{(o)} R_{I \alpha (e)}^{(1)-1} = P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} \in PM(Q, \circ)$$

Also as

$$R_z^{(1)} P_z^{(o)} R_z^{(1)-1} = (R_z^{(1)} P_z^{(o)} R_z^{(1)-1})^{-1} = (P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} P_{z \alpha}^{(o)})^{-1} \in PM(Q, \circ)$$

and

$$R_z^{(1)-1} P_z^{(o)} R_z^{(1)-1} = (R_z^{(1)-1} P_z^{(o)} R_z^{(1)-1})^{-1} = (P_{z \alpha}^{(o)} P_{z \alpha}^{(o)} P_{z \alpha}^{(o)})^{-1} \in PM(Q, \circ).$$

Here, we obtained that $\phi P_{z \alpha}^{(o)} \phi^{-1}, \phi^{-1} P_{z \alpha}^{(o)} \phi, \phi P_{z \alpha}^{(o)} \phi^{-1}, \phi^{-1} P_{z \alpha}^{(o)} \phi \in PM(Q, \circ)$ for each $\phi \in RM(Q, \circ)$ we have show that $RM(Q, \cdot) \subset PM(Q, \circ)$.

(v). Here, using equalities (2) and (3) with setting $I y = y$ we obtain $L_{z \alpha}^{(1)-1} = \beta^{-1} P_z^{(o)} \alpha \Rightarrow L_{z \alpha}^{(1)-1} = \beta^{-1} P_z^{(o)} \alpha$ since $\alpha = \beta$, gives

$$L_{z \alpha}^{(1)} = \alpha P_{z \alpha}^{(o)} \alpha^{-1}$$  \hspace{1cm} (10)

Let $e \in Q$ be the unit element of the middle Bol loop $(Q, \circ)$ and any fixed element $z \in Q$, then $z = e \circ z = \alpha (e) \circ \beta z \Rightarrow z = \alpha (e) / I \beta (z) \Rightarrow \alpha (e) = z \cdot I \beta (z) \Rightarrow \alpha (e) \setminus z = I \beta (z) \Rightarrow L_{I \beta (e)}^{(1)-1} z = I \beta (z)$ thus

$$\alpha = L_{I \beta (e)}^{(1)}$$  \hspace{1cm} (11)
Since we set $\alpha = \beta$. Here, equality (10) become $L_z^{(1)} = L_{\alpha(z)}^{(1)} P_z^{(0)} L_{\alpha(z)}^{(1)}$. Now, for any fixed element $z \in Q$, using (8) and (10), we want to show that for every $L_z^{(1)} \in LM(Q, \cdot)$ and $P_z^{(0)} \in PM(Q, \circ)$, we have $L_z^{(1)} P_z^{(0)} L_z^{(1)} \in PM(Q, \circ)$:

$$L_z^{(1)} P_z^{(0)} L_z^{(1)} = \alpha P_z^{(0)} \alpha^{-1} P_z^{(1)} \alpha = L_{\alpha(z)}^{(1)} P_{\alpha(z)}^{(0)} L_{\alpha(z)}^{(1)} = P_z^{(0)} P_z^{(0)} P_z^{(0)} \in PM(Q, \circ).$$

Also, for any fixed element $z \in Q$, using (8) and (10), we want to show that for every $L_z^{(1)} \in LM(Q, \cdot)$ and $P_z^{(0)} \in PM(Q, \circ)$, we have $L_z^{(1)} P_z^{(0)} L_z^{(1)} \in PM(Q, \circ)$:

$$L_z^{(1)} P_z^{(0)} L_z^{(1)} = \alpha^{-1} P_z^{(0)} \alpha P_z^{(1)} \alpha^{-1} = L_{\alpha(z)}^{(1)} P_{\alpha(z)}^{(0)} L_{\alpha(z)}^{(1)} = P_z^{(0)} P_z^{(0)} P_z^{(0)} \in PM(Q, \circ).$$

Also as

$$L_z^{(1)} P_z^{(0)} L_z^{(1)} = (L_z^{(1)} P_z^{(0)} L_z^{(1)})^{-1} = (P_z^{(0)} P_z^{(0)} P_z^{(0)})^{-1} \in PM(Q, \circ)$$

and

$$L_z^{(1)} P_z^{(0)} L_z^{(1)} = (L_z^{(1)} P_z^{(0)} L_z^{(1)})^{-1} = (P_z^{(0)} P_z^{(0)} P_z^{(0)})^{-1} \in PM(Q, \circ).$$

Here, we obtained that $\Phi P_z^{(0)} \Phi^{-1} \Phi^{-1} P_z^{(0)} \Phi, \Phi P_z^{(0)} \Phi^{-1} \Phi^{-1} P_z^{(0)} \Phi \in PM(Q, \circ)$ for each $\phi \in LM(Q, \circ)$ we have show that $LM(Q, \cdot) \subset PM(Q, \circ)$.

**Corollary 3.4.** Let $(Q, o)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If $(\alpha, \beta, i)$ is an autotopism of the middle Bol loop $(Q, o)$, then $PM(Q, o) = RM(Q, \cdot)$.

**Proof.** Suppose that $(\alpha, \beta, i) \in Aut(Q, o)$. Recalling the equality (5) and (6) in Proposition 1. $R_z^{(1)} = \alpha P_z^{(0)} \alpha^{-1} \in RM(Q, \cdot)$, this imply that $PM(Q, o) \subseteq RM(Q, \cdot)$. Also, using (5) and (6), we have $P_z^{(0)} = \alpha R_z^{(1)} \alpha^{-1} \in PM(Q, o)$ this imply that $RM(Q, \cdot) \subseteq PM(Q, o)$, so $PM(Q, o) = RM(Q, \cdot)$.

**Corollary 3.5.** Let $(Q, o)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If $(\alpha, \beta, i)$ is an autotopism of the middle Bol loop $(Q, o)$, then $PM(Q, o) = LM(Q, \cdot)$.

**Proof.** Suppose that $(\alpha, \beta, i) \in Aut(Q, o)$. Using this equality from Proposition 1, $P_z^{(0)} = \beta L_z^{(1)} \alpha^{-1} \in PM(Q, o)$ and using (10) with Corollary 3.1, gives the desire result.

**Corollary 3.6.** Let $(Q, o)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If $(\alpha, \beta, i)$ is an autotopism of the middle Bol loop $(Q, o)$, then $(Q, \cdot)$ is an indexed two.

**Proof.** Using Lemma 3.1, $L_z^{(1)} = R_z^{(1)} I \Rightarrow L_z^{(1)} \alpha = R_z^{(1)} I \alpha$ for any $a \in Q$ this follow the last equality $z \cdot a = Ia/Iz \Rightarrow Ia = (z \cdot a) \cdot Iz$ setting $a = y \cdot x$ gives $I(y \cdot x) = (z \cdot yz) \cdot Iz \Rightarrow I(y \cdot x) = (z \cdot (y \cdot x)) \cdot Iz$. setting $z = 1$ gives the desire result.

**Proposition 3.7.** Let $(Q, o)$ be a middle Bol loop and $(Q, \cdot)$ be the corresponding left Bol loop. If $\Phi$ is an automorphism of the middle Bol loop $(Q, o)$, then, the following equalities hold:

(i). $P_z^{(0)} = \Phi L_{\Phi^{-1} z} \Phi^{-1}$.

(ii). $P_z^{(0)} = \Phi R_{\Phi^{-1} z} I \Phi^{-1}$.

(iii). $L_z^{(1)} = R_{\Phi z}^{(1)} I$.

(iv). $PM(Q, o) \supseteq RM(Q, \cdot)$. 
Proof. 

(i). Suppose \( \Phi \) is an automorphism of the middle Bol loop, then \( \Phi x \circ \Phi y = \Phi(x \circ y) \) for all \( x, y \in Q \). Let \( \Phi x \circ \Phi y = \Phi(x \circ y) = z \) for any fixed element \( z \in Q \). This follow from last equality \( \Phi x \circ \Phi y = z \Rightarrow \Phi x \setminus z = \Phi y \) this imply that

\[
P_z^{(i)} \Phi x = \Phi y \tag{12}
\]

. Consider the equality above \( \Phi(x \circ y) = z \) for any fixed element \( z \in Q \), we have \( x \circ y = \Phi^{-1}z \Rightarrow x/Iy = \Phi^{-1}z \Rightarrow x = \Phi^{-1}z \cdot Iy \), where / is a right division in \((Q, \cdot)\), this follow \( \Phi^{-1}z \setminus x = Iy \Rightarrow Iy \Rightarrow y = L^{(i)}_{\Phi^{-1}z} \) and using (12) we have \( P_z^{(i)} \Phi x = \Phi L^{(i)}_{\Phi^{-1}z}, x \Leftrightarrow P_z^{(i)} = \Phi L^{(i)}_{\Phi^{-1}z} \Phi^{-1} \).

(ii). Consider the equality above \( \Phi(x \circ y) = z \) for any fixed element \( z \in Q \), we have \( x \circ y = \Phi^{-1}z \Rightarrow x/Iy = \Phi^{-1}z \Rightarrow x = \Phi^{-1}z \cdot Iy \Rightarrow Ix = y \cdot x \), where / is a right division in \((Q, \cdot)\), this last equality imply \( Ix/\Phi z = y \Rightarrow R_z^{(i)}Ix = y \), using (12) with the last equality give \( \Phi R_z^{(i)}Ix = P_z^{(i)} \Phi x \), hence \( P_z^{(i)} = \Phi R_z^{(i)}\Phi^{-1} \).

(iii). Using (i) and (ii) the proof is obvious.

(iv). Consider the equality in (ii), \( \Phi R_z^{(i)}Ix = P_z^{(i)} \Phi x \), setting \( x = Ix \) will give us

\[
\Phi R_z^{(i)} = P_z^{(i)} \Phi \Leftrightarrow R_z^{(i)} = \Phi^{-1} R_z^{(i)} \Phi \Leftrightarrow R_z^{(i)} = \Phi P_z^{(i)} \Phi^{-1} \tag{13}
\]

Setting \( z = \Phi^{-1}z \), (13) becomes \( R_z^{(i)} = \Phi P_z^{(i)} \Phi^{-1} \). Also \( \Phi(x \circ y) = z \) for any fixed \( z \in Q \) this follow the last equality

\[
x \circ y = \Phi^{-1}z \Leftrightarrow x \setminus y = \Phi \Phi^{-1}z = y,
\]

\[
P_z^{(i)} = \Phi^{-1} \Phi^{-1} \tag{14}
\]

Using (12) and (14), \( P_z^{(i)} \Phi(x) = \Phi P_z^{(i)}(x) \Leftrightarrow P_z^{(i)} = \Phi P_z^{(i)} \Phi^{-1} \Leftrightarrow
\]

\[
P_z^{(i)} = \Phi^{-1} P_z^{(i)} \Phi^{-1} \tag{15}
\]

Now, for any fixed element \( z \in Q \), using (13) and (15), we want to show that for every \( R_z^{(i)} \in RM(Q, \cdot) \) and \( P_z^{(i)} \in PM(Q, \circ) \), we have \( R_z^{(i)} P_z^{(i)} R_z^{(i)} = PM(Q, \circ) \):

\[
R_z^{(i)} P_z^{(i)} R_z^{(i)} = \Phi P_z^{(i)} \Phi^{-1} P_z^{(i)} \Phi^{-1} \Phi \Phi^{-1} P_z^{(i)} P_z^{(i)} \Phi = P_z^{(i)} \Phi P_z^{(i)} \Phi^{-1} P_z^{(i)} P_z^{(i)} \Phi \in PM(Q, \circ)
\]

and

\[
R_z^{(i)} P_z^{(i)} R_z^{(i)} = \Phi P_z^{(i)} \Phi^{-1} P_z^{(i)} \Phi^{-1} \Phi \Phi^{-1} P_z^{(i)} P_z^{(i)} \Phi = P_z^{(i)} \Phi P_z^{(i)} \Phi^{-1} P_z^{(i)} P_z^{(i)} \Phi \in PM(Q, \circ)
\]

So as

\[
R_z^{(i)} P_z^{(i)} R_z^{(i)} = (R_z^{(i)} P_z^{(i)} R_z^{(i)})^{-1} \Phi P_z^{(i)} P_z^{(i)} \Phi = (P_z^{(i)} P_z^{(i)} \Phi) \Phi \in PM(Q, \circ)
\]

and

\[
R_z^{(i)} P_z^{(i)} R_z^{(i)} = (R_z^{(i)} P_z^{(i)} R_z^{(i)})^{-1} \Phi P_z^{(i)} P_z^{(i)} \Phi = (P_z^{(i)} P_z^{(i)} \Phi) \Phi \in PM(Q, \circ)
\]

Now, we have obtained \( \phi P_z^{(i)} \Phi, \phi P_z^{(i)} \Phi, \phi P_z^{(i)} \Phi, \phi P_z^{(i)} \Phi \in PM(Q, \circ) \) for each \( \phi \in RM(Q, \cdot) \) we have show that \( RM(Q, \cdot) \triangleleft PM(Q, \circ) \).
Using equality in Proposition 3.7, the result is obvious. Let

\( L_{\phi^{-1}}^{(1)} = \Phi \Phi_{\Phi^{-1}}^{-1} \subseteq PM(Q, \cdot) \) and using the equality (13), \( L_{z}^{(1)} = \Phi \Phi_{\Phi^{-1}}^{-1} \subseteq RM(Q, \cdot) \). The two equalities imply that \( PM(Q, \cdot) = RM(Q, \cdot) \).

Corollary 3.9. Let \((Q, \circ)\) be a middle Bol loop and \((Q, \cdot)\) be the corresponding left Bol loop. If \( \Phi \) is an automorphism of the middle Bol loop \((Q, \circ)\), then, \( PM(Q, \circ) = LM(Q, \cdot) \).

Proof. Using equality (16) in Proposition 3.7, the result is obvious.

Corollary 3.10. Let \((Q, \circ)\) be a middle Bol loop and \((Q, \cdot)\) be the corresponding left Bol loop. If \( \Phi \) is an automorphism of the middle Bol loop \((Q, \circ)\), then \((Q, \cdot)\) is an indexed two.

Proof. Using Proposition 3.7, with this equality \( L_{z}^{(1)} = R_{\Phi_{z}}^{(1)} \), the proof is simple.

References


(v). Consider the equality above \( L_{\phi^{-1}}^{(1)} = Iy \), setting \( Iy = y \) and using (12), gives

\[
P_z^{(1)} = \Phi \Phi_{\Phi^{-1}}^{-1} \Rightarrow \Phi^{-1} P_{\Phi z}^{(1)} = L_z^{(1)} \Rightarrow L_z^{(1)} = \Phi^{-1} P_{\Phi z}^{(1)} \Phi
\] (16)

Now, for any fixed element \( z \in Q, \) using (15) and (16), we want to show that for every \( L_z^{(1)} \in LM(Q, \cdot) \) and \( P_z^{(1)} \in PM(Q, \circ) \), we have \( L_z^{(1)} P_z^{(1)} L_z^{(1)} = PM(Q, \circ) \):

\[
L_z^{(1)} P_z^{(1)} L_z^{(1)} = \Phi^{-1} P_{\Phi z}^{(1)} \Phi P_{\Phi z}^{(1)} \Phi^{-1} P_{\Phi z}^{(1)} = P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} \in PM(Q, \circ)
\]

and for every \( L_z^{(1)} \in LM(Q, \cdot) \) and \( P_z^{(1)} \in PM(Q, \circ) \), we have \( L_z^{(1)} P_z^{(1)} L_z^{(1)} = PM(Q, \circ) \):

\[
L_z^{(1)} P_z^{(1)} L_z^{(1)} = \Phi^{-1} P_{\Phi z}^{(1)} \Phi P_{\Phi z}^{(1)} \Phi^{-1} P_{\Phi z}^{(1)} = P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} \in PM(Q, \circ).
\]

So as

\[
L_z^{(1)} P_z^{(1)} L_z^{(1)} = (L_z^{(1)} P_z^{(1)} L_z^{(1)})^{-1} = (P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} P_{\Phi z}^{(1)})^{-1} \in PM(Q, \circ)
\]

and

\[
L_z^{(1)} P_z^{(1)} L_z^{(1)} = (L_z^{(1)} P_z^{(1)} L_z^{(1)})^{-1} = (P_{\Phi z}^{(1)} P_{\Phi z}^{(1)} P_{\Phi z}^{(1)})^{-1} \in PM(Q, \circ)
\]

Here, we have obtained \( \phi P_z^{(1)} \phi^{-1}, \phi^{-1} P_z^{(1)} \phi, \phi P_z^{(1)} \phi^{-1}, \phi^{-1} P_z^{(1)} \phi \in PM(Q, \circ) \) for each \( \phi \in LM(Q, \cdot) \), we have show that \( LM(Q, \cdot) \subseteq PM(Q, \circ) \).