Int. J. Math. And Appl., 6(4)(2018), 157–161
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

A Fixed Point Theorem for Kannan-type Maps in Metric Spaces

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Abstract: We prove a generalization of Kannan's fixed point theorem, based on a recent result of Vittorino Pata.

MSC: 47H10; 47H09.

Keywords: Kannan map, fixed point, convergence rate.

Accepted on: 08.06.2018

1. Introduction

Our starting point is Kannan's result in metric fixed point theory [1]. It has been shown that Kannan's theorem is independent of the Banach Fixed Point Theorem [2], and that it also characterizes the metric completeness concept [3].

Definition 1.1. *Let* (X, d) *be a metric space. Let us call* $T : X \to X$ *a* Kannan map *if there exists some* $\lambda \in [0, 1)$ *such that*

$$d(Tx, Ty) \leq \frac{\lambda}{2}\{d(x, Tx) + d(y, Ty)\} \tag{1}$$

for all $x, y \in X$.

For complete metric spaces, Kannan proved the following:

Theorem 1.2 ([1]). *If* (X, d) *is a complete metric space, and if* T *is a Kannan map on* X, *then there exists a unique* $x \in X$ *such that* $Tx = x$.

And Subrahmanyam ([3]) has proved the counterpart by showing that if all the Kannan maps on a metric space have fixed points then that space must necessarily be complete.

2. Generalization of Kannan's Fixed Point Theorem

As in [4], from this point onwards, let (X, d) stand for a complete metric space. Let us select arbitrarily a point $x_0 \in X$, and call it the "zero" of X. We denote

$$\|x\| := d(x, x_0) \forall x \in X.$$

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The first author is indebted to the ***UGC*** *(University Grants Commissions), India for awarding him a* ***JRF*** *(Junior Research Fellowship) during the tenure in which this paper was written.*

Let $\Lambda \geq 0$, $\alpha \geq 1$, and $\beta \in [0,\, \alpha]$ be fixed constants, and let $\psi : [0,\, 1] \to [0,\, \infty)$ denote a preassigned increasing function that vanishes (with continuity) at zero. Then, for a map $T : X \to X$, Pata goes on to show that the following theorem holds.

Theorem 2.1 ([4])**.** *If the inequality*

$$d(Tx,\, Ty) \leq (1-\varepsilon)d(x,\, y) + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)[1 + \|x\| + \|y\|]^{\beta} \tag{2}$$

is satisfied for every $\varepsilon \in [0,\, 1]$ *and every* $x,\, y \in X$*, then* T *possesses a unique fixed point* $x_* = Tx_*$ $(x_* \in X)$*.*

Motivated by this generalization of the Banach contraction principle, we can come up with an analogous generalized form of Kannan's fixed point theorem.

2.1. The Main Theorem

With everything else remaining the same except for a more general $\beta \geq 0$, our goal is to prove the following:

Theorem 2.2. *If the inequality*

$$d(Tx,\, Ty) \leq \frac{1-\varepsilon}{2}\{d(x,\, Tx) + d(y,\, Ty)\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left(\|x\| + \|Tx\| + \|y\| + \|Ty\|\right)^{\beta} \tag{3}$$

is satisfied $\forall\varepsilon \in [0,\, 1]$ *and* $\forall x,\, y \in X$*, then* T *possesses a unique fixed point*

$$x^* = Tx^* \; (x^* \in X).$$

Remark 2.3. *Since we can always redefine* Λ *to keep 3 valid no matter what initial* $x_0 \in X$ *we choose, we are in no way restricting ourselves by taking that "zero" instead of a generic* $x \in X$ *[4].*

2.2. Proofs

Uniqueness of x^*

Proof. If possible, let $\exists x^*,\, y^* \in X$ such that

$$x^* = Tx^*, y^* = Ty^*, \text{ and } x^* \neq y^*.$$

Then 3 implies, $\forall\varepsilon \in [0,\, 1]$,

$$d(x^*,\, y^*) \leq \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left(\|x^*\| + \|Tx^*\| + \|y^*\| + \|Ty^*\|\right)^{\beta}.$$

In particular, $\varepsilon = 0$ gives us $d(x^*,\, y^*) \leq 0 \implies x^* = y^*$, which is a contradiction. □

Existence of x^*

We now bring into play the two sequences

$$x_n = Tx_{n-1} = T^n x_0 \;\text{ and }\; c_n = \|x_n\| \; (n = 1,\, 2,\, 3,\, \cdots).$$

But before we proceed any further, we will need the following:

Lemma 2.4. $\{c_n\}$ *is bounded.*

Proof. From 3, considering again the case of $\varepsilon = 0$, we see that for $n = 1, 2, 3, \cdots$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \frac{1}{2}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} \\ \therefore d(x_{n+1}, x_n) &\leq d(x_n, x_{n-1}) \\ &\vdots \\ &\leq d(x_1, x_0) = c_1. \end{aligned} \tag{4}$$

Now

$$\begin{aligned} c_n &= d(x_n, x_0) \\ &\leq d(x_n, x_1) + d(x_1, x_0) \\ &= d(x_n, x_1) + c_1 \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_1) + c_1 \\ &\leq c_1 + d(x_{n+1}, x_1) + c_1 && [\text{using } 4] \\ &\leq d(Tx_n, Tx_0) + 2c_1 \\ &\leq \frac{1}{2}\{d(x_{n+1}, x_n) + d(x_1, x_0)\} + 2c_1 && [\text{using } 3 \text{ with } \varepsilon = 0] \\ &\leq \frac{1}{2}(c_1 + c_1) + 2c_1 && [4] \\ &= 3c_1 && \forall n \in \mathbb{N}. \end{aligned}$$

And hence the lemma is proved. □

Next we strive to show that:

Lemma 2.5. *$\{x_n\}$ is Cauchy.*

Proof. In light of 3, for $n = 1, 2, 3, \cdots$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \frac{1-\varepsilon}{2}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} + \Lambda\varepsilon^{\alpha}\psi(\varepsilon)\left(\|x_{n+1}\| + \|x_n\| + \|x_n\| + \|x_{n-1}\|\right)^{\beta} \\ &\leq \frac{1-\varepsilon}{2}\{d(x_{n+1}, x_n) + d(x_n, x_{n-1})\} + C\varepsilon^{\alpha}\psi(\varepsilon) \end{aligned}$$

for $C = sup_{j\in\mathbb{N}}\Lambda(4c_j)^{\beta} < \infty$ (on account of Lemma 2.4). But then, $\forall \varepsilon \in (0, 1]$,

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \frac{1-\varepsilon}{1+\varepsilon} d(x_n, x_{n-1}) + \frac{2C\varepsilon^{\alpha}}{1+\varepsilon}\psi(\varepsilon) \\ &\vdots \\ &\leq \cdots \\ &\vdots \\ &\leq \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n} d(x_1, x_0) + \frac{2C\varepsilon^{\alpha}}{1+\varepsilon}\psi(\varepsilon)\left[1 + \frac{1-\varepsilon}{1+\varepsilon} + \cdots + \left(\frac{1-\varepsilon}{1+\varepsilon}\right)^{n-1}\right] \end{aligned}$$

$$\begin{aligned}
&\le k^n d(x_1, x_0) + \frac{2C\varepsilon^\alpha}{1+\varepsilon}\psi(\varepsilon)(1+k+\cdots+k^{n-1}) && [\text{letting } k = \frac{1-\varepsilon}{1+\varepsilon} \ge 0]\\
&\le k^n d(x_1, x_0) + \frac{2C\varepsilon^\alpha}{1+\varepsilon}\psi(\varepsilon)(1+k+\cdots+k^{n-1}+\cdots) && [\because \text{k} \ge 0\,]\\
&\le k^n d(x_1, x_0) + \frac{2C\varepsilon^\alpha}{1+\varepsilon}\psi(\varepsilon)\frac{1}{1-k}\\
&= k^n d(x_1, x_0) + C\varepsilon^{\alpha-1}\psi(\varepsilon) && [\text{putting } k = \frac{1-\varepsilon}{1+\varepsilon}] \qquad (5)
\end{aligned}$$

for all $n \in \mathbb{N}$. At this point we note that if $\varepsilon \in (0, 1]$, then $k < 1$. Therefore, taking progressively lower values of ε that approach zero but never quite reach it, the R.H.S. of 5 can be made as small as one wishes it to be as $n \to \infty$. Indeed, since $C\varepsilon^{\alpha-1}\psi(\varepsilon) \to 0$ as $\varepsilon \to 0+$, for an arbitrary $\eta > 0$, $\exists \varepsilon = \varepsilon(\eta) > 0$ such that $C\varepsilon^{\alpha-1}\psi(\varepsilon) < \frac{\eta}{2}$. Again, for this ε $(= \varepsilon(\eta))$, $\exists N \in \mathbb{N}$ such that $k^n d(x_1, x_0) < \frac{\eta}{2}$ $\forall n \ge N$ because $k^n d(x_1, x_0) \to 0$ as $n \to \infty$. Together that gives us

$$k^n d(x_1, x_0) + C\varepsilon^{\alpha-1}\psi(\varepsilon) < \frac{\eta}{2} + \frac{\eta}{2} = \eta\, \forall n \ge N.$$

In other words,

$$d(x_n, x_{n+1}) \to 0 \,\text{as}\, n \to \infty,\ \varepsilon \to 0+. \qquad (6)$$

Hence, from 3, using the same $C = sup_{j\in\mathbb{N}}\Lambda(4c_j)^\beta$, letting $n \to \infty$ and $\varepsilon \to 0+$,

$$\begin{aligned}
d(x_n, x_{n+p}) &= d(Tx_{n-1}, Tx_{n+p-1})\\
&\le \frac{1-\varepsilon}{2}\{d(x_{n-1}, x_n) + d(x_{n+p-1}, x_{n+p})\} + C\varepsilon^\alpha\psi(\varepsilon)\\
&\to 0 && [\text{using } 6]
\end{aligned}$$

uniformly over $p = 1, 2, \cdots$, which basically assures us that $\{x_n\}$ is Cauchy. ☐

Equipped with Lemma 2.5 and taking into note the completeness of X, we can now safely guarantee the existence of some $x^* \in X$ to which $\{x_n\}$ converges.

Finally, all that remains to show is that:

x^* is a fixed point for T.

Proof. For this we observe that, $\forall n \in \mathbb{N}$,

$$\begin{aligned}
d(Tx^*, x^*) &\le d(Tx^*, x_{n+1}) + d(x_{n+1}, x^*)\\
&= d(Tx^*, Tx_n) + d(x_{n+1}, x^*)\\
&\le \frac{1}{2}\{d(Tx^*, x^*) + d(Tx_n, x_n)\} + d(x_{n+1}, x^*) && [\text{using } 3 \text{ with}\, \varepsilon = 0\, \text{again}]\\
\implies \frac{1}{2}d(Tx^*, x^*) &\le \frac{1}{2}d(x_n, x_{n+1}) + d(x_{n+1}, x^*) && (7)
\end{aligned}$$

As $n \to \infty$ (and $\varepsilon \to 0+$), we know that

$$\begin{aligned}
&d(x_n, x_{n+1}) \to 0 && [\text{from } 6],\\
&d(x_{n+1}, x^*) \to 0 && [\text{as } x_n \to x^*].
\end{aligned}$$

So 7 actually gives us that $d(x^*, Tx^*) \le 0 \implies Tx^* = x^*$, which is the required result. ☐

3. Comparison with Kannan's Original Result

The requirements of Theorem 2.2 are indeed weaker than those of Kannan's theorem. To see that, let us start from 1 with $\lambda \in (0,\, 1)$ (barring the trivial case where $\lambda = 0$).

We have, $\forall \varepsilon \in [0,\, 1]$,

$$
\begin{aligned}
d(Tx,\, Ty) &\le \frac{\lambda}{2}\,\{d(x,\, Tx) + d(y,\, Ty)\} \\
&\le \frac{1-\varepsilon}{2}\,\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda + \varepsilon - 1}{2}\,\{d(x,\, Tx) + d(y,\, Ty)\} \\
&= \frac{1-\varepsilon}{2}\,\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda}{2}\left(1 + \frac{\varepsilon - 1}{\lambda}\right)\{d(x,\, Tx) + d(y,\, Ty)\} \\
&\le \frac{1-\varepsilon}{2}\,\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda}{2}(1 + \overline{\varepsilon - 1})^{\frac{1}{\lambda}}\,\{d(x,\, Tx) + d(y,\, Ty)\} \\
&\text{[using Bernoulli's Inequality, since } \varepsilon - 1 \ge -1 \text{ \& } \frac{1}{\lambda} > 1] \\
&\le \frac{1-\varepsilon}{2}\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda}{2}\varepsilon^{\frac{1}{\lambda}}\,[\{d(x,\, x_0) + d(x_0,\, Tx)\} + \{d(y,\, x_0) + d(x_0,\, Ty)\}] \\
&= \frac{1-\varepsilon}{2}\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda}{2}\varepsilon^{1+\gamma}(\|x\| + \|Tx\| + \|y\| + \|Ty\|) \\
&\text{[taking } 1 < \frac{1}{\lambda} = 1 + \gamma \text{ for some } \gamma > 0] \\
&\le \frac{1-\varepsilon}{2}\{d(x,\, Tx) + d(y,\, Ty)\} + \frac{\lambda}{2}\varepsilon\varepsilon^{\gamma}(\|x\| + \|Tx\| + \|y\| + \|Ty\|). \qquad (8)
\end{aligned}
$$

Then a quick comparison between 3 and 8 with $\psi(\varepsilon) = \varepsilon^{\gamma}$ $(\gamma > 0)$ provides us with what we need.

4. An Estimate of the "Error"

Various ways exist to estimate the "error" in Banach's contraction principle (see, eg [5]). Here, to get an idea about the speed of convergence *a priori*, we note that

$$
\begin{aligned}
d(x_n,\, x^*) &= d(Tx_{n-1},\, Tx^*) \\
&\le \frac{1}{2}d(x_{n-1},\, Tx_{n-1}) && \text{[using 3 with } \varepsilon = 0] \\
&= \frac{1}{2}d(x_n,\, x_{n-1}) \\
&\le \frac{k^{n-1}d(x_1,\, x_0) + C\varepsilon^{\alpha-1}\psi(\varepsilon)}{2} && \text{[using 5].}
\end{aligned}
$$

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