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On Third Order Hankel Determinant for Some Special Class of Analytic Functions Related with Generalized Sakaguchi Functions

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Abstract: In this paper, we investigate the third order Hankel determinant for some special class of analytic functions related with generalized Sakaguchi functions in the open unit disk using subordination.

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1. Introduction

Let $\mathcal{A}$ denote the family of analytic functions in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ is of the form,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of all functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of convex functions in $\mathbb{U}$, if it satisfies the following inequality:

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{U}.$$

Recently Frasin [7] introduced and studied a generalized Sakaguchi type class $\mathcal{S}(\alpha, s, t)$ if it satisfies

$$\Re\left\{\frac{(s-t)zf'(z)}{f(sz) - f(tz)}\right\} > \alpha \tag{2}$$

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for some $0 \leq \alpha < 1$, $s, t \in \mathcal{C}$ with $s \neq t$ and for all z $\in \mathbb{U}$.

We also denote by the subclass $T(\alpha, s, t)$ the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ such that $zf'(z) \in \mathcal{S}(\alpha, s, t)$. The class $\mathcal{S}(\alpha, 1, t)$ was introduced and studied by Owa [15, 16], and the class $\mathcal{S}(\alpha, 1, -1) = \mathcal{S}_s(\alpha)$ was introduced by Sakaguchi [19]. Also we note that $\mathcal{S}(\alpha, 1, 0) \equiv \mathcal{S}^*(\alpha)$ and $T(\alpha, 1, 0) \equiv \mathcal{C}(\alpha)$ which are, respectively, the familiar classes of starlike and convex functions of order $\alpha(0 \leq \alpha < 1)$. With a view to recalling the principal of subordination between analytic functions, let the functions f and g be analytic in $\mathbb{U}$. Then we say that the function f is *subordinate* to g, if there exits a Schwarz function ω, analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that $f(z) = g(\omega(z)), z \in \mathbb{U}$. We denote this subordination by

$$f \prec g \quad or \quad f(z) \prec g(z).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let ϕ be analytic, and let the Maclaurin series of ϕ be given by

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots. \tag{3}$$

where all coefficients are real and $B_1 > 0$. Mathur & Mathur [21] investigated the class $\mathcal{S}_s^*(\phi, s, t)$ as follows,

Definition 1.1. *The function $f \in \mathcal{A}$ is in the class $\mathcal{S}_s^*(\phi, s, t)$ if*

$$\left\{ \frac{(s-t)zf'(z)}{f(sz) - f(tz)} \right\} \prec \phi(z), \qquad s \neq t. \tag{4}$$

and if $\mathcal{C}_s(\phi, s, t)$ denotes the subclasses of $\mathcal{A}$ consisting functions $f(z)$ such that $zf'(z) \in \mathcal{S}_s^*(\phi, s, t)$.

Remark 1.2. *By the suitable choices of s and t,we obtain the following subclasses*

- $\mathcal{S}_s^*(\phi, 1, 0) \equiv \mathcal{S}^*(\phi)$ and $\mathcal{C}_s(\phi, 1, 0) \equiv \mathcal{C}(\phi)$ which is the class introduced and studied by Ma and Minda [13].
- $\mathcal{S}_s^*(\phi, 1, -1) \equiv \mathcal{S}_s^*(\phi)$, which is the class introduced and studied by Shanmugam [20].

For $s = 1, t = 0$ and $\phi(z) = \dfrac{1 + A(z)}{1 + B(z)}$,$(-1 \leq B < A \leq 1)$, the subclass $\mathcal{S}_s^*(\phi, 1, 0)$ reduces to the class $\mathcal{S}^*[A, B]$ studied by Janowski [9].

1.1. Hankal Determinant

The Hankel determinant $H_q(n)$ of Taylor's coefficients of function $f \in \mathcal{A}$ of the form (1), is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \qquad (n, q \in \mathbb{N} = 1, 2, 3 \ldots). \tag{5}$$

The Hankel determinant is useful,in showing that a function of bounded characteristic in $\mathbb{U}$, *i.e.,* a function which is a ratio of two bounded analytic functions with its Laurent series around the origin having integral coefficients, is rational [4]. Pommerenke [17] proved that the Hankel determinants of univalent functions satisfy $|H_q(n)| < Kn^{-(\frac{1}{2}+\beta)q+\frac{3}{2}}$, where $\beta > 1/4000$ and K depends only on q. Later Hayman [8] proved that $|H_q(n)| < An^{1/2}$ (A is an absolute constant) for areally mean univalent functions. A classical theorem of Fekete-Szegö [6] considered the second Hankel determinant $|H_2(1)| = |a_3 - a_2^2|$ for univalent functions. They made an early study for the estimate of well known Fekete-Szegö functional $|a_3 - \mu a_2^2|$ when μ is real. Janteng [10] investigated the sharp upper bound for second Hankel determinant $|H_2(2)| = |a_2a_4 - a_3^2|$ for univalent functions whose derivative has positive real part.

Recently, Babalola [1], Raza and Malik [18], Bansal [3] and Mishra [14] have studied third Hankel determinant $H_3(1)$, for various classes of analytic and univalent functions.

In this paper, we consider the Hankel determinant for the case $q = 3$ and $n = 1$,

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}.$$

For $f \in \mathcal{A}$, $a_1 = 1$ so that,

$$H_3(1) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and by using the triangle inequality, we have

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{6}$$

2. Preliminary Results

Let P denote the class of functions

$$p(z) = 1 + c_1z + c_2^2 + \cdots \tag{7}$$

which are regular in $\mathbb{U}$ and satisfy $\Re\,[p(z)] > 0$, $z \in \mathbb{U}$. To prove the main results we shall require the following lemmas.

Lemma 2.1 ([19])**.** *If $f \in \mathcal{S}_s^*(\phi, s, t)$ of the form (1), then $|a_n| \leq 1, n \geq 2$.*

Lemma 2.2 ([5])**.** *$f \in \mathcal{C}_s(\phi, s, t)$ of the form (1), then $|a_n| \leq \dfrac{1}{n}, n \geq 2$.*

Lemma 2.3 ([11, 12])**.** *Let $p \in P$. Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{8}$$

and

$$4c_3 = c_1^3 + 2xc_1(4 - c_1^2) - x^2c_1(4 - c_1^2) + 2z(1 - |x|^2)(4 - c_1^2) \tag{9}$$

for some x, z such that $|x| \leq 1$ and $|z| \leq 1$.

Lemma 2.4 ([2])**.** *Let $p \in P$. Then*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| = \begin{cases} 2(1-\sigma) & \text{if } \sigma \leq 0, \\ 2 & \text{if } 0 \leq \sigma \leq 2 \\ 2(\sigma - 1) & \text{if } \sigma \geq 2. \end{cases}$$

3. Main Results

Theorem 3.1. *If $f \in \mathcal{S}_s^*(\phi, s, t)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{B_1^2}{R_2^2}$$

where

$$R_2 = 3 - s^2 - st - t^2.$$

Proof. Let $f \in \mathcal{S}_s^*(\phi, s, t)$, then there exists a Schwarz function $w(z) \in A$ such that

$$\frac{(s-t)zf'(z)}{f(sz) - f(tz)} = \phi(w(z)), \quad (z \in \mathbb{U}, s \neq t) \tag{10}$$

If $P_1(z)$ is analytic and has positive real part in $\mathbb{U}$ and $p_1(0) = 1$, then define the functions $p_1(z)$ as

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \ldots$$

From the above equation we obtain

$$w(z) = \frac{p_1(z) - 1}{p_1(z) + 1} = \frac{c_1}{2}z + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right)z^2 + \ldots \tag{11}$$

Then p_1 is analytic in $\mathbb{U}$ with $p_1(0) = 1$ and has a positive real part in $\mathbb{U}$. By using (11) and (3), it is clear that

$$\phi\left(\frac{p_1(z) - 1}{p_2(z) + 1}\right) = 1 + \frac{B_1c_1}{2}z + \left\{\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right\}z^2 \ldots \tag{12}$$

From (4) and (12), we can get

$$\begin{aligned}(s-t)(z + 2a_2z^2 + 3a_3z^3 + 4a_4z^4 \ldots) = \{(s-t)z + a_2(s^2 - t^2)z^2 + a_3(s^3 - t^3)z^3 \ldots\} \\ \left\{1 + \left(\frac{B_1c_1}{2}\right)z + \left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2c_1^2}{4}\right]z^2 \ldots\right\}\end{aligned} \tag{13}$$

Equating the co-efficients of like powers of z in (13) we get

$$a_2 = \frac{B_1c_1}{2R_1} \tag{14}$$

$$a_3 = \frac{1}{4R_2}\left[2B_1c_2 + Q_1c_1^2\right] \tag{15}$$

$$\begin{aligned}a_4 = &\frac{1}{R_3}\left[\frac{B_1c_3}{2} + \left\{\frac{B_2 - B_1}{2} + \frac{B_1^2(s+t)}{4R_1} + \frac{B_1^2(s^2 + st + t^2)}{4R_2}\right\}c_1c_2\right. \\ &\left. + \left\{\frac{B_1 - 2B_2 + B_3}{8} + \frac{B_1(B_2 - B_1)(s+t)}{8R_1} + \frac{(s^2 + st + t^2)Q_1B_1}{8R_2}\right\}c_1^3\right].\end{aligned} \tag{16}$$

where

$$R_1 = 2 - s - t, R_2 = 3 - s^2 - st - t^2, R_3 = 4 - s^3 - s^2t - st^2 - t^3 \tag{17}$$

and

$$Q_1 = B_2 - B_1 + \frac{(s+t)B_1^2}{R_1} \tag{18}$$

From equation (14), (15) and (16), we obtain

$$|a_2a_4 - a_3^2| = \left|\frac{B_1^2}{4R_1R_3}c_1c_3 + \left\{\frac{B_1(B_2 - B_1)}{4R_1R_3} + \frac{B_1^3(s+t)}{8R_1^2R_3} + \frac{B_1^3(s^2 + st + t^2)}{8R_1R_2R_3}\right\}c_1^2c_2\right.$$

$$\begin{aligned}
&+\left\{\frac{B_1(B_1-2B_2+B_3)}{16R_1R_3}+\frac{B_1^2(B_2-B_1)(s+t)}{16R_1^2R_3}+\frac{B_1^2Q_1(s^2+st+t^2)}{16R_1R_2^2R_3}\right\}c_1^4-\frac{1}{16R_2^2}\left(2c_2B_1+Q_1c_1^2\right)^2\Bigg|\\
|a_2a_4-a_3^2| = &\left|\frac{B_1^2}{4R_1R_3}c_1c_3+\left\{\frac{B_1(B_2-B_1)}{4R_1R_3}+\frac{B_1^3(s+t)}{8R_1^2R_3}+\frac{B_1^3(s^2+st+t^2)}{8R_1R_2R_3}-\frac{B_1Q_1}{4R_2^2}\right\}c_1^2c_2\right.\\
&\left.+\left\{\frac{B_1(B_1-2B_2+B_3)}{16R_1R_3}+\frac{B_1^2(B_2-B_1)(s+t)}{16R_1^2R_3}+\frac{B_1^2Q_1(s^2+st+t^2)}{16R_1R_2^2R_3}-\frac{Q_1^2}{16R_2^2}\right\}c_1^4-\frac{B_1^2}{4R_2^2}c_2^2\right| \qquad (19)
\end{aligned}$$

Putting the values of c_2 and c_3 from equations (8) and (9) in (19), we assume that $c_1 = c \in [0,2]$. With elementary calculations, we get

$$\begin{aligned}
|a_2a_4-a_3^2| = &\left|\left\{\frac{\beta}{2}+\eta+\frac{B_1^2}{16R_1R_3}-\frac{B_1^2}{16R_2^2}\right\}c^4+\left\{\frac{B_1^2}{8R_1R_3}+\frac{\beta}{2}-\frac{B_1^2}{8R_2^2}\right\}c^2x(4-c^2)\right.\\
&\left.-\frac{B_1^2}{16R_1R_3}c^2x^2(4-c^2)+\frac{B_1^2}{8R_1R_3}c(4-c^2)(1-|x|^2)z-\frac{B_1^2}{16R_2^2}x^2(4-c^2)^2\right|
\end{aligned}$$

where

$$\beta = \frac{B_1(B_2-B_1)}{4R_1R_3}+\frac{B_1^3(s+t)}{8R_1^2R_3}+\frac{B_1^3(s^2+st+t^2)}{8R_1R_2R_3}-\frac{B_1Q_1}{4R_2^2}$$

and

$$\eta = \frac{B_1(B_1-2B_2+B_3)}{16R_1R_3}+\frac{B_1^2(B_2-B_1)(s+t)}{16R_1^2R_3}+\frac{B_1^2Q_1(s^2+st+t^2)}{16R_1R_2^2R_3}-\frac{Q_1^2}{16R_2^2}$$

Now applying the triangle inequality and replacing $|x|$ by ρ, we obtain,

$$\begin{aligned}
|a_2a_4-a_3^2| \le &\left|\left\{\beta+\eta+\frac{B_1^2}{16R_1R_3}-\frac{B_1^2}{16R_2^2}\right\}c^4\right|+\left\{\frac{B_1^2}{8R_1R_3}+\frac{\beta}{2}-\frac{B_1^2}{8R_2^2}\right\}c^2\rho(4-c^2)\\
&+\frac{B_1^2}{16R_1R_3}c^2\rho^2(4-c^2)+\frac{B_1^2}{8R_1R_3}c(4-c^2)(1-\rho^2)+\frac{B_1^2}{16R_2^2}\rho^2(4-c^2)^2. \qquad (20)\\
=&F(c,\rho).
\end{aligned}$$

We assume that the upper bound occurs at the interior point of the rectangle $[0,2]\times[0,1]$. Differentiating (20) with respect to ρ, we get

$$\frac{\partial F}{\partial\rho} = \left\{\frac{B_1^2}{8R_1R_3}+\frac{\beta}{2}-\frac{B_1^2}{8R_2^2}\right\}c^2(4-c^2)+\frac{B_1^2}{8R_1R_3}c^2\rho(4-c^2)-\frac{B_1^2}{4R_1R_3}c(4-c^2)\rho+\frac{B_1^2}{8R_2^2}\rho(4-c^2)^2$$

For $0<\rho<1$ and fixed $c\in[0,2]$, it can be easily seen that $\dfrac{\partial F}{\partial\rho} > 0$. This shows that $F(c,\rho)$ is an increasing function of ρ. Therefore, $\max F(c,\rho) = F(c,1) = G(c)$

$$F(c,1) = G(c) = \left\{\beta+\eta+\frac{B_1^2}{16R_1R_3}-\frac{B_1^2}{16R_2^2}\right\}c^4+\left\{\frac{B_1^2}{8R_1R_3}+\frac{\beta}{2}-\frac{B_1^2}{8R_2^2}\right\}c^2(4-c^2)+\frac{B_1^2}{16R_1R_3}c^2(4-c^2)+\frac{B_1^2}{16R_2^2}(4-c^2)^2.$$

By elementary calculus we have $G''(c)\le 0$ for $0\le c\le 2$ and $G(c)$ has real critical point at $c=0$. Thus the upper bound of $F(\rho)$ corresponds to $\rho=1$ and $c=0$. Thus the maximum of $G(c)$ occurs at $c=0$. Hence,

$$|a_2a_4-a_3^2| \le \frac{B_1^2}{R_2^2}$$

□

Remark 3.2 ([14]). *When $B_1=B_2=B_3=2, s=1$ and $t=-1$, Theorem 3.1 reduces to $|a_2a_4-a_3^2|\le 1$.*

Theorem 3.3. *If $f\in\mathcal{S}_s^*(\phi,s,t)$, then*

$$|a_2a_3-a_4| \le 2A_3, \quad \textit{where } A_3 = \frac{B_1}{2R_3}.$$

Proof. From equation (14), (15) and (16), we obtain

$$|a_2a_3 - a_4| = \left| \left\{ \frac{B_1Q_1}{8R_1R_2} + \frac{B_1 - 2B_2 + B_3}{8} + \frac{B_1(B_2 - B_1)(s+t)}{8R_1} + \frac{B_1Q_1(s^2+st+t^2)}{8R_2} \right\} c_1^3 \right.$$
$$\left. + \left\{ \frac{B_1^2}{4R_1R_2} - \frac{B_2 - B_1}{2R_3} - \frac{B_1^2(s+t)}{4R_1R_3} - \frac{B_1^2(s^2+st+t^2)}{4R_2R_3} \right\} c_1c_2 - \frac{B_1c_3}{2R_3} \right| \tag{21}$$

Putting the values of c_2 and c_3 from equations (8) and (9) in (21) , we assume that $c_1 = c \in [0, 2]$ we get,

$$|a_2a_3 - a_4| = \left\{ A_1 + \frac{A_2}{2} - \frac{A_3}{4} \right\} c^3 + \left\{ \frac{A_2}{2} - \frac{A_3}{2} \right\} cx(4 - c^2) + \frac{A_3}{4} cx^2(4 - c^2) - \frac{A_3}{2}(4 - c^2)(1 - |x|^2)z,$$

where

$$A_1 = \left\{ \frac{B_1Q_1}{8R_1R_2} + \frac{B_1 - 2B_2 + B_3}{8} + \frac{B_1(B_2 - B_1)(s+t)}{8R_1} + \frac{B_1Q_1(s^2+st+t^2)}{8R_2} \right\} \tag{22}$$

$$A_2 = \left\{ \frac{B_1^2}{4R_1R_2} - \frac{B_2 - B_1}{2R_3} - \frac{B_1^2(s+t)}{4R_1R_3} - \frac{B_1^2(s^2+st+t^2)}{4R_2R_3} \right\} \tag{23}$$

$$\text{and} \quad A_3 = \frac{B_1}{2R_3} \tag{24}$$

Now applying the triangle inequality and replacing $|x|$ by ρ, we obtain,

$$|a_2a_3 - a_4| \le \left| \left\{ A_1 + \frac{A_2}{2} - \frac{A_3}{4} \right\} c^3 \right| + \left\{ \frac{A_2}{2} - \frac{A_3}{2} \right\} c\rho(4 - c^2) + \frac{A_3}{4} c\rho^2(4 - c^2) + \frac{A_3}{2}(4 - c^2)(1 - \rho^2) = F(c, \rho). \tag{25}$$

We assume that the upper bound occurs at the interior point of the rectangle $[0, 2] \times [0, 1]$. Differentiating (25) with respect to ρ, we get

$$\frac{\partial F}{\partial \rho} = \left\{ \frac{A_2}{2} - \frac{A_3}{2} \right\} c(4 - c^2) + \frac{A_3}{2} \rho(4 - c^2)(c - 2)$$

For $0 < \rho < 1$ and fixed $c \in [0, 2]$, it can be easily seen that $\frac{\partial F}{\partial \rho} < 0$. This shows that $F(c, \rho)$ is an decreasing function of ρ. Therefore, $\max F(c, \rho) = F(c, 0) = G(c)$

$$G(c) = \left\{ \frac{4A_1 + 2A_2 - A_3}{4} \right\} c^3 + \frac{A_3}{2}(4 - c^2)$$

For optimum value of $G(c)$, consider $G'(c) = 0$ this implies that $c = 0$ or

$$c = \frac{4A_3}{3(4A_1 + 2A_2 - A_3)}$$

Thus G attained its maximum value at $c = 0$. Hence the maximum of the functional $|a_2a_3 - a_4|$ are given by the inequalities of the theorem. This completes the proof. □

Remark 3.4 ([14]). *When $B_1 = B_2 = B_3 = 2, s = 1$ and $t = -1$, Theorem 3.3 reduces to $|a_2a_3 - a_4| \le \frac{1}{2}$.*

Theorem 3.5. *If $f \in \mathcal{S}_s^*(\phi, s, t)$, then*

$$|a_3 - a_2^2| \le \frac{B_1}{R_2}.$$

Where R_2 is given by (17)

Proof. Since $f \in S_s^*(\phi, s, t)$,then using equations (14) and (15) we obtain

$$\begin{aligned}|a_3 - a_2^2| &= \left| \frac{B_1}{2R_2} c_2 - \left\{ \frac{B_1^2}{4R_1^2} - \frac{Q_1}{4R_2} \right\} \frac{c_1^2}{2} \right| \\ &= \frac{B_1}{2R_2} \left| c_2 - \left\{ \frac{B_1 R_2}{2R_1^2} - \frac{Q_1}{2B_1} \right\} \frac{c_1^2}{2} \right|\end{aligned}$$

where R_1, R_2 is given by (17) and setting $\sigma = \dfrac{B_1 R_2}{2R_1^2} - \dfrac{Q_1}{2B_1}$, by using Lemma 2.4, we have $|a_3 - a_2^2| \leq \dfrac{B_1}{R_2}$. □

Theorem 3.6. *Let the function f given by (1) be in the class $\mathcal{S}_s^*(\phi, s, t)$. Then*

$$|H_3(1)| \leq \frac{B_1^2}{R_2^2} + 2A_3 + \frac{B_1}{R_2}.$$

Where R_2 and A_3 is given by (17) and (24).

Proof.

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_4 - a_2 a_3| + |a_5||a_3 - a_2^2| \tag{26}$$

Using Lemma 2.1, Theorem 3.1, Theorem 3.3 and Theorem 3.5 in (26), the above results can be easily obtained. □

Remark 3.7 ([14]). *When $B_1 = B_2 = B_3 = 2, s = 1$ and $t = -1$, Theorem 3.6 reduces to $|H_3(1)| \leq \dfrac{5}{2}$.*

Theorem 3.8. *If $f \in \mathcal{C}_s(\phi, s, t)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{9R_2^2}$$

Where R_2 is given by (17).

Proof. From the definitions of the classes $\mathcal{S}_s^*$ and $\mathcal{C}_s$, it follows that the function $f \in \mathcal{C}_s$ if and only if $zf' \in \mathcal{S}_s^*$. Thus replacing a_n by na_n in (14), (15) and (16), we obtain

$$a_2 = \frac{B_1 c_1}{4R_1} \tag{27}$$

$$a_3 = \frac{B_1 c_2}{6R_2} + \frac{Q_1 c_1^2}{12R_2} \tag{28}$$

$$\begin{aligned}a_4 = \frac{B_1 c_3}{8R_3} &+ \left\{ \frac{B_2 - B_1}{8R_3} + \frac{B_1^2(s+t)}{16R_1R_3} + \frac{B_1^2(s^2+st+t^2)}{16R_2R_3} \right\} c_1 c_2 \\ &+ \left\{ \frac{B_1 - 2B_2 + B_3}{32R_3} + \frac{B_1(s+t)(B_2 - B_1)}{32R_1R_3} + \frac{Q_1 B_1(s^2+st+t^2)}{32R_2R_3} \right\} c_1^3\end{aligned} \tag{29}$$

Where R_1, R_2, R_3 and Q_1 is given by (17) and (18). From the equations (27), (28) and (29),

$$|a_2 a_4 - a_3^2| = \left| \frac{B_1^2}{32R_1R_3} c_1 c_3 + \beta_1 c_1^2 c_2 + \eta_1 c_1^4 - \left\{ \frac{B_1^2}{36R_2^2} c_2^2 + \frac{Q_1^2 c_1^4}{144R_2^2} \right\} \right| \tag{30}$$

Where

$$\beta_1 = \frac{B_1(B_2 - B_1 + B_1^3(s+t))}{32R_1R_3} + \frac{B_1^3(s^2+st+t^2)}{64R_1R_2R_3} + \frac{B_1(B_2 - B_1) + B_1 Q_1}{36R_2^2}$$

and

$$\eta_1 = \frac{B_1(B_1 - 2B_2 + B_3) + B_1^2(s+t)(B_2 - B_1)}{128R_1R_3} + \frac{(s^2+st+t^2)Q_1 B_1}{128R_2R_3} + Q_1^2$$

Putting the values of c_2 and c_3 from equations (8) and (9) in (30), we assume that $c_1 = c \in [0,2]$. With elementary calculations, we get

$$|a_2a_4 - a_3^2| = \Bigg|\left\{\frac{\beta_1}{2} + \eta_1 + \frac{B_1^2}{128R_1R_3} - \frac{B_1^2}{128R_2^2}\right\}c^4 + \left\{\frac{B_1^2}{64R_1R_3} + \frac{\beta_1}{2} - \frac{B_1^2}{64R_2^2}\right\}c^2x(4-c^2) \\ -\frac{B_1^2}{128R_1R_3}c^2x^2(4-c^2) + \frac{B_1^2}{64R_1R_3}c(4-c^2)(1-|x|^2)z - \frac{B_1^2}{144R_2^2}x^2(4-c^2)^2\Bigg|$$

Now applying the triangle inequality and replacing $|x|$ by ρ, we obtain,

$$\begin{aligned} |a_2a_4 - a_3^2| \le& \left|\left\{\frac{\beta_1}{2} + \eta_1 + \frac{B_1^2}{128R_1R_3} - \frac{B_1^2}{128R_2^2}\right\}c^4\right| + \left\{\frac{B_1^2}{64R_1R_3} + \frac{\beta_1}{2} - \frac{B_1^2}{64R_2^2}\right\}c^2\rho(4-c^2) \\ &+\frac{B_1^2}{128R_1R_3}c^2\rho^2(4-c^2) + \frac{B_1^2}{64R_1R_3}c(4-c^2)(1-\rho^2) + \frac{B_1^2}{144R_2^2}\rho^2(4-c^2)^2 \\ =& F_1(c,\rho). \end{aligned} \tag{31}$$

We assume that the upper bound occurs at the interior point of the rectangle $[0,2]\times[0,1]$. Differentiating (31) with respect to ρ, we get

$$\frac{\partial F_1}{\partial \rho} = \left\{\frac{B_1^2}{64R_1R_3} + \frac{\beta_1}{2} - \frac{B_1^2}{64R_2^2}\right\}c^2(4-c^2) + \frac{B_1^2}{64R_1R_3}c^2\rho(4-c^2) - \frac{B_1^2}{32R_1R_3}c\rho(4-c^2) + \frac{B_1^2}{72R_2^2}\rho(4-c^2)^2$$

For $0<\rho<1$ and fixed $c\in[0,2]$, it can be easily seen that $\dfrac{\partial F_1}{\partial \rho} > 0$. This shows that $F_1(c,\rho)$ is an increasing function of ρ. Therefore, $\max F_1(c,\rho) = F(c,1) = G(c)$

$$F_1(c,1) = G(c) = \left\{\frac{\beta_1}{2} + \eta_1 + \frac{B_1^2}{128R_1R_3} - \frac{B_1^2}{128R_2^2}\right\}c^4 + \left\{\frac{B_1^2}{64R_1R_3} + \frac{\beta_1}{2} - \frac{B_1^2}{64R_2^2}\right\}c^2(4-c^2) \\ +\frac{B_1^2}{128R_1R_3}c^2(4-c^2) + \frac{B_1^2}{144R_2^2}(4-c^2)^2$$

By elementary calculus we have $G''(c) \le 0$ for $0 \le c \le 2$ and $G(c)$ has real critical point at $c=0$. Thus the upper bound of $F_1(\rho)$ corresponds to $\rho = 1$ and $c=0$. The maximum of $G(c)$ occurs at $c=0$. Hence,

$$|a_2a_4 - a_3^2| \le \frac{B_1^2}{9R_2^2}$$

□

Remark 3.9 ([14]). *When* $B_1 = B_2 = B_3 = 2, s = 1$ *and* $t=-1$, *Theorem 3.8 reduces to* $|a_2a_4 - a_3^2| \le \frac{1}{9}$.

Theorem 3.10. *If* $f \in \mathcal{C}_s(\phi, s, t)$, *then*

$$|a_2a_3 - a_4| \le \sqrt{\frac{4(3A_3 - 2A_2)}{3(4A_1 + 2A_3)}}\left[\frac{6A_3 - 4A_2}{3}\right].$$

Where

$$A_1 = \frac{B_1Q_1}{48R_2} - \frac{B_1 - 2B_2 + B_3}{32R_3} + \frac{B_1^2(s+t)}{32R_1R_3} + \frac{B_1(s+t)(B_2-B_1)}{32R_1R_3} + \frac{(s^2+st+t^2Q_1B_1)}{32R_2R_3} \tag{32}$$

$$A_2 = \frac{B_1^2}{24R_1R_2} - \frac{B_2 - B_1}{8R_3} - \frac{B_1^2(s+t)}{16R_1R_3} - \frac{B_1^2(s^2++st+t^2)}{16R_2R_3} \tag{33}$$

$$A_3 = \frac{B_1}{8R_3} \tag{34}$$

Proof. From equations (27), (28)and (29), we get

$$|a_2a_3-a_4| = \left|\frac{B_1^2}{24R_1R_2}c_1c_2+\frac{B_1Q_1}{48R_2}c_1^3-\frac{B_1}{8R_3}c_3-\left\{\frac{B_2-B_1}{8R_3}+\frac{B_1^2(s+t)}{16R_1R_3}+\frac{B_1^2(s^2+st+t^2)}{16R_2R_3}\right\}c_1c_2\right.$$
$$\left.-\left\{\frac{B_1-2B_2+B_3}{32R_3}+\frac{B_1(s+t)(B_2-B_1)}{32R_1R_3}+\frac{(s^2+st+t^2)Q_1B_1}{32R_2R_3}\right\}c_1^3\right|$$
$$|a_2a_3-a_4| = A_1c_1^3+A_2c_1c_2-A_3c_3 \tag{35}$$

Where A_1, A_2 and A_3 are given in (32), (33) and (34). Putting the values of c_2 and c_3 from equations (8) and (9) in (35), we assume that $c_1 = c \in [0,2]$. With elementary calculations, we get

$$|a_2a_3-a_4| = \left\{\frac{A_3}{4}-\frac{A_2}{2}-A_1\right\}c^3-\left\{\frac{A_3-A_2}{2}\right\}cx(4-c^2)+\frac{A_3}{4}cx^2(4-c^2)-\frac{2(4-c^2)(1-|x|^2)z}{4}$$

Now applying the triangle inequality and replacing $|x|$ by ρ, we obtain,

$$|a_2a_3-a_4| \le \left|\left\{\frac{A_3}{4}-\frac{A_2}{2}-A_1\right\}c^3\right|+\left\{\frac{A_3-A_2}{2}\right\}c\rho(4-c^2)+\frac{A_3}{4}c\rho^2(4-c^2)+\frac{2(4-c^2)(1-\rho^2)}{4} = F_1(c,\rho). \tag{36}$$

We assume that the upper bound occurs at the interior point of the rectangle $[0,2]\times[0,1]$. Differentiating (36) with respect to ρ, we get

$$\frac{\partial F_1}{\partial \rho} = \left\{\frac{A_3-A_2}{2}\right\}c(4-c^2)+\frac{A_3}{2}c\rho(4-c^2)-(4-c^2)\rho.$$

For $0<\rho<1$ and fixed $c\in[0,2]$, it can be easily seen that $\frac{\partial F_1}{\partial \rho} > 0$. This shows that $F_1(c,\rho)$ is an increasing function of ρ. Therefore, $\max F(c,\rho) = F_1(c,1) = G(c)$

$$F_1(c,1) = G(c) = \left\{\frac{A_3}{4}-\frac{A_2}{2}-A_1\right\}c^3+\left\{\frac{A_3-A_2}{2}\right\}c(4-c^2)+\frac{A_3}{4}c(4-c^2)$$
$$G'(c) = \left\{\frac{A_3}{4}-\frac{A_2}{2}-A_1\right\}3c^2+\left\{\frac{3A_3-2A_2}{4}\right\}(4-3c^2)$$

By elementary calculus, $G(c)$ is maximum at $c = \sqrt{\frac{4(3A_3-2A_2)}{3(4A_1+2A_3)}}$ and is given by

$$G(c) \le \sqrt{\frac{4(3A_3-2A_2)}{3(4A_1+2A_3)}}\left[\frac{6A_3-4A_2}{3}\right]$$

Thus for all admissible $c\in[0,2]$, the maximum of the functional $|a_2a_3-a_4|$ are given by the inequalities of the theorem. This completes the proof. □

Remark 3.11 ([14]). *When $B_1 = B_2 = B_3 = 2, s = 1$ and $t = -1$, Theorem 3.10 reduces to* $|a_2a_3-a_4| \le \frac{4}{27}$.

Theorem 3.12. *If $f \in \mathcal{C}_s(\phi, s, t)$, then*

$$|a_3-a_2^2| \le \frac{B_1}{3R_2},$$

where R_2 is given by (17).

Proof. Since $f\in\mathcal{C}_s(\phi,s,t)$,then using equations (27) and (28), we obtain

$$|a_3-a_2^2| = \left|\frac{B_1}{6R_2}c_2-\left[\frac{B_1^2}{16R_1^2}-\frac{Q_1}{12R_2}\right]\frac{c_1^2}{2}\right|$$
$$= \frac{B_1}{6R_2}\left|c_2-\left\{\frac{3B_1R_2}{8R_1^2}-\frac{Q_1}{2R_2B_1}\right\}\frac{c_1^2}{2}\right|$$

Where R_1, R_2 is given by (17) and setting $\sigma = \frac{3B_1R_2}{8R_1^2}-\frac{Q_1}{2R_2B_1}$, by using Lemma 2.4, we have $|a_3-a_2^2| \le \frac{B_1}{3R_2}$. □

Theorem 3.13. *Let the function f given by (1) be in the class $\mathcal{C}_s(\phi, s, t)$. Then*

$$|H_3(1)| \le \frac{B_1^2}{27R_2^2} + \sqrt{\frac{4(3A_3 - 2A_2)}{3(4A_1 + 2A_3)}}\left[\frac{6A_3 - 4A_2}{12}\right] + \frac{B_1}{15R_2}.$$

Where A_1, A_2, A_3 is given by (32), (33) and (34).

Proof.

$$|H_3(1)| \le |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2| \tag{37}$$

Using Lemma 2.2, Theorem 3.8, Theorem 3.10 and Theorem 3.12 in (37), the above results can be easily obtained. □

Remark 3.14 ([14]). *When $B_1 = B_2 = B_3 = 2, s = 1$ and $t = -1$, Theorem 3.13 reduces to $|H_3(1)| \le \frac{19}{135}$.*

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