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An M/M/1 Queuing System with Three Types Input Sources

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Abstract: This paper considers an M/M/1 queuing system with three types input sources, where the rate of arrival and service capacity follow Poisson distribution. The arrival process consists of three stages said to be active state, sick state and passive state. The system remains in three state for a random time which is exponentially distributed. The queue discipline is first-come-first-serve (FCFS). Laplace transforms of the various probability generating functions are obtained and the steady state results are derived. The probability that the arrival process (input) will be in active state, sick state and passive state, is also analyzed.

Keywords: Queuing Theory, Markovian Process, Laplace Transform, Exponential Distribution, Poisson distribution and Probability Generating Function.

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1. Introduction

Queuing theory originated when a Danish mathematician A.K. Erlang published in 1909 his pioneering paper "The theory of probabilities and telephone conversations" on the study of congestion of telephone traffic. His studies are now classics in queuing theory. Until about 1940 the development of the new branch of applied probability was directed by the needs encountered in the design of automatic telephone exchanges. After the Second World War when applications of mathematical models and methods in technology and other applied areas rose to a level previously unknown, it was realized that queuing theory too had a very broad field of applicability to various scientific and organizational phenomena. Queuing theory plays an important role in modeling real life problems involving congentions in wide areas of science, technology and management. In queuing theory, a model is constructed so that queue lengths and waiting times can be predicted [1]. Queuing theory is generally considered a branch of operations research because the results are often used when making business decisions about the resources needed to provide service. Queuing theory is a mathematical access in Operations Research applied to the analysis of queue.

A queuing system may be described as one having a service facility at which units of some kind arrive for service , and where, whenever there are more units in the system than the service facility can handle simultaneously, a queue or waiting line is formed. These units take their turn for service according to a preassigned rule and after service they leave the system. By units we mean those demanding service , e.g. customers at a bank counter or at a reservation counter, calls arriving arriving at a telephone exchange, vehicles at a traffic intersection, machines for repair before a repairman, airplanes waiting

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for take-off or landing at a busy airport, merchandise awaiting shipment at a yard, computer programs waiting to be run on a time-sharing basis, etc. Thus, the input to the system consists of the customers demanding service and the output is the serviced customers. Generally, a queuing system is characterized by the following:

(i). The input process

(ii). The queue discipline

(iii). The service mechanism

In queuing system, we will discuss two common concepts:.

1.1. Utilization factor

Utilization plays the crucial role and is defined as the proportion of the system‘s resources which is used by the traffic which arrives at it. It should be strictly less than one for the system to function well. It is usually denoted by the symbol ρ. If $\rho \geq 1$, then the queue will continue to grow as time goes on. In the simplest case of an M/M/1 queue (Poisson arrivals and a single Poisson server) then it is given by the mean arrival rate over the mean service rate, that is, $\rho = \frac{\lambda}{\mu}$, where λ is the mean arrival rate and μ is the mean service rate.
More generally, $\rho = \frac{\lambda}{\mu \times c}$, where λ is the mean arrival rate, μ is the mean service rate and c is the number of servers, such as in an M/M/c queue. In general, a lower utilization corresponds to less queuing for customers but means that the system is more idle, which may be considered inefficient.

1.2. Little's theorem

Little's theorem [2] describes the relationship between throughput rate (i.e. arrival and service rate), cycle time and work in process (i.e. number of customers/jobs in the system). The theorem states that the expected number of customers (N) for a system in steady state can be determined using the following equation:

$$L = \lambda T$$

Here, λ is the average customer arrival rate and T is the average service time for a customer. Three fundamental relationships can be derived from Little's theorem [3]:

- L increases if λ or T increases.
- λ increases if$L$ increases or T decreases.
- T increases if L increases or λ decreases.

A useful queuing model represents a real life system with sufficient accuracy and is analytically tractable. A queuing model based on the Poisson process and its companion exponential probability distribution often meets two requirements. A Poisson process models random events as maintaining from a memoryless process. That is, the length of time interval from the current time to the occurrence of the next event does not depend upon the time occurrence of the last event. Heterogeneous queuing systems have been studied by Neuts [4], Yechiali and Naor [5], and Murari and Agrawal [6]. In queuing model studied in [5], the arrival process at a service station is Poisson; the service time distribution is taken as negative exponential and the parameter depends on the environment. In the queuing problem studied by Murari and Agarwal [6], the arrival process breaks down with two arrival intensities viz., λ and 0. Krishanamoorthy [7] considers a

Poisson queue with two-heterogeneous servers with modified queue disciplines. The steady-state solution, transient solution and busy period distribution for the first discipline and the steady-state solution for the second discipline are obtained. Heterogeneous researches in the area of queuing theory have been studied be Yechiali and Naor [5], Murari and Agarwal [6], Madan K. C. [8], Saraswat and Agarwal [10], Rakesh Kumar and Sumeet Kumar Sharma [11]. Singh [12] extends the work of Krishanamoorthy [7] on heterogeneous servers by incorporating balking to compares results with homogeneous servers queue to show that the conditions under which the heterogeneous system is better than corresponding homogeneous system. Kumar and Sharma [11] study the two-heterogeneous server Markovian queuing model with discouraged arrivals, reneging and retention of reneged customers. The steady-state probabilities of system size are obtained explicitly using iterative method and also discussed some useful measures of effectiveness. In recent research, a queuing system studied by Saraswat G. K. [13] with multiple inputs.

In the ensuing problem, we study a queuing system with three arrival rates. It is assumed that service rate is same for all three states of the input. To be more clear, once units join the queue from different states of the input, they do not wear labels, viz., active, sick or passive. This queuing model closely adheres with so many practical situations. For example, consider, production unit of a factory engaged in the production of some food products like, canned juices, jam, spices, etc. The ready stocks of these food items are sent to a sales depot, where depot manager supervises the sale of these items. If we consider sales depot as a service channel and the supply from the factory as arrivals to sales depot then that factory must maintain a normal supply of the item to the sales depot in order to keep depot manager busy continuously and thereby fulfill the demands of customers. But, for various difficulties, like power shortage, go-slow practice by the workers or shortage of raw material, the factory is bound to reduce its output and sometimes, in cases of power failure, strike by the workers, breakdown of machine, it stops sending items to the sales depot. However, after an elapse of some time when it (factory) becomes normal, it renews the supply. It seems reasonable to assume that:

(i). The time during which normal supply of the items is maintained,

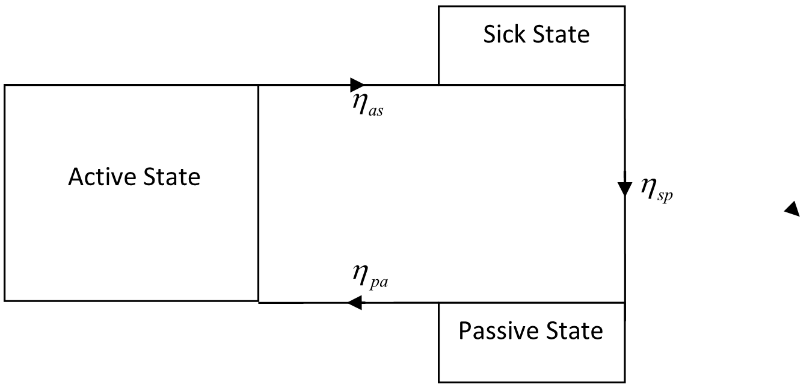
(ii). The time during which the reduced supply is maintained,

(iii). The time during which production stops and nothing is supplied,

are random variables which exponentially distributed.

2. Queuing Model Description

A stream of Poisson-type unit arrives at a single service station. The arrival pattern is multifarious, i.e., there exists three different arrival rates λ_a (when input source is active), λ_s (when input source is sick) and zero (when input source is passive). The input source is operative in one state at a time. The service time of customers is exponentially distributed with Poissonian service rate μ corresponding to arrival rates λ_a, λ_s and 0 (zero). The state of the system, operating with arrival rate λ_a is designated as P, operating with arrival rate λ_s is designated as Q, and operating with arrival rate zero is designated as R. The system starts with input source in active state. The time duration for which it remains in active state is a random variable which is exponentially distributed with parameter η_{ap}. After the active state the input source moves to the sick state, that is, the rate of arrival of units decreases considerably. The time period which is spent in sick state is also a random variable with parameter η_{sp}, which is different from active state. After the sick state, the input source shifts to passive state. In this state, units stop arriving at a service facility. The input remains in the passive state for a random time with exponential rate η_{pa}, which is different from those of active and sick states. After the passive state the input source again moves to active state and process continues in this way. The transition rate from one state to another state is as shown in the following figure:



Further, service time is assumed to be exponentially distributed with parameter μ for all states of the input. The stochastic processes involved, viz., interarrival time of units and service time of customers are independent of each other. The following results have been analyzed.

(i). $L.T.'^{S}$ of the probability generating function of the distribution of the number of units in the system for different states of the input.

(ii). $L.T.'^{S}$ of the probabilities for different states of the input.

(iii). A particular case, when input does not move to sick state.

(iv). The explicit steady state results corresponding to (i).

(v). The explicit steady state probabilities corresponding to (ii).

3. Solution of Queuing Model

In this section, the mathematical framework of the queuing model is presented. The time dependent and steady state solution of the problem have been discussed.

3.1. Time dependent solution

Define

$P_n(t) \equiv$ Probability that at time t, the input is in the active state P and n units are in the system.

$Q_n(t) \equiv$ Probability that at time t, the input is in the sick state Q and n units are in the system.

$R_n(t) \equiv$ Probability that at time t, the input in the passive state R and n units are in the system.

$S_n(t) \equiv$ Probability that at time t, there are n units is in the system.

Clearly,

$$S_n(t) \equiv P_n(t) + Q_n(t) + R_n(t).$$

Reckon time from the instant when the queue length is zero and the system is in the alive state. Initial conditions become

$$P_n(0) = \begin{cases} 1, & n = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$Q_n(0) = 0, \quad \forall \;\; n \geq 0$$
$$R_n(0) = 0, \quad \forall \;\; n \geq 0$$

Now define the following probability generating functions:

$$P(z,t) = \sum_{n=0}^{\infty} z^n p_n(t),$$
$$Q(z,t) = \sum_{n=0}^{\infty} z^n Q_n(t),$$
$$R(z,t) = \sum_{n=0}^{\infty} z^n R_n(t),$$
$$S(z,t) = \sum_{n=0}^{\infty} z^n S_n(t),$$

These must coverage with in the unit circle $|z| = 1$. Elementary probability reasoning lead to the following differential equations:

$$\frac{d}{dt}P_0(t) = -(\lambda_a + \eta_{as}) P_0(t) + \mu P_1(t) + \eta_{sa} R_0(t) \tag{1}$$
$$\frac{d}{dt}P_n(t) = -(\lambda_a + \mu + \eta_{as}) P_n(t) + \lambda_a P_{n-1}(t) + \mu P_{n+1}(t) + \eta_{pa} R_n(t), \quad n \geq 1 \tag{2}$$
$$\frac{d}{dt}Q_0(t) = -(\lambda_s + \eta_{sp}) Q_0(t) + \mu Q_1(t) + \eta_{as} P_0(t) \tag{3}$$
$$\frac{d}{dt}Q_n(t) = -(\lambda_s + \mu + \eta_{sp}) Q_n(t) + \lambda_s Q_{n-1}(t) + \mu Q_{n+1}(t) + \eta_{as} P_n(t), \quad n \geq 1 \tag{4}$$
$$\frac{d}{dt}R_0(t) = -\eta_{pa} R_0(t) + \mu R_1(t) + \eta_{sp} Q_0(t) \tag{5}$$
$$\frac{d}{dt}R_n(t) = -(\mu + \eta_{pa}) R_n(t) + \mu R_{n+1}(t) + \eta_{ap} Q_n(t), \quad n \geq 1 \tag{6}$$

Multiplying (1) - (6) by appropriate process of z, using their respective probability generating functions, taking L.T's and using initial conditions. We have

$$K_a(z,s)\overline{P}(z,s) = z + \mu(z-1)\overline{P_0}(s) + z\eta_{pa}\overline{R}(z,s) \tag{7}$$
$$K_s(z,s)\overline{Q}(z,s) = \mu(z-1)\overline{Q_0}(s) + z\eta_{as}\overline{P}(z,s) \tag{8}$$
$$K_p(z,s)\overline{R}(z,s) = \mu(z-1)\overline{R_0}(z,s) + z\eta_{sp}\overline{Q}(z,s) \tag{9}$$

s Where

$$K_a(z,s) = [z\{s + \lambda_a(1-z) + \mu + \eta_{as}\} - \mu]$$
$$K_s(z,s) = [z\{s + \lambda_s(1-z) + \mu + \eta_{sp}\} - \mu]$$
$$K_p(z,s) = [z(s + \mu + \eta_{pa}) - \mu]$$

On solving equations (7) - (9)

$$\overline{P}(z,s) = \frac{zK_s(z,s)K_p(z,s) + \mu(z-1)\left[K_s(z,s)K_p(z,s)\overline{P_0}(s) + z\eta_{pa} + K_s(z,s)\overline{R_0}(s) + z^2\eta_{sp}\eta_{pa}\overline{Q_0}(s)\right]}{K_a(z,s)K_s(z,s)K_p(z,s) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \tag{10}$$
$$\overline{Q}(z,s) = \frac{z^2\eta_{as}K_p(z,s) + \mu(z-1)\left[K_a(z,s)K_p(z,s)\overline{Q_0}(s) + z\eta_{as}K_p(z,s)\overline{P_0}(s) + z^2\eta_{as}\eta_{pa}\overline{R_0}(s)\right]}{K_a(z,s)K_s(z,s)K_p(z,s) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \tag{11}$$

$$\overline{R}(z,s)=\frac{z^3\eta_{as}\eta_{sp}+\mu(z-1)\left[K_a(z,s)K_s(z,s)\overline{R_0}(s)+z\eta_{sp}K_a(z,s)\overline{Q_0}(s)+z^2\eta_{as}\eta_{sp}\overline{P_0}(s)\right]}{K_a(z,s)K_s(z,s)K_p(z,s)-z^3\eta_{as}\eta_{sp}\eta_{pa}} \tag{12}$$

$$\overline{S}(z,s)=\overline{P}(z,s)+\overline{Q}(z,s)+\overline{R}(z,s) \tag{13}$$

Substituting the values of $\overline{P}(z,s)$, $\overline{Q}(z,s)$ and $\overline{R}(z,s)$ from equations (10) - (12) in equation (13). We obtain

$$\bar{S}(z,s)=\frac{\begin{array}{c}zK_s(z,s)K_p(z,s)+z^2\eta_{as}K_p(z,s)+z^3\eta_{as}\eta_{sp}+\mu(z-1)\\ \sum_{s,p,a\text{ and P,Q,R}}[K_s(z,s)K_p(z,s)\bar{P}_0(s)+z\eta_{pa}K_s(z,s)\bar{R}_0(s)+z^2\eta_{sp}\eta_{pa}\bar{Q}_0(s)]\end{array}}{K_a(z,s)K_s(z,s)K_p(z,s)-z^3\eta_{as}\eta_{sp}\eta_{pa}} \tag{14}$$

Where $\sum$ runs cyclically over a,s,p and P,Q,R. $\bar{S}(z,s)$ is known in terms of three unknowns, viz., $\overline{P_0}(s)$, $\overline{Q_0}(s)$ and $\overline{R}_0(s)$. We proceed to obtain these unknowns. We first prove that each of $K_a(z,s), K_s(z,s)$ and $K_p(z,s)$ has a zero inside the unit circle $|z|=1$. Write,

$$g(z)=z\{s+\lambda_a(1-z)+\mu+\eta_{as}\}$$
$$f(z)=\mu$$

(i). Both $g(z)$ and $f(z)$ are analytic inside and on the contour $|z|=1$,

(ii). On $|z|=1$,

$$\begin{aligned}|g(z)|&=|z\{s+\lambda_a(1-z)+\mu+\eta_{as}\}|\\ &>\mu=|f(z)|,\quad\text{for}\quad Rl(s)>0\end{aligned}$$

Hence, $|g(z)|>|f(z)|$. Both the conditions of Rouche's theorem are satisfied. Therefore, $g(z)$ and $g(z)-f(z)$, which is $K_a(z,s)$, have the same number of zeros inside the unit circle $|z|=1$. Since $g(z)$ has one zero inside the unit circle, $K_a(z,s)$ will also have one zero inside the unit circle. Similarly, it can be shown that $K_s(z,s)$ and $K_p(z,s)$ also have one zero inside the unit circle $|z|=1$. Consider the denominator of $\bar{S}(z,s)$ which is given by $K_a(z,s)K_s(z,s)K_p(z,s)-z^3\eta_{as}\eta_{sp}\eta_{pa}$. Write,

$$G(z)=K_a(z,s)K_s(z,s)K_p(z,s)$$
$$F(z)=z^3\eta_{as}\eta_{sp}\eta_{pa}$$

(i). Both $G(z)$ and $F(z)$ are analytic inside and on the contour $|z|=1$.

(ii). On $|z|=1$

$$\begin{aligned}|G(z)|&=|K_a(z,s)|\,|K_s(z,s)|\,|K_p(z,s)|\\ &\geq[|z\{s+\lambda_a(1-z)+\mu+\eta_{as}\}|-|\mu|][|z\{s+\lambda_s(1-z)+\mu+\eta_{sp}\}|-|\mu|][|z(s+\mu+\eta_{pa})|-|\mu|]\\ &>[|\mu+\eta_{as}|-|\mu|][|\mu+\eta_{sp}|-|\mu|][|\mu+\eta_{pa}|-|\mu|],\quad\text{for}\quad RI(s)>0\\ &>\eta_{as}\eta_{sp}\eta_{pa}\\ &=|F(z)|\end{aligned}$$

Hence, $|G(z)|>|F(z)|$. Rouche's theorem is satisfied. Therefore $G(z)$ and $G(z)-F(z)$, which is the denominator of $\bar{S}(z,s)$, have the same number of zeros inside the unit circle $|z|=1$. As we have already proved, that $G(z)$ has three zeros inside the $|z|=1$, so the denominator of $\bar{S}(z,s)$ will also have three zeros inside the unit circle. Since $\bar{S}(z,s)$ is differentiable inside the unit circle, then three zeros must vanish the numerator of $\bar{S}(z,s)$, which give rise to three equations in three unknowns, viz., $\overline{P_0}(s)$, $\overline{Q_0}(s)$ and $\overline{R}_0(s)$. Solving this set of three equations, the values of three unknowns are obtained. Thus $\bar{S}(z,s)$ can be determined completely.

Different States of Input Source

Laplace transform of various probabilities is in the active state, sick state and passive state can be got by setting Set $z = 1$ in equation (10) - (12).

$$\overline{P}(1, s) = \frac{(s + \eta_{sp})(s + \eta_{pa})}{(s + \eta_{as})(s + \eta_{sp})(s + \eta_{pa}) - \eta_{as}\eta_{sp}\eta_{pa}} \tag{15}$$

$$\overline{Q}(1, s) = \frac{\eta_{as}(s + \eta_{pa})}{(s + \eta_{as})(s + \eta_{sp})(s + \eta_{pa}) - \eta_{as}\eta_{sp}\eta_{pa}} \tag{16}$$

$$\overline{R}(1, s) = \frac{\eta_{as}\eta_{sp}}{(s + \eta_{as})(s + \eta_{sp})(s + \eta_{pa}) - \eta_{as}\eta_{sp}\eta_{pa}} \tag{17}$$

Particular Case

When the input does not have sick state, that is, it goes to directly from active state to passive state and from passive state to active state. The corresponding solution can be obtained by making η_{sp} tend to infinity, η_{as} tend to η_{ap} and $\bar{Q}_n(s)$ tend to zero, $(n = 0, 1, 2, 3, \ldots)$. Hence from equations (10) - (12).

$$\left.\begin{aligned} \overline{P}(z, s) &= \frac{zK_p(z,s) + \mu(z-1)\left[K_p(z,s)\overline{P_0}(s) + z\eta_{pa}K_s(z,s)\overline{R_0}(s)\right]}{K_a(z,s)K_p(z,s) - z^2\eta_{ap}\eta_{pa}} \\ \overline{Q}(z, s) &= 0 \\ \overline{R}(z, s) &= \frac{z^2\eta_{ap} + \mu(z-1)\left[K_a(z,s)\overline{R_0}(s) + z\eta_{ap}\overline{P_0}(s)\right]}{K_a(z,s)K_p(z,s) - z^2\eta_{ap}\eta_{pa}} \end{aligned}\right] \tag{18}$$

3.2. Steady State Solution

The steady state solution can be obtained by the well-known properly of the L, T., viz.,

$$\underset{s\to 0}{Lim}\, s\bar{F}(s) = \underset{t\to\infty}{Lim}\, F(t) \tag{19}$$

If the limit on the right exists. Thus, if

$$\underset{t\to\infty}{Lim}\, P_n(t) = P_n$$

We have,

$$\underset{s\to 0}{Lim}\, s\bar{P}_n(s) = P_n \quad etc.$$

Using property (19) to equations (7) - (9), we have.

$$K_a(z)\, P(z) = \mu(z-1)\, P_0 + z\eta_{pa}R(z) \tag{20}$$

$$K_s(z)\, Q(z) = \mu(z-1)\, Q_0 + z\eta_{as}P(z) \tag{21}$$

$$K_p(z)\, R(z) = \mu(z-1)\, R_0 + z\eta_{sp}Q(z) \tag{22}$$

Where

$$K_a(z) = [z\{\lambda_a(1-z) + \mu + \eta_{as}\} - \mu]$$

$$K_s(z) = [z\{\lambda_s(1-z) + \mu + \eta_{sp}\} - \mu]$$

$$K_p(z) = [z(\mu + \eta_{sp}) - \mu]$$

Solving equations (19) - (21)

$$P(z) = \frac{\mu(z-1)\left[K_s(z)\, K_p(z)\, P_0 + z\eta_{pa}K_s(z)\, R_0 + z^2\eta_{sp}\eta_{pa}Q_0\right]}{K_a(z)\, K_s(z)\, K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa}} \tag{23}$$

$$Q\left(z\right)=\frac{\mu\left(z-1\right)\left[K_{a}\left(z\right)K_{p}\left(z\right)Q_{0}+z\eta_{as}K_{p}\left(z\right)P_{0}+z^{2}\eta_{as}\eta_{pa}R_{0}\right]}{K_{a}\left(z\right)K_{s}\left(z\right)K_{p}\left(z\right)-z^{3}\eta_{as}\eta_{sp}\eta_{pa}} \tag{24}$$

$$R\left(z\right)=\frac{\mu\left(z-1\right)\left[K_{a}\left(z\right)K_{s}\left(z\right)R_{0}+z\eta_{sp}K_{a}\left(z\right)Q_{0}+z^{2}\eta_{as}\eta_{sp}P_{0}\right]}{K_{a}\left(z\right)K_{s}\left(z\right)K_{p}\left(z\right)-z^{3}\eta_{as}\eta_{sp}\eta_{pa}} \tag{25}$$

$$S\left(z\right)=P(z)+Q(z)+R(z)$$

Hence, from equations (23) - (25)

$$S(z)=\frac{\mu(z-1)\sum_{s,p,a\text{ and P,Q,R}}[K_{s}(z)K_{p}(z)P_{0}+z\eta_{pa}K_{s}(z)R_{0}(s)+z^{2}\eta_{sp}\eta_{pa}Q_{0}]}{K_{a}(z)K_{s}(z)K_{p}(z)-z^{3}\eta_{as}\eta_{sp}\eta_{pa}} \tag{26}$$

Where $\sum$ runs cyclically over a, s, p and P, Q, R. $S(z)$ is known in terms of three unknowns, viz., P_0, Q_0 and R_0. We proceed to obtain these unknowns. Setting $z = 1$ in equations (20) - (22)

$$\eta_{as}P\left(1\right)=\eta_{pa}R\left(1\right) \tag{27}$$

$$\eta_{sp}Q\left(1\right)=\eta_{as}P\left(1\right) \tag{28}$$

$$\eta_{pa}R\left(1\right)=\eta_{sp}Q\left(1\right) \tag{29}$$

Equations (27) - (29) lead to the following

$P(1) \equiv$ The Steady state probability for which input will remain in active state.

$$=\frac{\eta_{sp}\eta_{pa}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \tag{30}$$

$Q(1) \equiv$ The Steady state probability for which input will remain in sick state.

$$=\frac{\eta_{pa}\eta_{as}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \tag{31}$$

$R(1) \equiv$ The Steady state probability for which input will remain in passive state.

$$=\frac{\eta_{as}\eta_{sp}}{\left(\eta_{sp}\eta_{pa}+\eta_{pa}\eta_{as}+\eta_{as}\eta_{sp}\right)} \tag{32}$$

The denominator of $P(z)$, $Q(z)$ and $R(z)$, $[K_a\left(z\right)K_s\left(z\right)K_p\left(z\right)-z^3\eta_{as}\eta_{sp}\eta_{pa}]$ is of 5^{th} degree in z. So this must have five zeros. We now prove that it has three zeros inside and two zeros outside the unit circle. $K_a(z)=[z\{\lambda_a(1-z)+\mu+\eta_{as}\}-\mu]$ has two zeros, viz., α_1 and α_2, whose values are given by

$$\alpha_1=\frac{1}{2\lambda_a}[(\lambda_a+\mu+\eta_{as})-\sqrt{\{(\lambda_a+\mu+\eta_{as})^2-4\lambda_a\mu\}}]$$
$$\alpha_2=\frac{1}{2\lambda_a}[(\lambda_a+\mu+\eta_{as})+\sqrt{\{(\lambda_a+\mu+\eta_{as})^2-4\lambda_a\mu\}}]$$

As proved earlier $K_a(z,s)$ has two real zeros, one inside and other outside unit circle $|z|=1$. Therefore, we say α_1 is inside and α_2 is outside of unit circle$|z|=1$. $K_s(z)=[z\{\lambda_s(1-z)+\mu+\eta_{sp}\}-\mu]$, has two real zeros, viz., α_3 and α_4, whose values are given by

$$\alpha_3=\frac{1}{2\lambda_s}[(\lambda_s+\mu+\eta_{sp})-\sqrt{\{(\lambda_s+\mu+\eta_{sp})^2-4\lambda_s\mu\}}]$$
$$\alpha_4=\frac{1}{2\lambda_s}[(\lambda_s+\mu+\eta_{sp})+\sqrt{\{(\lambda_s+\mu+\eta_{sp})^2-4\lambda_s\mu\}}]$$

By the reasoning given earlier α_3 is inside and α_4 is outside of unit circle$|z| = 1$. $K_p(z) = \{z(\mu + \eta_{pa}) - \mu\}$ has one zero viz., α_5

$$\alpha_5 = \frac{\mu}{(\mu + \eta_{pa})},$$

which is clearly inside of unit circle $|z| = 1$. So, we conclude that factor $K_a(z, s)K_s(z)K_p(z)$ has three zeros, α_1, α_3 and α_5 inside and two zeros α_2 and α_4 are outside of unit circle $|z| = 1$. Studying 1.23, a factor $(z-1)$ is common in numerator and denominator of $P(z)$. We cancel this factor. The denominator of $P(z)$ will now have four zeros, two inside and two outside of unit circle. We now proceed to prove that denominator of $P(z)$ has two real zeros outside the unit circle $|z| = 1$. Let

$$\begin{aligned} f(z) &= K_a(z)\, K_s(z)\, K_p(z) - z^3\eta_{as}\eta_{sp}\eta_{pa} \\ &\equiv (z-\alpha_1)(z-\alpha_2)(z-\alpha_3)(z-\alpha_4)(z-\alpha_5) - \frac{z^3\eta_{as}\eta_{sp}\eta_{pa}}{\lambda_a\lambda_s(\mu+\eta_{pa})} \end{aligned} \tag{33}$$

Dividing $f(z)$ by $(z-1)$ and taking limit as z tends to infinity, we find that

$$\lim_{z\to\infty} \frac{f(z)}{z-1} > 0$$

If we take limit as z tends to 1. Then

$$\lim_{z\to 1} \frac{f(z)}{z-1} = \{\mu(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as}) - \eta_{pa}(\lambda_s\eta_{as} + \lambda_a\eta_{sp})\}$$

This is obtained by using L' Hospital's rule. For $\frac{f(z)}{(z-1)}$ to have even number of real zeros between 1 and ∞, $\lim_{z\to 1}\frac{f(z)}{z-1} > 0$, i.e.,

$$\{\mu(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as}) - \eta_{pa}(\lambda_s\eta_{as} + \lambda_a\eta_{sp})\} > 0 \tag{34}$$

and this must be true, as this is the condition of ergodicity, which is proved as below?

Effective arrival rate of units is $\{\lambda_a P(1) + \lambda_s Q(1)\}$, as it represents the total number of arrivals in one unit of time when the input is in working stage (active state and sick state). Total number of units served by the system in one unit of time are $[\mu\,\{P(1) + Q(1) + R(1)\}]$. Condition of ergodicity demands that effective arrival rate be less them effective service rate. Therefore,

$$\{\lambda_a P(1) + \lambda_s Q(1)\} < \mu\,\{P(1) + Q(1) + R(1)\}\,.$$

Substituting the values of $P(1), Q(1)$ and $R(1)$ from equations (30) - (32) respectively. We obtain,

$$\{\mu(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as}) - \eta_{pa}(\lambda_s\eta_{as} + \lambda_a\eta_{sp})\} > 0 \tag{35}$$

We find that (34) and (35) are identical and this gives the condition of ergodicity. This Concludes that $\lim_{z\to\infty}\frac{f(z)}{z-1}$ and $\lim_{z\to 1}\frac{f(z)}{z-1}$ have like signs, so an even number of zeros of $f(z)$ lie in between 1 and ∞. We proceed to prove that $\frac{f(z)}{z-1}$ has two zeros say z_1 and z_2, which lie outside $|z| = 1$. Considering $\alpha_4 > \alpha_2$, we have from (33).

$$\lim_{z\to\alpha_2} \frac{f(z)}{(z-1)} = -\frac{\alpha_2^3\eta_{as}\eta_{sp}\eta_{pa}}{(\alpha_2 - 1)\,\lambda_a\lambda_s\,(\mu+\eta_{pa})} < 0$$

Sign changes between 1 and α_2. So there is a real zero, say z_1, in between 1 and α_2.

$$\lim_{z\to\alpha_4} \frac{f(z)}{(z-1)} = -\frac{\alpha_4^3\eta_{as}\eta_{sp}\eta_{pa}}{(\alpha_4 - 1)\,\lambda_a\lambda_s\,(\mu+\eta_{pa})} < 0$$

Like sign between α_2 and α_4. But there is a change of sign in between α_4 and ∞. So there is a real zero, say z_2, is between α_4 and ∞. This conclude that two real zeros of $\frac{f(z)}{z-1}$, z_1 and z_2 lie in the interval $[1, \alpha_2)$ and $[\alpha_4, \infty)$ respectively. The two zeros of the denominator in (23) which are inside $|z| = 1$ must vanish its numerator, because $P(z)$ is a well defined functions inside the unit circle. Thus, cancelling two factors in the numerator and in the denominator corresponding to these zeros, then equations (23) reduces to the following form:

$$P(z) = \frac{A}{(z - z_1)} + \frac{B}{(z - z_2)} \tag{36}$$

Where A and B are to determined. Setting $z = 1$.

$$P(1) = \frac{A}{1 - z_1} + \frac{B}{1 - z_2}$$

Using (30),

$$A = -\left[\frac{B\,(z_1 - 1)}{(z_2 - 1)} + \frac{(z_1 - 1)\,\eta_{sp}\eta_{pa}}{(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}\right]$$

Therefore, $P(z)$ in term of B is

$$P(z) = \frac{B\,(z_2 - z_1)\,(z - 1)}{(z_2 - 1)\,(z - z_2)\,(z - z_1)} - \frac{(z_1 - 1)\,\eta_{sp}\eta_{pa}}{(z - z_1)\,(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})} \tag{37}$$

$$P_n = B\left[\frac{(z_1 - 1)}{(z_2 - 1)\,z_1^{n+1}} - \frac{1}{z_2^{n+1}}\right] + \frac{(z_1 - 1)\,\eta_{sp}\eta_{pa}}{z_1^{n+1}(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}, \quad n \geq 0 \tag{38}$$

$$P_0 = \frac{B\,(z_1 - z_2)}{z_1 z_2\,(z_2 - 1)} + \frac{(z_1 - 1)\,\eta_{sp}\eta_{pa}}{z_1(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})} \tag{39}$$

Substituting the values of $P(z)$ and P_0 from (37) and (39) in equation (20).

$$\begin{aligned} R(z) = {} & \frac{B\,(z_2 - z_1)\,(z - 1)\,\{z(\mu - z_1 z_2\lambda_a) + z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}}{\eta_{pa}\,(z_2 - 1)\,z_1 z_2\,(z - z_1)\,(z - z_2)} \\ & - \frac{(z_1 - 1)\,\eta_{sp}\{z(\mu - z_1\lambda_a) + z_1(\lambda_a + \eta_{as}) - \mu\}}{(z - z_1)\,z_1(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})} \end{aligned} \tag{40}$$

$$\begin{aligned} R_n = {} & \frac{B\,(\mu - z_1 z_2\lambda_a)}{z_1 z_2\eta_{pa}}\left[\frac{(z_1 - 1)}{(z_2 - 1)\,z_1^n} - \frac{1}{z_2^n}\right] + \frac{B}{z_1 z_2\eta_{pa}}\left[\frac{(z_1 - 1)}{(z_2 - 1)\,z_1^{n+1}} - \frac{1}{z_2^{n+1}}\right]\{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\} \\ & + \frac{(z_1 - 1)\,\eta_{sp}[z_1\{\lambda_a((1 - z_1) + \mu + \eta_{as}\} - \mu]}{z_1^{n+1}(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}, \quad n \geq 0 \end{aligned} \tag{41}$$

$$R_0 = \frac{(z_1 - 1)\,\eta_{sp}\{z_1(\lambda_a + \eta_{as}) - \mu\}}{z_1^2(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})} - \frac{B\,(z_2 - z_1)\,\{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}}{z_1^2 z_2^2\eta_{pa}\,(z_2 - 1)} \tag{42}$$

Substituting the values of R(z) and R_0 from (40) and (42) in equation (22)

$$\begin{aligned} Q(z) = {} & \frac{B\,(z_2 - z_1)\,(z - 1)}{\eta_{pa}\eta_{sp}z_1^2 z_2^2\,(z_2 - 1)\,(z - z_1)\,(z - z_2)}[z_1 z_2\{z(\mu + \eta_{pa}) - \mu\}\{\mu - z_1 z_2\lambda_a\} \\ & + \{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}\{z_1 z_2(\mu + \eta_{pa}) + z\mu - \mu(z_1 + z_2)\}] \\ & - \frac{(z_1 - 1)}{z_1^2\,(z - z_1)\,(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}[\{z(\mu + \eta_{pa}) - \mu\}(\mu - z_1\lambda_a)z_1 + (z\mu + z_1\eta_{pa} - \mu)\{z_1(\lambda_a + \eta_{as}) - \mu\}] \end{aligned} \tag{43}$$

$$\begin{aligned} Q_n = {} & \frac{B}{\eta_{sp}\eta_{pa}}\left[\begin{array}{l} \frac{(z_1 - 1)(\mu - z_1 z_2\lambda_a)\{z_1(\mu + \lambda_a) - \mu\}}{z_1^{n+2}z_2(z_2 - 1)} - \frac{(\mu - z_1 z_2\lambda_a)\{z_2(\mu + \eta_{pa}) - \mu\}}{z_1 z_2^{n+2}} \\ + \frac{(z_1 - 1)\{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}\{z_1\mu + z_1 z_2(\mu + \eta_{pa}) - \mu(z_1 + z_2)\}}{z_1^{n+3}z_2^2(z_2 - 1)} \\ - \frac{\{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}\{z_2\mu + z_1 z_2(\mu + \eta_{pa}) - \mu(z_1 + z_2)\}}{z_1^2 z_2^{n+3}} \end{array}\right] \\ & + \frac{(z_1 - 1)\,\{z_1(\mu + \eta_{pa}) - \mu\}}{z_1^{n+3}(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}[\{z_1(\lambda_a + \eta_{as}) - \mu\} + z_1(\mu - z_1\lambda_a)], \quad n \geq 0 \end{aligned} \tag{44}$$

$$Q_0 = \frac{B\,(z_2 - z_1)}{z_1^3 z_2^3\,(z_2 - 1)\,\eta_{sp}\eta_{pa}} \begin{bmatrix} z_1 z_2 \mu(\mu - z_1 z_2 \lambda_a) - \{z_1 z_2(\lambda_a + \mu + \eta_{as}) \\ -\mu(z_1 + z_2)\}\{z_1 z_2(\mu + \eta_{pa}) - \mu(z_1 + z_2)\} \end{bmatrix}$$
$$+ \frac{(z_1 - 1)}{z_1^3(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}[(z_1\eta_{pa} - \mu)\{z_1(\lambda_a + \eta_{as}) - \mu\} - z_1\mu(\mu - z_1\lambda_a)] \tag{45}$$

Equations (37) - (45) give the values of $P(z), P_n, P_0; R(z), R_n, R_0$ and $Q(z), Q_n, Q_0$ respectively in terms of B. If B is known, these are all obtained explicitly. Setting $z = \alpha_5$ in equation(22), we get

$$Q(\alpha_5) = \frac{\eta_{pa}}{\eta_{sp}} R_0$$

Substituting the value of R_0 from (42)

$$Q\,(\alpha_5) = \frac{B\,(z_1 - z_2)\,\{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}}{z_1^2 z_2^2\,(z_2 - 1)\,\eta_{sp}} + \frac{\eta_{pa}\,(z_1 - 1)\,\{z_1(\lambda_a + \eta_{as}) - \mu\}}{z_1^2(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}$$

Substituting $z = \alpha_5$ in $Q(z)$ gives by (43) and equating two values of $Q\,(\alpha_5)$, thus obtained, we get

$$\frac{B\,(z_1 - z_2)}{\eta_{sp} z_2^2\,(z_2 - 1)\,\{\mu - z_2(\mu + \eta_{pa})\}} \begin{bmatrix} \{z_1 z_2(\lambda_a + \mu + \eta_{as}) - \mu(z_1 + z_2)\}[z_1 z_2(\mu + \eta_{pa})^2 + \mu^2 \\ -\mu(z_1 + z_2)(\mu + \eta_{pa})\} - \{\mu - z_1(\mu + \eta_{pa})\}\{\mu - z_2(\mu + \eta_{pa})\}] \end{bmatrix}$$
$$= \frac{(z_1 - 1)\,\{z_1(\lambda_a + \eta_{as}) - \mu\}}{(\eta_{as}\eta_{sp} + \eta_{sp}\eta_{pa} + \eta_{pa}\eta_{as})}[\{\mu^2 + z_1\eta_{pa} - \mu)(\mu + \eta_{pa})\} + \eta_{pa}\{\mu - z_1(\mu + \eta_{pa})\}] \tag{46}$$

Equation (46) gives value of Bin term of known quantities.

4. Conclusion

This research paper has discussed the queuing model with three different type inputs. From the result, we have obtained that, the time dependent and steady state probabilities formulae explicitly. To extend work in future, we consider three types service rates.

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