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Homotopy Perturbation Approach to the Solution of Non-linear Burger's Equation

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Abstract: The aim of this paper is to build another effective intermittent connection to explain nonlinear Burgers' equation. The homotopy perturbation technique is utilized to explain this equation. Burger's equation is a popular reaction diffusion equation in the biomathematics on the grounds that Burgers equations emerge in numerous applications, it is worth trying new solution methods. In this method, the solution is considered as an infinite series expansion where it converges rapidly to the exact solution. Numerical experimentation demonstrates the precision of a minimum error of order third for different space steps and coefficient of kinematic viscosity. The technique is viewed as high in accuracy. All calculations has been completed utilizing MATLAB.

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1. Introduction

Reaction-diffusion equations are helpful in numerous areas of science and engineering. In applications to populace science, the reaction term models development, and the diffusion term represents relocation. The study of nonlinear problems is of essential significance in all areas of Physics and Engineering, as well as in other disciplines. It is exceptionally hard to solve non-linear problems in general, it is frequently more tricky to get an analytic or approximation solution than a numerical one to a given systems of differential equations.

Burger's equation is well-known to demonstrate stun arrangement. It is the 1D Navier-Stokes equation without the pressure term and the volume forces. Because of its closeness to the Navier-Stokes equation, Burgers' equation regularly emerges in the mathematical modeling used to solve problems in fluid dynamics involving turbulence. Bateman [2] has first presented Burgers' equation as worthy of study and gave its steady solutions. After that Burgers [5] improved the Navier- Stokes equation by just dropping the pressure term. It was later regarded as a mathematical model for turbulence and such an equation is broadly alluded to as Burgers' equation. Since then the equation has found applications in field as diverse as number theory, gas dynamics, heat conduction, elasticity, etc. However, Cole [6] and Hopf [12] have demonstrated that the homogeneous Burger's equation lacks the most important property attributed to turbulence. The exact solutions of the one dimensional Burgers' equation have been overviewed by Benton and Platzman [4]. The homotopy perturbation method (HPM) was proposed by He [8] in 1999. This method has been utilized by numerous mathematicians and engineers

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to solve various functional equations. The homotopy perturbation technique does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an imbedding parameter p-set membership, variant [0, 1], which is considered as a "small parameter". The approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters. When p = 0, the system of equations usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p is gradually increased to 1, the system goes through a sequence of deformations, the solution for each of which is close to that at the previous stage of deformation. Eventually at p = 1, the system takes the original form of the equation and the final stage of deformation gives the desired solution. One of the most remarkable features of the HPM is that usually just few perturbation terms are sufficient for obtaining a reasonably accurate solution. Considerable research works have been conducted recently in applying this method to a class of linear and nonlinear equations. Homotopy method was further developed and improved by He and applied to nonlinear problems [9, 10], and boundary value problem [11]. It can be said that He's HPM is a universal one and is able to solve various kinds of nonlinear functional equations. For example, it was applied to hyperbolic partial differential equations [4]. In this method, the solution is considered as the summation of an infinite series which usually converges rapidly to exact solutions. Using the homotopy perturbation method Kumar et al [14, 15] gave the solution of reaction diffusion equation. This homotopy perturbation method will become a much more interesting method to solve nonlinear differential equations in science and engineering. Singh [17] compare the homotopy perturbation method with differential transformation method to solution of a reaction diffusion equation.In this paper, presentation a numerical solution of nonlinear reaction-diffusion Burger's equation using homotopy perturbation approach. The organization of this paper is as follows: in Section 2 the theoretical approach is presented. In Section 3 the homotopy perturbation method by He and homotopy perturbation method for the Burger's equation are determined by Section 4. In Section 5 computational results for the Burger's equation using homotopy perturbation method are presented and compare the result with exact solution by Wood [18] and last conclusion is presented in Section 6.

2. Theoretical Approach

Consider the one-dimensional Burgers' equation of the following form

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - \mu v\frac{\partial v}{\partial x} \qquad 0 \leq x \leq 1,\ 0 \leq t \leq T \tag{1}$$

With the initial condition

$$v(x, 0) = f(x), \quad 0 \leq x \leq 1 \tag{2}$$

And the boundary conditions

$$v(0, t) = g(t) \;\; and \;\; v(1, t) = h(t) \tag{3}$$

where $v = v(x, t)$ be the unknown function of x and t we are looking some domain and k be a parameter $(k > 0)$ and $v\frac{\partial v}{\partial x}$ is the nonlinear term. The Burgers model of turbulence is a very significant fluid dynamic model and the study of this model and the theory of shock waves has been measured by many authors, and to find a conceptual thoughtful of a class of physical flows and for testing various numerical methods. The characteristic feature of Equation (1) is that it is the simplest mathematical formulation of the competition between nonlinear advection and viscous diffusion. It contains the simplest forms of the the dissipation term $k\frac{\partial^2 v}{\partial x^2}$ and nonlinear advection term $\mu v\frac{\partial v}{\partial x}$, where $\mu \in R$, and $k = 1/R$ is an arbitrary constant (R: Reynolds number and k: kinematic viscosity) for simulating the physical phenomena of wave motion, and thus find the behavior of the solution.

3. Basic Ideas of He's Homotopy Perturbation Method

To show the basic idea of homotopy perturbation techniques, let us assume the following non-linear differential equation

$$A(v) = f(r),\ r \in \Omega \tag{4}$$

With boundary condition

$$B\left(v, \frac{\partial v}{\partial n}\right) = 0,\ r \in \Gamma, \tag{5}$$

Where A be a general differential operator, B is represent a boundary operator, $f(r)$ is a known analytical function, and Γ is the boundary of the domain Ω. Generally, the operator a break into two parts L and N, where L is called linear and N is called non-linear operator therefore equation (1) can be rewrite the following form.

$$L(v) + N(v) - f(r) = 0 \tag{6}$$

We constructor a homotopy $u(r,p) : \Omega \times [0,1] \to R$, that satisfies

$$H(u,p) = (1-p)\left[L(u) - L(v_0)\right] + p\left[A\left(u\right) - f\left(r\right)\right] = 0, \quad p \in [0,1], \quad r \in \Omega, \tag{7}$$

Or, equivalently,

$$H(u,p) = L(u) - L(v_0) + pL(v_0) + p[N(u) - f(r)] = 0, \tag{8}$$

Where v_0 is an initial estimate value to the solution of Equation (1) In this method we use the homotopy parameter p to develop v as the power series

$$u = u_0 + pu_1 + p^2 u_2 + \dots \tag{9}$$

The approximate solution can be obtain by putting $p = 1$,

$$v = \lim_{p\to 1} u = u_0 + u_1 + u_2 + \dots \tag{10}$$

The combination of the homotopy method and perturbation method are known as homotopy perturbation method (HPM), which lessens the restrictions of the traditional perturbation methods. On the other hand, this technique can have complete benefits of the traditional perturbation techniques.

4. Homotopy Perturbation Method for Burger's Equation

In this section, we Consider Burgers' equation of the following form

$$\frac{\partial v}{\partial t} = k\frac{\partial^2 v}{\partial x^2} - v\frac{\partial v}{\partial x} \tag{11}$$

Subject to the initial and boundary conditions are

$$v\left(x,0\right) = f\left(x\right), \quad 0 \le x \le 1 \tag{12}$$

$$v(0,t) = g(t) \quad and \quad v(1,t) = h(t) \tag{13}$$

To solving Equation (11) with the conditions by homotopy perturbation method, we construct a homotopy as follows:

$$(1-p)\left(\frac{\partial u}{\partial t}-\frac{\partial v_0}{\partial t}\right)+p\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-k\frac{\partial^2 u}{\partial x^2}\right)=0 \tag{14}$$

or

$$\frac{\partial u}{\partial t}-\frac{\partial v_0}{\partial t}=p\left(k\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial t}-u\frac{\partial u}{\partial x}\right) \tag{15}$$

Assume the solution of equation (15) in the form:

$$v=v_0+pv_1+p^2v_2+p^3v_3+\dots \tag{16}$$

Substituting equation (16) into equation (15) and equating the term of same power of p, it follows that

$$p^0:\ \frac{\partial u_0}{\partial t}-\frac{\partial u_0}{\partial t}=0 \tag{17}$$

$$p^1:\ \frac{\partial u_1}{\partial t}=k\frac{\partial^2 u_0}{\partial x^2}-u_0\frac{\partial u_0}{\partial x}-\frac{\partial u_0}{\partial t}, u_1\left(x,0\right)=0 \tag{18}$$

$$p^2:\ \frac{\partial u_2}{\partial t}=k\frac{\partial^2 u_1}{\partial x^2}-u_0\frac{\partial u_1}{\partial x}-u_1\frac{\partial v_0}{\partial x}, u_2\left(x,0\right)=0 \tag{19}$$

$$p^3:\ \frac{\partial u_3}{\partial t}=\varepsilon\frac{\partial^2 u_2}{\partial x^2}-u_0\frac{\partial u_2}{\partial x}-u_1\frac{\partial u_1}{\partial x}-u_2\frac{\partial u_0}{\partial x}, u_3\left(x,0\right)=0 \tag{20}$$

$$\vdots \tag{21}$$

$$p^i:\ \frac{\partial u_i}{\partial t}=k\frac{\partial^2 u_{i-1}}{\partial x^2}-\sum_{j=0}^{i-1}u_j\frac{\partial u_{i-j-1}}{\partial x}, u_i\left(x,0\right)=0 \tag{22}$$

We can start with $u_0\left(x,0\right)=v_0\left(x,0\right)=f\left(x\right)$, and all above non-linear equations can be solve of the following recurring relation, and we will have all the solutions.

$$u_i=\int_0^t\left(\frac{\partial^2 u_{i-1}}{\partial x^2}-\sum_{j=0}^{i-1}u_j\frac{\partial u_{i-j-1}}{\partial x}\right)dt \tag{23}$$

The solution of equation (11) can be obtain by putting $p=1$ in the equation (16).

$$v=\lim_{p\to 1}u=u_0+u_1+u_2+\dots \tag{24}$$

5. Numerical Experiment and Discussion

We present numerical results to illustrate the effectiveness of the proposed method. Consider Burger's Equation (1) with the following initial and boundary conditions

$$v\left(x,0\right)=\frac{2k\pi\sin(\pi x)}{\alpha+\cos(\pi x)} \tag{25}$$

$$v\left(0,t\right)=v\left(1,t\right)=0 \tag{26}$$

The exact solution of Equation (1) with the above conditions was given in [1] as

$$v\left(x,t\right)=\frac{2\pi k e^{-\pi^2 kt}\sin(\pi x)}{\alpha+e^{-\pi^2 kt}\cos(\pi x)} \tag{27}$$

Computations of Absolute and Percentage Errors:

Absolute errors of the method were computed by use of the formula:

$$|v_{ij} - v(x_i, t_j)|$$

Where v_{ij} be the approximate value at the grid point (x_i, t_j) and $v(x_i, t_j)$ be the exact value of the grid point (x_i, t_j). And the percentage errors compute by

$$\left(\frac{|v_{ij} - v(x_i, t_j)|}{v(x_i, t_j)}\right) \times 100$$

In this paper, Homotopy perturbation method is used to solve the Burger's equation numerically. Figure 1 is represent a 3-D plot of the surface $v\,(x,t) = \frac{2\pi k e^{-\pi^2 kt} \sin(\pi x)}{\alpha + e^{-\pi^2 kt} \cos(\pi x)}$ for $0 \leq x \leq 1$, $0 \leq t \leq 1$, $\alpha = 2$, $k = 0.1$, while the Figure 2 is a 3 dimensional plot of the surface $v\,(x,t) = \frac{2\pi k e^{-\pi^2 kt} \sin(\pi x)}{\alpha + e^{-\pi^2 kt} \cos(\pi x)}$ for $0 \leq x \leq 1$, $0 \leq t \leq 1$, $\alpha \leq 2$, $k = 1$. Table 1 represent the approximate solution of $v\,(x,t)$ by HPM and calculate the Error and Percentage error at $t = 0.2$, $a = 2$ and kinematic viscosity $k = 0.1$, while the Table 2 be the approximate solution of $v\,(x,t)$ by HPM at $t = 0.2$, $a = 2$ and kinematic viscosity $k = 1$, and represent the absolute Error and Percentage error for different value of x.

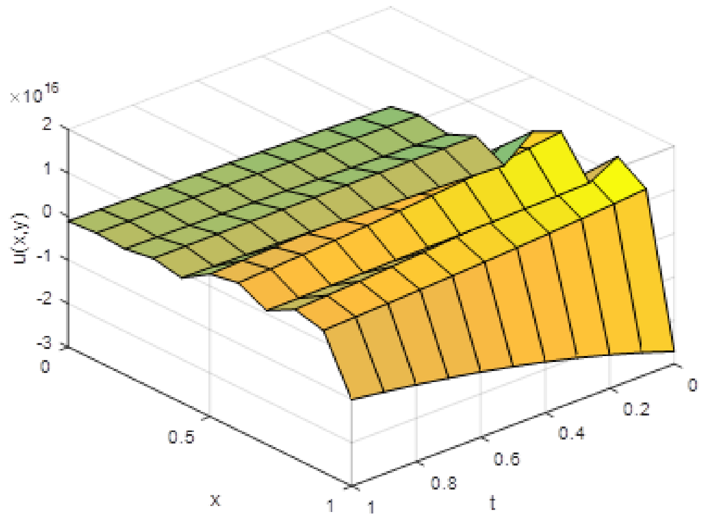

Figure 1. **A 3 dimensional plot of the surface** $v\,(x,t) = \frac{2\pi k e^{-\pi^2 kt} \sin(\pi x)}{\alpha + e^{-\pi^2 kt} \cos(\pi x)}$ **for** $0 \leq x \leq 1$, $0 \leq t \leq 1$, $\alpha = 2$, $k = 0.1$

x	By HPM	Exact Value	Error	Relative Error	Percentage Error
0.1	0.05731694	0.057357632	0.0000406962	0.000709516	0.070951645
0.2	0.11379476	0.113879096	0.0000843353	0.000740569	0.074056887
0.3	0.16808266	0.168216351	0.000133689	0.000794744	0.079474379
0.4	0.2176562	0.217846912	0.000190712	0.00087544	0.08754404
0.5	0.25788351	0.25813816	0.000254646	0.000986474	0.098647355
0.6	0.28088703	0.281204727	0.000317695	0.001129763	0.112976271
0.7	0.27496724	0.275325196	0.000357957	0.001300125	0.13001252
0.8	0.22693279	0.227268433	0.000335645	0.001476864	0.147686376
0.9	0.13071418	0.130926031	0.00021185	0.001618088	0.161808828

Table 1. **Approximate solution by HPM Error and Percentage error at** $t = 0.2$, $k = 0.1$ **and** $a = 2$

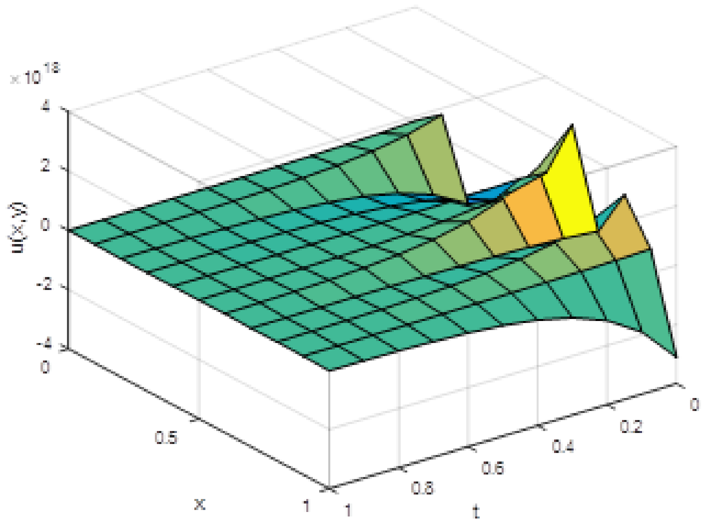

Figure 2. **A 3 dimensional plot of the surface** $v\left(x,t\right)=\frac{2\pi ke^{-\pi^2kt}\sin(\pi x)}{\alpha+e^{-\pi^2kt}\cos(\pi x)}$ **for** $0\leq x\leq 1$**,** $0\leq t\leq 1$**,** $\alpha=2$**,** $k=1$

x	By HPM	Exact Value	Error	Relative Error	Percentage Error
0.1	0.12649961	0.12661678	0.000117165	0.000925349	0.092534857
0.2	0.24286406	0.24309111	0.000227045	0.000933992	0.093399187
0.3	0.33920862	0.33953042	0.0003218	0.00094778	0.094778046
0.4	0.40632228	0.40671506	0.000392783	0.000965746	0.09657458
0.5	0.4364022	0.43683312	0.000430924	0.000986474	0.098647355
0.6	0.42414659	0.42457461	0.000428018	0.001008111	0.100811058
0.7	0.3680838	0.36846275	0.000378949	0.00102846	0.102846041
0.8	0.27178243	0.27206679	0.000284365	0.001045204	0.104520435
0.9	0.1443938	0.14454647	0.000152676	0.001056245	0.105624506

Table 2. **Approximate solution by HPM Error and Percentage error at** $t=0.2$**,** $k=1$ **and** $a=2$

Table 3 summarized the numerical result of $v(x,0.001)$ obtain by HPM for the different value of the kinematic viscosity $k=0.1$, $k=0.2$, $k=0.5$, $k=1$, $a=2$ and compare with exact solution Wood, W.L. (2006). Figure 3 represent the 2dimension plot of $v(x,0.001)$ at different value of the kinematic viscosity $k=0.1$, $k=0.2$, $k=0.5$, $k=1$. If x is increase the corresponding value of v is also increase and after some interval v is decrease for all value of kinematic viscosity.

x	$k=0.1$		$k=0.2$		$k=0.5$		$k=1$	
	HPM	Exact	HPM	exact	HPM	exact	HPM	Exact
0.1	0.06575	0.06575	0.13141	0.13141	0.32787	0.32787	0.65354	0.65355
0.2	0.13138	0.13138	0.26258	0.26258	0.65507	0.65507	1.30553	1.30554
0.3	0.19628	0.19628	0.39226	0.39226	0.97841	0.97842	1.94936	1.94938
0.4	0.25858	0.25858	0.51671	0.51671	1.28846	1.28847	2.56592	2.56595
0.5	0.31385	0.31385	0.62708	0.62708	1.56306	1.56307	3.11074	3.11077
0.6	0.35297	0.35297	0.70512	0.70512	1.75664	1.75665	3.49287	3.49291
0.7	0.35944	0.35944	0.71788	0.71788	1.78721	1.78722	3.5496	3.54964
0.8	0.30958	0.30958	0.61814	0.61814	1.53769	1.53771	3.05013	3.05018
0.9	0.18475	0.18475	0.36881	0.36882	0.91686	0.91687	1.81666	1.81669

Table 3. **Approximate solution by HPM and exact Solution** $v\left(x,0.001\right)$ **at different value of** k **and** $a=2$

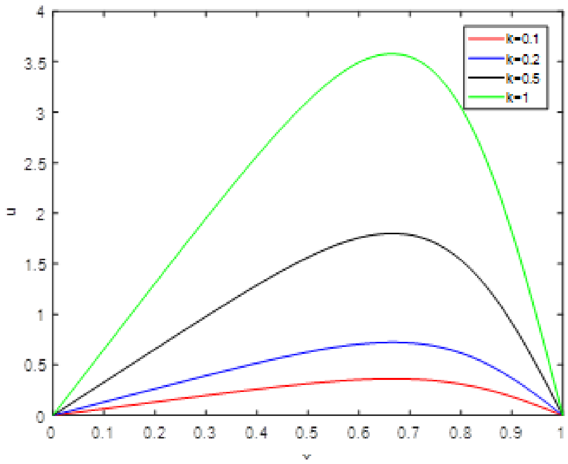

Figure 3. **A 2-D plot of the surface** $v(x,t) = \frac{2\pi k e^{-\pi^2 kt}\sin(\pi x)}{\alpha + e^{-\pi^2 kt}\cos(\pi x)}$ **for** $0 \le x \le 1$, $t = 0.001$, $\alpha = 2$, **for different value of** k**.**

Table 4 summarized the numerical result of $v(x,t)$ at $t = 0.1$ obtain by HPM for the different value of the kinematic viscosity $k = 0.1$, $k = 0.2$, $k = 0.5$, $k = 1$, $a = 2$ and compare with exact solution Wood, W.L. (2006). Figure 3 represent the 2 dimension plot of $v(x, 0.001)$ at different value of the kinematic viscosity $k = 0.1$, $k = 0.2$, $k = 0.5$, $k = 1$. If x is increase the corresponding value of v is also increase and after some interval v is decrease for all value of kinematic viscosity.

x	$k = 0.1$		$k = 0.2$		$k = 0.5$		$k = 1$	
	present	Exact	Present	exact	present	exact	present	exact
0.1	0.06147	0.06151	0.11463	0.1148	0.22966	0.22975	0.30735	0.30761
0.2	0.12243	0.12252	0.22759	0.22793	0.45204	0.45222	0.59807	0.59858
0.3	0.18185	0.18199	0.33617	0.3367	0.6578	0.65807	0.85376	0.85452
0.4	0.23746	0.23767	0.43531	0.43608	0.83342	0.83379	1.05295	1.05394
0.5	0.28463	0.28492	0.51577	0.51679	0.95897	0.95944	1.1709	1.17205
0.6	0.31477	0.31513	0.56177	0.56305	1.00702	1.00757	1.18163	1.18287
0.7	0.31384	0.31426	0.54993	0.55137	0.94546	0.94602	1.0638	1.06498
0.8	0.26409	0.2645	0.45387	0.45521	0.74851	0.749	0.81042	0.81136
0.9	0.15454	0.15481	0.26143	0.26228	0.41756	0.41785	0.43977	0.4403

Table 4. **Approximate solution by HPM and exact solution** $v(x, 0.1)$ **at different value of** k **and** $a = 2$

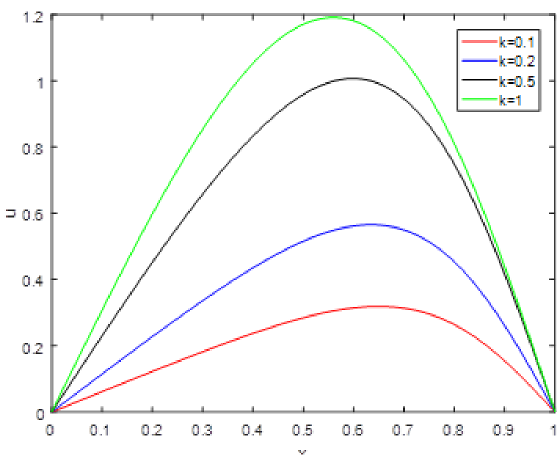

Figure 4. **A 2-D plot of the surface** $v(x,t) = \frac{2\pi k e^{-\pi^2 kt}\sin(\pi x)}{\alpha + e^{-\pi^2 kt}\cos(\pi x)}$ **for** $0 \le x \le 1$, $t = 0.1$, $\alpha = 2$, **for different value of** k

6. Conclusions

In this paper, the Homotopy perturbation Method is proposed for solutions of nonlinear Burgers equations. The small amount of computation compared to that required in numerical methods [19], and the rapid convergence show that the method is reliable and provides a significant improvement in solving partial differential equations over existing methods. The present method reduces the computational difficulties of the other methods and all the calculations can be made with simple manipulations. The solutions introduced in this study can be used to obtain the closed form of the solutions if they are required. Thus, it can be concluded that the HPM is very powerful and efficient in finding analytical as well as numerical solutions for wide classes of partial differential equations. The computations related work of this paper was performed by MATLAB.

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