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Strong Implicative and Fuzzy Strong Implicative Filters of Lattice Wajsberg Algebras

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Abstract: In this paper, we consider and discuss some properties of strong implicative filter of lattice Wajsberg algebra. Also, we introduce the notion of fuzzy strong implicative filter and investigate some properties with interesting illustrations. Moreover, we obtain the relation between a strong implicative filter and fuzzy strong implicative filter in lattice Wajsberg algebra.

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1. Introduction

Zadeh [7] introduced the concept and fuzzy set in 1965. Since then this idea has been applied to other algebraic structure such as groups, semigroups, rings, modules, vector spaces and topologies. With development of fuzzy set, it is widely used in many fields. Filters correspond to sets of formulae closed with respect to Modus Ponens. So filters are important tools in studying non-classical algebras. From the logical point view, various filters correspond to various sets of provable formulae. At present, there are two way to generalize the types of filters: one is fuzzy set theory and the other is folding theory. In the fuzzy approach, a lot of work has been done fuzzy filters which are particular cases of fuzzy set. Mordchaj Wajsbreg [6] introduced the concept of Wajsberg algebras in 1935 and studied by Font, Rodriguez and Torrens in [5]. They [5] defined lattice structure of Wajsberg algebras. Further, they [5] introduced the notion of an implicative filter of lattice Wajsberg algebras and discussed some of their properties. Basheer Ahamed and Ibrahim [1, 2] introduced the definitions of fuzzy implicative filter and an anti fuzzy implicative filter of lattice Wajsberg algebras and obtained some properties with illustrations. Recently, the authors [3] introduced the notions of implicative and strong implicative filters of lattice Wajsberg algebra, and investigated some properties of them. In the present paper, we discuss the properties of strong implicative filter. We introduce the notion of fuzzy strong implicative filter of lattice Wajsberg algebra, and investigate some properties with illustrations. Further, we discuss some related properties of strong implicative and fuzzy strong implicative filters of lattice Wajsberg algebra with useful examples.

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2. Preliminaries

In this section, we review some basic definitions and properties which are helpful our subsequent discussions.

Definition 2.1 ([5]). *Let $(A, \rightarrow, *, 1)$ be an algebra with a quasi complement "$*$" and a binary operation "$\rightarrow$" is called Wajsberg algebra if and only if it satisfies the following axioms for all $x, y, z \in A$,*

(1). $1 \rightarrow x = x$

(2). $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1.$

(3). $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x.$

(4). $(x^* \rightarrow y^*) \rightarrow (y \rightarrow x) = 1.$

Proposition 2.2 ([5]). *The Wajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,*

(1). $x \rightarrow x = 1.$

(2). If $x \rightarrow y = y \rightarrow x = 1$ *then* $x = y.$

(3). $x \rightarrow 1 = 1.$

(4). $x \rightarrow (y \rightarrow x) = 1.$

(5). If $x \rightarrow y = y \rightarrow z = 1$ *then* $x \rightarrow z = 1$

(6). $(x \rightarrow y) \rightarrow ((z \rightarrow x) \rightarrow (z \rightarrow y)) = 1$

(7). $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z).$

(8). $x \rightarrow 0 = x \rightarrow 1^* = x^*.$

(9). $(x^*)^* = x.$

(10). $x^* \rightarrow y^* = y \rightarrow x.$

Definition 2.3 ([5]). *The Wajsberg algebra A is called a lattice Wajsberg algebra if it satisfies the following axioms for all $x, y \in A$,*

(1). A partial ordering "$\leq$" on a lattice Wajsberg algebra A, such that $x \leq y$ if and only if $x \rightarrow y = 1$.

(2). $(x \vee y) = (x \rightarrow y) \rightarrow y.$

(3). $(x \wedge y) = ((x^* \rightarrow y^*) \rightarrow y^*)^*$. *Thus, we have $(A, \vee, \wedge, *, 0, 1)$ is a lattice Wajsberg algebra with lower bound 0 and upper bound 1.*

Proposition 2.4 ([5]). *The Wajsberg algebra $(A, \rightarrow, *, 1)$ satisfies the following axioms for all $x, y, z \in A$,*

(1). If $x \leq y$ *then* $x \rightarrow z \geq y \rightarrow z.$

(2). If $x \leq y$ *then* $z \rightarrow x \leq z \rightarrow y.$

(3). $x \leq y \rightarrow z$ *if and only if* $y \leq x \rightarrow z.$

(4). $(x \vee y)^* = (x^* \wedge y^*).$

(5). $(x \wedge y)^* = (x^* \vee y^*)$.

(6). $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

(7). $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$.

(8). $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

(9). $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$.

(10). $(x \wedge y) \rightarrow z = (x \rightarrow y) \vee (x \rightarrow z)$.

(11). $(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)$.

(12). $(x \wedge y) \rightarrow z = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.5 ([4]). *A lattice Wajsberg algebra A is called a lattice H-Wajsberg algebra, if it satisfies $x \vee y \vee ((x \wedge y) \rightarrow z) = 1$ for all $x, y, z \in A$. In a lattice H-Wajsberg algebra A, the following hold*

(1). $x \rightarrow (x \rightarrow y) = (x \rightarrow y)$.

(2). $x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z)$.

Definition 2.6 ([5]). *Let A be a lattice Wajsberg algebra. A subset F of A is called an implicative filter of A if it satisfies the following axioms for all $x, y \in A$.*

(1). $1 \in F$.

(2). $x \in F$ *and* $x \rightarrow y \in F$ *imply* $y \in F$.

Definition 2.7 ([3]). *Let A be a lattice Wajsberg algebra. A subset F of A is called a strong implicative filter of A if it satisfies the following axioms for all $x, y, z \in A$.*

(1). $1 \in F$.

(2). $x \rightarrow (y \rightarrow z) \in F$ *and* $x \rightarrow y \in F$ *imply* $x \rightarrow z \in F$.

Definition 2.8 ([7]). *Let X be a set. A function $\mu : X \rightarrow [0, 1]$ is called a fuzzy subset on X, for each $x \in X$, the value of $\mu(x)$ describes a degree of membership of x in μ.*

Definition 2.9 ([7]). *Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu_t = \{x \in A/\mu(x) \geq t\}$ is called level subset of μ.*

Definition 2.10 ([7]). *Let μ be a fuzzy set in a set A. Then for $t \in [0, 1]$, the set $\mu^t = \{x \in A/\mu(x) \leq t\}$ is called the lower t-level cut of μ.*

Definition 2.11 ([1]). *Let A be a lattice Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy implicative filter of A if it satisfies the following axioms for all $x, y \in A$.*

(1). $\mu(1) \geq \mu(x)$.

(2). $\mu(z) \geq \min\{\mu(y), \mu(y \rightarrow z)\}$.

Proposition 2.12 ([1]). *Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A, then for all $x, y \in A$, $x \leq y$ implies $\mu(x) \leq \mu(y)$.*

Proposition 2.13 ([3]). *Let A be a lattice Wajsberg algebra, F_1 and F_2 are any two implicative filters of A, $F_1 \subseteq F_2$. If F_1 is a strong implicative filter, so is F_2.*

Definition 2.14 ([4]). *Let A_1 and A_2 be lattice Wajsberg algebras, $f : A_1 \to A_2$ be a mapping from A_1 to A_2, if for any $x, y \in A_1$, $f(x \to y) = f(x) \to f(y)$ holds, then f is called an implication homomorphism from A_1 to A_2.*

Definition 2.15 ([4]). *Let A_1 and A_2 be lattice Wajsberg algebras, $f : A_1 \to A_2$ be an implication homomorphism from A_1 to A_2, is called a lattice implication homomorphism from A_1 to A_2 if it satisfies the following axioms for all $x, y \in A_1$,*

(1). $f(x \wedge y) = f(x) \wedge f(y)$.

(2). $f(x \vee y) = f(x) \vee f(y)$.

(3). $f(x^*) = [f(x)]^*$.

3. Main Results

3.1. Fuzzy implicative filter

In this section, we obtain some properties of fuzzy implicative filters.

Proposition 3.1. *Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A if and only if it satisfies the following for any $x, y, z \in A$,*

(1). $\mu(1) \geq \mu(x)$.

(2). $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(y \to z)\}$.

Proof. Let μ be a fuzzy implicative filter of A, from (i) of Definition 2.11, we get (1) and for any $x, y, z \in A$, we have

$$\mu(x \to z) \geq \min\{\mu(y \to z), \mu((y \to z) \to (x \to z))\} \tag{1}$$

Now,

$$\begin{aligned}(x \to y) \to ((y \to z) \to (x \to z)) &= (x \to y) \to ((z \to y) \to (x \to y)) \\ &= (z \to y) \to ((x \to y) \to (x \to y)) \\ &= (z \to y) \to 1 = 1.\end{aligned}$$

Then, from (1) of Definition 2.3, we have $(x \to y) \leq (y \to z) \to (x \to z)$. From the Proposition 2.12, we get

$$\mu((y \to z) \to (x \to z)) \geq \mu(x \to y) \tag{2}$$

Using (2) in (1), we have $\mu(x \to z) \geq \min\{\mu(y \to z), \mu((y \to z) \to (x \to z))\} \geq \min\{\mu(x \to y), \mu(y \to z)\}$.
Conversely, if μ satisfies (1) and (2), for any $x \in A$, then from (1), we have $\mu(1) \geq \mu(x)$. For any $x, y, z \in A$, put $x = 1$ in (2), we get $\mu(1 \to z) \geq \min\{\mu(1 \to y), \mu(y \to z)\}$. Then $\mu(z) \geq \min\{\mu(y), \mu(y \to z)\}$. Thus μ is a fuzzy implicative filter of a lattice Wajsberg algebra A. □

Proposition 3.2. *Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A if and only if it satisfies the following for any $x, y, z \in A$,* *(1).* $\mu(1) \geq \mu(x)$.

(2). $\mu(z) \geq \min\{\mu(x), \mu(y), \mu(x \to (y \to z))\}$.

Proof. Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A, then (1) is obvious and for any $x, y, z \in A$, $\mu(z) \geq \min\{\mu(x), \mu(x \to z)\}$, $\mu(x \to z) \geq \min\{\mu(y), \mu(y \to (x \to z))\} = \min\{\mu(y), \mu(x \to (y \to z))\}$. (By (7) of Proposition 2.2). Hence, we have $\mu(z) \geq \min\{\mu(x), \mu(y), \mu(x \to (y \to z))\}$.

Conversely, μ satisfies (1) and (2), for any $x \in A$, from (1), we have $\mu(1) \geq \mu(x)$. For any $x, y, z \in A$, put $x = y$ in (2) then we have $\mu(z) \geq \min\{\mu(x), \mu(x \to (x \to z)\}$. From $(x \to z) \to (x \to (x \to z)) = 1$ and from (1) of Definition 2.3 and Proposition 2.12, we get $x \to z \leq x \to (x \to z)$ and $\mu(x \to z) \leq \mu((x \to (x \to z))$. Thus $\mu(z) \geq \min\{\mu(x), \mu(x \to (x \to z)\} \geq \min\{\mu(x), \mu(x \to z)\}$. Therefore $\mu(z) \geq \min\{\mu(x), \mu(x \to z)\}$. Hence μ is a fuzzy implicative filter of a lattice Wajsberg algebra A. □

Proposition 3.3. *Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A if and only if it satisfies the following for any $x, y, z \in A$,*

(1). $\mu(1) \geq \mu(x)$.

(2). $\mu(z \to x) \geq \min\{\mu((z \to y) \to x), \mu(y)\}$.

Proof. Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A, then (1) is obvious,

To Prove: $\mu(z \to x) \geq \min\{\mu((z \to y) \to x), \mu(y)\}$.

For any $x, y, z \in A$, we have

$$\mu(z \to x) \geq \min\{\mu(y \to (z \to x)), \mu(y)\} \tag{3}$$

Now for any $x, y, z \in A$,

$$\begin{aligned}((z \to y) \to x) \to ((z \to y) \to (z \to x)) &= ((z \to y) \to x) \to (z \to ((z \to y) \to x)) \\ &= z \to (((z \to y) \to x) \to ((z \to y) \to x)) \\ &= z \to 1 = 1.\end{aligned}$$

From (1) of Definition 2.3, we get $((z \to y) \to x) \leq (z \to y) \to (z \to x)$. Then from the Proposition 2.12, we have $\mu((z \to y) \to x) \leq \mu((z \to y) \to (z \to x))$. We know that $z \to (y \to x) \geq (z \to y) \to (z \to x)$. Then from the Proposition 2.12, we get $\mu(z \to (y \to x)) \geq \mu((z \to y) \to (z \to x))$. Thus $\mu(z \to (y \to x)) \geq \mu((z \to y) \to x)$. From (7) of Proposition 2.2, we have

$$\mu(y \to (z \to x)) \geq \mu((z \to y) \to x) \tag{4}$$

Using (4) in (3), we have $\mu(z \to x) \geq \min\{\mu((z \to y) \to x), \mu(y)\}$.

Conversely, μ satisfies (1) and (2). Obviously, $\mu(1) \geq \mu(x)$. Taking $z = 1$ in (2) and from the (1) of Definition 2.1, we get $\mu(x) \geq \min\{\mu(y), \mu(y \to x)\}$. Hence μ is a fuzzy implicative filter of A. □

Proposition 3.4. *Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A. If $x \leq y \to z$ for any $x, y, z \in A$, then $\mu(z) \geq \min\{\mu(x), \mu(y)\}$.*

Proof. Let μ be a fuzzy implicative filter of a lattice Wajsberg algebra A, for any $x, y, z \in A$, $\mu(z) \geq \min\{\mu(y), \mu(y \to z)\}$. Given that $x \leq y \to z$ then from the Proposition 2.12, we have $\mu(y \to z) \geq \mu(x)$. Hence $\mu(z) \geq \min\{\mu(y), \mu(y \to z)\} \geq \min\{\mu(y), \mu(x)\}$. That is, $\mu(z) \geq \min\{\mu(x), \mu(y)\}$. □

3.2. Fuzzy strong implicative filter

In this section, we introduce fuzzy strong implicative filter of lattice Wajsberg algebra and investigate some properties with illustrations.

Definition 3.5. *Let A be a lattice Wajsberg algebra. A fuzzy subset μ of A is called a fuzzy strong implicative filter of A if it satisfies the following for all $x, y, z \in A$,*

(1). $\mu(1) \geq \mu(x)$.

(2). $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$.

Example 3.6. *Consider a set $A = \{0, a, b, c, d, 1\}$ with Figure 1 as a partial ordering. Define a quasi complement "$*$" and a binary operation "$\to$" on A as in Tables 1 and 2.*

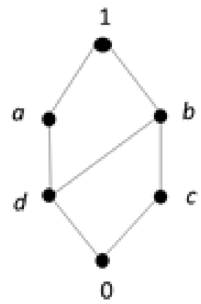

Figure 1:

x	x^*
0	1
a	c
b	d
c	a
d	b
1	0

Table 1: Complement

$\longrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	a	c	c	1
b	c	1	1	c	c	1
c	b	a	b	1	a	1
d	a	1	a	1	1	1
1	0	a	b	c	d	1

Table 2: Implication

Define $\vee$ and $\wedge$ operations on A as follow, $(x \vee y) = (x \to y) \to y, (x \wedge y) = ((x^* \to y^*) \to y^*)^*$ for all $x, y \in A$. Then, A is a lattice Wajsberg algebra. Consider the fuzzy subset μ on A as,

$$\mu(x) = \begin{cases} 0.7 & \textit{if } x = 1 \\ 0.3 & \textit{otherwise} \end{cases}$$

for all $x \in A$. Then μ is fuzzy strong implicative filter. In the same Example 3.2, consider the fuzzy subset μ on A as,

$$\mu(x) = \begin{cases} 0.8 & \textit{if } x \in \{1, a, c\} \\ 0.4 & \textit{if } x \in \{0, b, d\} \end{cases}$$

Then, we have μ is not fuzzy strong implicative filter of lattice Wajsberg algebra A. Since, we have $\mu(c \to b) = 0.4$. But, $\min\{\mu(c \to d), \mu(c \to (d \to b)\} = \min\{\mu(a), \mu(1)\} = 0.8$. *Therefore, we get $\mu(c \to b) \ngeq \min\{\mu(c \to d), \mu(c \to (d \to b)\}$.*

Proposition 3.7. *Let A be a lattice Wajsberg algebra, and let μ be a fuzzy strong implicative filter of A, then μ is a fuzzy implicative filter of A.*

Proof. Let μ be a fuzzy strong implicative filter of a lattice Wajsberg algebra A then $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$. Take $x = 1$, then from (1) of Definition 2.1, we have $\mu(z) \geq \min\{\mu(1 \to y), \mu(1 \to (y \to z))\} = \min\{\mu(y), \mu(y \to z)\}$. Thus μ is a fuzzy implicative filter of A. □

Proposition 3.8. *Let A be a lattice Wajsberg algebra. Then A is a lattice H-Wajsberg algebra if and only if each fuzzy filter of A is a fuzzy strong implicative filter.*

Proof. Let A be a lattice H-Wajsberg algebra and μ be a fuzzy filter of A, then from (2) of Definition 2.5, we have $x \to (y \to z) = (x \to y) \to (x \to z)$. It follows that $\min\{\mu(x \to y), \mu(x \to (y \to z))\} = \min\{\mu(x \to y), \mu((x \to y) \to (x \to z))\} \leq \mu(x \to z)$. Hence μ is a fuzzy strong implicative filter.

Conversely, assume that each fuzzy filter of A is a fuzzy strong implicative filter. Define a mapping $\mu : A \to [0, 1]$ as

$$\mu(x) = \begin{cases} 0.8 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$$

for all $x \in A$. Then μ is a fuzzy filter of A and hence it is a fuzzy strong implicative filter. It follows that $\{1\} = \mu_{0.8}$ is an implicative filter of A. This implies that A is a lattice H-Wajsberg algebra. □

Proposition 3.9. *A fuzzy subset μ of a lattice Wajsberg algebra A is a fuzzy strong implicative filter of A if and only if, for each $t \in [0, 1]$, μ_t is either empty or strong implicative filter of A.*

Proof. Let μ be a fuzzy strong implicative filter of A and $\mu_t \neq \phi$ for all $t \in [0, 1]$. It is clear that $1 \in \mu_t$ since $\mu(1) \geq t$. Let $x, y, z \in A$ be such that $x \to y \in \mu_t$ and $x \to (y \to z) \in \mu_t$. Then, we have $\mu(x \to y) \geq t$ and $\mu(x \to (y \to z)) \geq t$. Thus, we get $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\} \geq t$, that is, $x \to z \in \mu_t$. Thus μ_t is a strong implicative filter of A.

Conversely, suppose that for each $t \in [0, 1]$, μ_t is either empty or a strong implicative filter of A. For any $x \in A$, let $\mu(x) = t$. Then, we get $x \in \mu_t$. Since $\mu_t \neq \phi$ is a strong implicative filter of A, therefore $1 \in \mu_t$ and hence $\mu(1) \geq t = \mu(x)$. Thus $\mu(1) \geq \mu(x)$ for all $x \in A$. Now, it is enough to show μ satisfies $\mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\}$. If not, then there exist $x', y', z' \in A$ such that $\mu(x' \to z') < \min\{\mu(x' \to y'), \mu(x' \to (y' \to z'))\}$. Taking $t_0 = \frac{1}{2}\{\mu(x' \to z') + \min\{\mu(x' \to y'), \mu(x' \to (y' \to z'))\}\}$. Then, we have that $\mu(x' \to z') < t_0 < \min\{\mu(x\prime \to y'), \mu(y' \to (x' \to z'))\}$. Hence, $x' \to z' \notin \mu_{t_0}, x' \to y' \in \mu_{t_0}$ and $y' \to (x' \to z') \in \mu_{t_0}$, that is, μ_{t_0} is not a fuzzy implicative filter of A, which is a contradiction. Therefore, μ is a fuzzy strong implicative filter of A. □

Proposition 3.10. *Let μ and ρ be fuzzy implicative filters of a lattice Wajsberg algebra A with $\mu \leq \rho$ and $\mu(1) = \rho(1)$. If μ is a fuzzy strong implicative filter of A, then so is ρ.*

Proof. To prove: ρ is a fuzzy strong implicative filter of A. It is enough to show that for any $t \in [0, 1]$, ρ_t is either empty or a strong implicative filter of A. If the level subset ρ_t is nonempty, then $\mu_t \neq \phi$ and $\mu_t \subseteq \rho_t$, if $x \in \mu_t$, then $t \leq \mu(x)$ and so $t \leq \rho(x)$ that is, $x \in \nu_t$. Hence $\mu_t \subseteq \rho_t$. Since μ is a fuzzy strong implicative filter of A, by the hypothesis it follows from Proposition 3.4 that μ_t is an implicative filter of A. From the Proposition 2.13, we have ρ_t is also an implicative filter of A. Hence ρ is a fuzzy strong implicative filter of A. □

Proposition 3.11. *Let A be a lattice Wajsberg algebra and μ be a fuzzy subset of A, if μ is a fuzzy strong implicative filter, then the following are satisfied and equivalent:*

(1). μ is a fuzzy implicative filter and for any $x, y \in A$, $\mu(x \to y) \geq \mu(x \to (x \to y))$;

(2). μ is a fuzzy implicative filter and for any $x, y, z \in A$, $\mu((x \to y) \to (x \to z)) \geq \mu(x \to (y \to z))$;

(3). $\mu(x) \leq \mu(1)$ and for any $x, y, z \in A$, $\mu(x \to y) \geq \min\{\mu(z \to (x \to (x \to y))), \mu(z)\}$.

Proof. (1) ⇒ (2) Let μ beafuzzy strong implicative filter of A. Then from the Definition 3.5, we have

$$\mu(1) \geq \mu(x) \;\; \textit{and} \;\; \mu(x \to z) \geq \min\{\mu(x \to y), \mu(x \to (y \to z))\} \tag{5}$$

Put $x = 1$ in (5), we get $\mu(z) \geq \min\{\mu(y), \mu(y \to z)\}$ for all $x, y, z \in A$. Therefore μ is a fuzzy implicative filter of A. From $\mu(x \to z) \geq \mu(x \to (x \to z)$, put $z = y$, we have $\mu(x \to y) \geq \mu(x \to (x \to y))$. If for any $x, y, z \in A$, $\mu(x \to y) \geq \mu(x \to (x \to y))$. Then, we have $\mu((x \to y) \to (x \to z)) = \mu(x \to ((x \to y) \to z)) \geq \mu(x \to (x \to ((x \to y) \to z)))$, and $x \to (x \to ((x \to y) \to z)) = x \to ((x \to y) \to (x \to z)) = x \to ((y \to x) \to (y \to z)) \geq x \to (y \to z)$. From the Proposition 2.12, we have $\mu(x \to (x \to ((x \to y) \to z))) \geq \mu(x \to (y \to z))$. Therefore, $\mu((x \to y) \to (x \to z)) \geq \mu(x \to (y \to z))$.

(2) $\Rightarrow$ (3). Let (2) be hold. Then, it is clear that $\mu(1) \geq \mu(x)$. If for any $x, y, z \in A$, we have $\mu((x \to y) \to (x \to z)) \geq \mu(x \to (y \to z))$. Put $y = x$, then, we get

$$\begin{aligned} \mu((x \to x) \to (x \to z)) &\geq \mu(x \to (x \to z)) \\ \mu(x \to z) &\geq \mu(x \to (x \to z)) \quad \text{(From (i) of Proposition 2.2 and Definition 2.1)} \end{aligned}$$

Hence for any $x, y \in A$, $\mu(x \to y) \geq \mu(x \to (x \to y))$. Since μ is a fuzzy implicative filter and then, we have $\mu(x \to (x \to y)) \geq \min\{\mu(z \to (x \to (x \to y))), \mu(z)\}$. Hence $\mu(x \to y) \geq \min\{\mu(z \to (x \to (x \to y))), \mu(z)\}$.

(3) $\Rightarrow$ (1) Let (3) be hold. Then for any $x, y, z \in A$, we have $\mu(x \to y) \geq \min\{\mu(z \to (x \to (x \to y))), \mu(z)\}$. Put $x = 1$ then, we get $\mu(y) \geq \min\{\mu(z \to y), \mu(z)\}$ (From (1) of Proposition 2.2 and Definition 2.1). Also, we have $\mu(1) \geq \mu(x)$. Hence μ is a fuzzy implicative filter. Since for any $x, y, z \in A$, $\mu(x \to y) \geq \min\{\mu(z \to (x \to (x \to y))), \mu(z)\}$. Put $z = 1$ then, we get $\mu(x \to y) \geq \mu(x \to (x \to y))$ (From (1) of Definition 2.1 and $\mu(1) \geq \mu(x)$). □

Proposition 3.12. *Let A_1 and A_2 be any two lattice Wajsberg algebras, f is a lattice implication homomorphism from A_1 to A_2. Let μ_1 and μ_2 be the fuzzy subsets of A_1 and A_2 respectively. If μ_2 is fuzzy strong implicative filter of A_2, then $f^{-1}(\mu_2)$ is a fuzzy strong implicative filter of A_1.*

Proof. For any $x \in A_1$, from $f(1) = 1$, we have $f^{-1}(\mu_2)(x) = \mu_2(f(x)) \leq \mu_2(1) = \mu_2(f(1)) = f^{-1}(\mu_2)(1)$. That is, $f^{-1}(\mu_2)(1) \geq f^{-1}(\mu_2)(x)$ for any $x \in A_1$. For any $x, y, z \in A_1$ and $\mu_2 \subseteq A_2$ is a fuzzy strong implicative filter, we have

$$\begin{aligned} f^{-1}(\mu_2)(x \to z) &= \mu_2(f(x \to z)) \\ &= \mu_2(f(x) \to f(z)) \geq \min\{\mu_2(f(x) \to f(y)), \mu_2(f(x) \to (f(y) \to f(z)))\} \\ &= \min\{\mu_2(f(x \to y)), \mu_2(f(x) \to f(y \to z))\} \\ &= \min\{\mu_2(f(x \to y)), \mu_2(f(x \to (y \to z)))\} \\ &= \min\{f^{-1}(\mu_2)(x \to z), f^{-1}(\mu_2)(x \to (y \to z))\}. \end{aligned}$$

Hence $f^{-1}(\mu_2)$ is a fuzzy strong implicative filter of A_1. □

4. Conclusion

In this paper, we have introduced the notion of fuzzy strong implicative filter of lattice Wajsberg algebra, and discussed some of their properties with examples. Also, we have obtained a characterization of fuzzy implicative filters and fuzzy strong implicative filters in lattice Wajsberg algebras. Further, we have shown that some related properties of fuzzy implicative and fuzzy strong implicative filters of lattice Wajsberg algebra.

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