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Strictly Locating Sets in a Graph

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Abstract: Let G be a connected graph. A subset S of $V(G)$ is a locating set in G if for all $u, v \in V(G)\backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. A subset S of $V(G)$ is a strictly locating set in G if S is a locating set in G and $N_G(w) \cap S \neq S$ $\forall w \in V(G)\backslash S$. The minimum cardinality of a strictly locating set in G, denoted by $sln(G)$, is called the strictly locating number of G. In this paper, the concept of strictly locating set in a graph is investigated. Moreover, the strictly locating sets in the join and corona of graphs are characterized and the strictly locating numbers of these graphs are determined.

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1. Introduction

Let $G = (V, E)$ be a simple graph. The *open neighborhood* of a vertex v of G is defined as the set $N_G(v) = \{u \in V(G) | uv \in E(G)\}$, while the *closed neighborhood* of v in G is defined as $N_G[v] = N_G(v) \cup \{v\}$. Any vertex $u \in N_G(v)$ is called a *neighbor* of v. The *open neighborhood of a set* $S \subseteq V(G)$ is defined as $N_G(S) = \bigcup_{v \in S} N_G(v)$, while the *closed neighborhood of a set* S is defined as $N_G[S] = N_G(S) \cup S$. The *distance* $d_G(u, v)$ in G of two vertices u and v is the length of the shortest $u-v$ path in G. A subset S of $V(G)$ is a *locating set* in a connected graph G if for every two vertices u and v of $V(G)\backslash S$, $N_G(u) \cap S \neq N_G(v) \cap S$. It is a *strictly locating set* if it is a locating set and $N_G(u) \cap S \neq S$ for all $u \in V(G)\backslash S$. The minimum cardinality of a locating set in G, denoted by $ln(G)$ is called the *locating number* of G. The minimum cardinality of a strictly locating set in G, denoted by $sln(G)$, is called the *strictly locating number* of G. A locating set of minimum cardinality is called an *ln-set* in G and a strictly locating set of minimum cardinality is called an *sln-set* in G.

2. Results

The following results characterizes the strictly locating number of some graphs.

Remark 2.1. *For any connected graph G of order $n \geq 1$, $1 \leq sln(G) \leq n$.*

Theorem 2.2 ([1]). *Let G be a connected graph of order $n \geq 2$. If $ln(G) < sln(G)$, then $1 + ln(G) = sln(G)$.*

Lemma 2.3. *For any complete graph K_n of order $n \geq 1$, $sln(K_n) = n$.*

Lemma 2.4. *Let G be a connected non-trivial graph. Then $sln(G) = 1$ if and only if $G \cong K_1$.*

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Theorem 2.5. *Let G be a connected graph of order $n \geq 2$. If $sln(G) = 2$, then $2 \leq |V(G)| \leq 5$.*

Proof. Suppose that $sln(G) = 2$. By Lemma 2.4, $|V(G)| \geq 2$. Suppose that $|V(G)| > 5$. Let $S = \{x, y\}$ be a sln-set of G and let $w_i \in V(G)\backslash S$, where $i = 1, 2, 3, 4$. Since $N_G(w_i) \cap S$ is either $\varnothing, \{x\}$ or $\{y\}$ for each $i = 1, 2, 3, 4$, there exist distinct vertices $k, j \in \{1, 2, 3, 4\}$ such that $N_G(w_k) \cap S = N_G(w_j) \cap S$, contrary to the assumption that S is a strictly locating set in G. Thus, $|V(G)| \leq 5$. Therefore, $2 \leq |V(G)| \leq 5$. □

Theorem 2.6. *Let G be a non-trivial connected graph. Then $sln(G) = n$ if and only if $G = K_n$.*

Proof. Suppose that $sln(G) = n$ and suppose that $G \neq K_n$. Then $\exists\ w, v \in V(G)$ such that $d_G(w, v) = 2$. Let $y \in N_G(w) \cap N_G(v)$ and let $S = V(G)\backslash \{w\}$. Then S is a locating set in G. Since $wv \notin E(G)$, it follows that $N_G(w) \cap S \neq S$. Thus, S is a strictly locating set in G. Hence, $sln(G) \leq |S| = n - 1$, contrary to the assumption. Therefore, $G = K_n$. The converse follows from Lemma 2.3. □

Theorem 2.7. *Let G be a connected graph of order $n = 4$. Then $sln(G) = 2$ if and only if G is triangle free.*

Theorem 2.8. *Let G be a connected graph of order $n = 5$. Then $sln(G) = 2$ if and only if there exist distinct vertices x and y of G such that $|N_G(x) \cap N_G(y)| = 0$ and $|N_G(x)\backslash \{y\}| = |N_G(y)\backslash \{x\}| = 1$.*

Proof. Suppose $sln(G) = 2$. Then there exists distinct vertices x and y of G such that $S = \{x, y\}$ is a sln-set in G. Then $|N_G(x) \cap N_G(y)| = 0$. Suppose that $|N_G(x)\backslash \{y\}| = 0$. Since S is a sln-set, it follows that $|N_G(y)\backslash \{x\}| = 1$. Thus, $\exists\ u, w \in V(G)\backslash \{x, y\}$ such that $u, w \notin N_G(x) \cup N_G(y)$. Consequently, $N_G(w) \cap S = \varnothing = N_G(u) \cap S$. This is a contradiction to the assumption that S is a locating set. Therefore, $|N_G(x)\backslash \{y\}| = 1$. Similarly, $|N_G(y)\backslash \{x\}| = 1$. □

2.1. Strictly Locating Sets in the Join of Graphs

The *join* $G + H$ of two graphs G and H is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$.

Theorem 2.9. *Let G and H be connected non-trivial graphs. A set $S \subseteq V(G + H)$ is a strictly locating set in $G + H$ if and only if $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$ are strictly locating sets in G and H, respectively.*

Proof. Let $S \subseteq V(G+H)$ be a strictly locating set in $G+H$. Let $S_1 = V(G) \cap S$. Suppose that $S_1 = \varnothing$. Then for any two distinct vertices $a, b \in V(G)$, $N_{G+H}(a) \cap S = N_{G+H}(b) \cap S = S$, contrary to the assumption that S is a strictly locating set. Thus, $S_1 \neq \varnothing$. Similarly, $S_2 = V(H) \cap S \neq \varnothing$. Next, suppose that S_1 or S_2, say S_1 is not a locating set in G. Then there exist distinct vertices $u, v \in V(G)$ such that $N_G(u) \cap S_1 = N_G(v) \cap S_1$. Since $S_2 \subseteq N_{G+H}(u)$ and $S_2 \subseteq N_{G+H}(v)$, it follows that $N_{G+H}(u) \cap S = (N_G(u) \cap S_1) \cup S_2 = (N_G(v) \cap S_1) \cup S_2 = N_{G+H}(v) \cap S$. Hence, $N_{G+H}(u) \cap S = N_{G+H}(v) \cap S$. This is a contradiction since S is a strictly locating set in $G + H$. Therefore, S_1 and S_2 are locating sets in G and H, respectively. Now, suppose that S_1 or S_2 is not a strictly locating set in G and H, respectively, say S_1 is not a strictly locating set in G. Then $\exists\ y \in V(G)\backslash S_1$ such that $N_G(y) \cap S_1 = S_1$. Since $S_2 \subseteq N_{G+H}(y) \cap S$, it follows that $N_{G+H}(y) \cap S = S_1 \cup S_2 = S$. This is a contradiction since S is a strictly locating set in $G + H$. Hence, S_1 and S_2 are strictly locating sets in G and H, respectively.

For the converse, suppose that S_1 and S_2 are strictly locating sets in G and H, respectively. Let $S = S_1 \cup S_2$ and let $a, b \in V(G + H)\backslash S$ with $a \neq b$. If $a, b \in V(G)$, then $N_G(a) \cap S_1 \neq N_G(b) \cap S_1$. Thus, $N_{G+H}(a) \cap S = (N_G(a) \cap S_1) \cup S_2 \neq (N_G(b) \cap S_1) \cup S_2 = N_{G+H}(b) \cap S$. Similarly, if $a, b \in V(H)$, then $N_{G+H}(a) \cap S \neq N_{G+H}(b) \cap S$. Suppose that $a \in V(G)$ and $b \in V(H)$. Since S_1 is a strictly locating set in G, it follows that $S_1 \nsubseteq N_{G+H}(a)$. Thus, $S_1 \subseteq N_{G+H}(b)$ implies that $N_{G+H}(a) \cap S \neq N_{G+H}(b) \cap S$. Hence, $S = S_1 \cup S_2$ is a locating set in $G + H$. Finally, let $x \in V(G + H)\backslash S$. Suppose that

$x \in V(G)$. Since S_1 is a strictly locating set in G, it follows that $N_G(x) \cap S_1 \neq S_1$. Thus, $N_{G+H}(x) \cap S = (N_G(x) \cap S_1) \cup S_2 \neq S$. Similarly, if $x \in V(H)$, then $N_{G+H}(x) \cap S \neq S$. Therefore, S is a strictly locating set in $G + H$. □

Corollary 2.10. *Let G and H be connected non-trivial graphs. Then $sln(G + H) = sln(G) + sln(H)$.*

Proof. Let S be a sln-set in $G + H$ and let $S_1 = V(G) \cap S$ and $S_2 = V(H) \cap S$. By Theorem 2.9, S_1 and S_2 are strictly locating sets in G and H, respectively. Thus, $sln(G) + sln(H) \leq |S_1| + |S_2| = |S| = sln(G + H)$. Next, let S_1 be a sln-set in G and S_2 be a sln-set in H. Then $S = S_1 \cup S_2$ is a strictly locating set in $G + H$ by Theorem 2.9. Hence, $sln(G + H) \leq |S| = |S_1| + |S_2| = sln(G) + sln(H)$. Therefore, $sln(G + H) = sln(G) + sln(H)$. □

Theorem 2.11. *Let H be a connected non-trivial graph and let $K_1 = \langle v \rangle$. Then $S \subseteq V(H + K_1)$ is a strictly locating set in $H + K_1$ if and only if $v \in S$ and $S_1 = V(H) \cap S$ is a strictly locating set in H.*

Proof. Let $S \subseteq V(H + K_1)$ be a strictly locating set in $H + K_1$. Suppose that $v \notin S$. Then $N_{H+K_1}(v) \cap S = S$. This is a contradiction since S is a strictly locating set in $H + K_1$. Hence, $v \in S$. Next let $x, y \in V(H + K_1) \backslash S$. Then $x, y \in V(H) \backslash S_1$ where $S_1 = V(H) \cap S$. Since S is a strictly locating set in $H + K_1$,

$$N_{H+K_1}(x) \cap S = (N_{H+K_1}(x) \cap S_1) \cup \{v\} \neq N_{H+K_1}(y) \cap S = (N_{H+K_1}(y) \cap S_1) \cup \{v\}.$$

Hence, $N_H(x) \cap S_1 \neq N_H(y) \cap S_1$. Therefore, S_1 is a locating set in H. Now, suppose there exists $u \in V(H) \backslash S_1$ such that $N_H(u) \cap S_1 = S_1$. Then $N_{H+K_1}(u) \cap S = (N_H(u) \cap S_1) \cup \{v\} = S_1 \cup \{v\} = S$. This is a contradiction since S is a strictly locating set in $H + K_1$. Therefore, S_1 is a strictly locating set in H.

For the converse, suppose that $S = S_1 \cup \{v\}$ and $S_1 = V(H) \cap S$ is a strictly locating set in H. Let $x, y \in V(H + K_1) \backslash S = V(H) \backslash S_1$. Then $N_H(x) \cap S_1 \neq N_H(y) \cap S_1$. Thus,

$$N_{H+K_1}(x) \cap S = (N_H(x) \cap S_1) \cup \{v\} \neq (N_H(y) \cap S_1) \cup \{v\} = N_{H+K_1}(y) \cap S$$

Hence, S is a locating set in $H + K_1$. Finally, let $u \in V(H + K_1) \backslash S = V(H) \backslash S_1$. Since S_1 is a strictly locating set in H, it follows that $N_H(u) \cap S_1 \neq S_1$. Hence, $N_{H+K_1}(u) \cap S = (N_H(u) \cap S_1) \cup \{v\} \neq S$. Therefore, S is a strictly locating set in $H + K_1$. □

Corollary 2.12. *Let H be a connected non-trivial graph and $K_1 = \langle v \rangle$. Then $sln(H + K_1) = sln(H) + 1$.*

Proof. Follows from Theorem 2.11. □

Corollary 2.13. *Let G be a connected graph of order $n \geq 1$ and let K_n be a complete graph of order $n \geq 1$. Then $sln(G + K_n) = sln(G) + n$.*

Proof. Follows from Theorem 2.9 and Theorem 2.11. □

2.2. Strictly Locating Sets in the Corona of Graphs

Let G and H be graphs of order m and n, respectively. The *corona* of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, then joining the ith vertex of G to every vertex of the ith copy of H. For every $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding the join $\langle \{v\} \rangle + H^v$.

Theorem 2.14. Let G and H be non-trivial connected graphs. Then $S \subseteq V(G \circ H)$ is a strictly locating set in $G \circ H$ if and only if $V(G \circ H) \backslash S$ admits at most a single element x with $N_{G \circ H}(x) \cap S = \varnothing$ and $S = A \cup B \cup C \cup D$, where, $A \subseteq V(G)$, $B = \cup\{B_v : v \in A$ and B_v is a locating set in $H^v\}$, $C = \cup\{E_w : w \notin A, N_G(w) \cap A \neq \varnothing$ and E_w is a locating set in $H^w\}$ and $D = \cup\{D_w : w \neq A, N_G(w) \cap A = \varnothing$ and D_w is strictly locating set in $H^w\}$.

Proof. Suppose that S is a strictly locating set in $G \circ H$. Let $A = V(G) \cap S$ and let $v \in A$. Let $B_v = V(H^v) \cap S$ and let $x, y \in V(H^v) \backslash B_v$ with $x \neq y$. Then $x, y \notin S$. Since S is a locating set in $G \circ H$, $(N_{H^v}(x) \cap B_v) \cup \{v\} = N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S = (N_{H^v}(y) \cap B_v) \cup \{v\}$. Hence, B_v is a locating set in H^v. Next, let $w \notin A$. Consider the following cases:

Case 1. Suppose that $N_G(w) \cap A \neq \varnothing$.

Let $E_w = V(H^w) \cap S$ and $x, y \in V(H^w) \backslash E_w$ with $x \neq y$. Then $x, y \notin S$. Since S is a strictly locating set and $w \notin S$, $N_{H^w}(x) \cap E_w = N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S = N_{H^w}(y) \cap E_w$. Thus, E_w is a locating set in H^w.

Case 2. Suppose that $N_G(w) \cap A = \varnothing$.

Let $D_w = V(H^w) \cap S$. As in Case 1, D_w is a locating set in H^w. Suppose there exists $x \in V(H^w)$ such that $N_{H^w}(x) \cap D_w = N_{G \circ H}(x) \cap S = D_w$. Since $w \notin S$ and $N_G(w) \cap A = \varnothing$, $N_{G \circ H}(x) \cap S = N_{G \circ H}(w) \cap S = D_w$. Thus, S is not a locating set in $G \circ H$, contrary to our assumption. Thus, D_w is a strictly locating set in H^w.

Let $B = \bigcup\{B_v : v \in A$ and B_v is a locating set in $H^v\}$, $C = \bigcup\{E_w : w \notin A, N_G(w) \cap A \neq \varnothing$ and E_w is a locating set in $H^w\}$ and $D = \bigcup\{D_w : w \notin A, N_G(w) \cap A = \varnothing$ and D_w is strictly locating set in $H^w\}$. Then $S = A \cup B \cup C \cup D$. Moreover, since S is a strictly locating set, $V(G \circ H) \backslash S$ admits at most a single element whose neighborhood does not intersect with S.

For the converse, suppose that $S = A \cup B \cup C \cup D$, where A, B, C and D are the sets possessing the properties described. Let $x, y \in V(G \circ H) \backslash S$ with $x \neq y$ and let $u, v \in V(G)$ such that $x \in V(u + H^u)$ and $y \in V(v + H^v)$. Suppose $u = v$. Consider the following cases:

Case 1. Suppose that $v \in S$.

Then $x, y \in V(H^v) \backslash B_v$, where B_v is a locating set in H^v. Hence, $N_{H^v}(x) \cap B_v \neq N_{H^v}(y) \cap B_v$. Thus, $N_{G \circ H}(x) \cap S = (N_{H^v}(x) \cap B_v) \cup \{v\} \neq (N_{H^v}(y) \cap B_v) \cup \{v\} = N_{G \circ H}(y) \cap S$.

Case 2. Suppose that $v \notin S$.

If $x, y \in V(H^v)$, then $x, y \notin S_v = V(H^v) \cap S$, where S_v (E_v or D_v) is a locating set in H^v by assumption. Thus, $N_{G \circ H}(x) \cap S = N_{H^v}(x) \cap S_v \neq N_{H^v}(y) \cap S_v = N_{G \circ H}(y) \cap S$. Suppose that $x = v$ and $y \in V(H^v)$. If $N_G(v) \cap S \neq \varnothing$, say $w \in N_G(v) \cap S$, then $w \in [N_{G \circ H}(x) \cap S] \backslash [N_{G \circ H}(y) \cap S]$. Thus, $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$. If $N_G(v) \cap S = \varnothing$, then $S_v = V(H^v) \cap S = D_v$ is a strictly locating set by assumption. Thus, $N_{G \circ H}(x) \cap S = D_v \neq N_{H^v}(y) \cap S_v = N_{G \circ H}(y) \cap S$.

Suppose now that $u \neq v$. Consider the following cases:

Case 1. Suppose that $u, v \in S$.

Then $x \neq u$ and $y \neq v$. Since $u \in N_{G \circ H}(x)$ and $v \in N_{G \circ H}(y)$, we have $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$.

Case 2. Supppose that $u \notin S$ or $v \notin S$.

We may suppose that $u \notin S$. Then $S_u = V(H^u) \cap S$ is (equal to E_u or D_u). If $x = u$, then there exists $c \in S_u$ such that $c \in N_{G \circ H}(x) \backslash N_{G \circ H}(y)$. Suppose $x \neq u$. If $v \in S$, then $v \in N_{G \circ H}(y) \backslash N_{G \circ H}(x)$. Suppose $v \notin S$. If $N_{G \circ H}(x) \cap S \neq \varnothing$ and $N_{G \circ H}(y) \cap S \neq \varnothing$, then $N_{G \circ H}(x) \cap S \neq N_{G \circ H}(y) \cap S$. If one of $N_{G \circ H}(x) \cap S$ and $N_{G \circ H}(y) \cap S$ is empty, then the other is non-empty by assumption. Therefore, in all cases S is a locating set in $G \circ H$.

Now, let $t \in V(G \circ H) \backslash S$ and let $v \in V(G)$ such that $t \in V(v + H^v)$. Suppose $t = v$. Then $v \notin S$. Thus, $S_v = V(H^v) \cap S$ is a locating set in H^v where S_v is (E_v or D_v). Consider the following cases:

Case 1. Suppose $S_v = E_v$.

Then G non-trivial implies that $N_G(v) \cap A \neq \varnothing$, say $w \in N_G(v) \cap A = N_G(t) \cap A$. Since H is non-trivial, it follows that $V(H^w) \cap S = E_w \neq \varnothing$. Let $y \in E_w \subseteq S$. Then $y \notin N_{G \circ H}(t) \cap S$. Hence, $N_{G \circ H}(t) \cap S \neq S$.

Case 2. Suppose $S_v = D_v$.

Then $N_G(v) \cap A = N_G(t) \cap A = \varnothing$. Since G is non-trivial, $\exists\ w \in V(G)$ with $w \neq t$ and $wt \in E(G)$. Also, H non-trivial implies that $V(H^w) \cap S = D_w \neq \varnothing$. Let $y \in D_w \subseteq S$. Then $y \notin N_{G \circ H}(t) \cap S$. Therefore, $N_{G \circ H}(t) \cap S \neq S$.

Suppose that $t \neq v$. Consider the following cases:

Case 1. Suppose $v \in S$.

Then $t \in V(H^v) \backslash B_v$, where $B_v = V(H^v) \cap S$. Since G is non-trivial, $\exists\ w \in V(G)$ with $w \neq v$ and $wv \in E(G)$. Since H is non-trivial, it follows that B_w, D_w or E_w are non-empty. Hence, $\exists\ y \in B_w$ (D_w or E_w) such that $y \notin N_{G \circ H}(t) \cap S$. Therefore, $N_{G \circ H}(t) \cap S \neq S$.

Case 2. Suppose $v \notin S$.

Then $t \in V(H^v) \backslash S_v$, where S_v is either D_v or E_v. Suppose that $S_v = E_v$. Then $N_G(v) \cap A \neq \varnothing$. Let $y \in N_G(v) \cap A$. Since H is non-trivial, $B_y = V(H^v) \cap S \neq \varnothing$. Let $u \in B_y$. Then $u \notin N_{G \circ H}(t) \cap S$. Hence, $N_{G \circ H}(t) \cap S \neq S$. Suppose that $S_v = D_v$. Let $w \in V(G)$ such that $vw \in E(G)$. Then D_w or E_w is non-empty. Thus, $\exists\ r \in D_w$ or E_w such that $r \notin V_{G \circ H}(t) \cap S$. Therefore, $N_{G \circ H}(t) \cap S \neq S$.

Therefore, in all cases, S is a strictly locating set in $G \circ H$. □

Corollary 2.15. *Let G and H be non-trivial connected graphs. Then $|V(G)|\, ln(G) \leq sln(G \circ H) \leq |V(G)|\, sln(G)$.*

Proof. Let S be a minimum strictly locating set in $G \circ H$. Then $S = A \cup B \cup C \cup D$, where A, B, C and D are the sets described in Theorem 2.14. By Theorem 2.2, $sln(H) \leq ln(H) + 1$. Since $ln(H) \leq sln(H)$, it follows that

$$\begin{aligned} sln(G \circ H) &= |S| \\ &= |A| + |B| + |C| + |D| \\ &\geq |A| + |A|\, ln(H) + (|V(G)| - |A|) ln(H) \\ &= |A|\,(1 + ln(H)) + (|V(G)| - |A|) ln(H) \\ &\geq |A|\, sln(H) + (|V(G)| - |A|) ln(H) \\ &\geq |A|\, ln(H) + (|V(G)| - |A|) ln(H) \\ &= |V(G)|\, ln(H). \end{aligned}$$

Next, let T be a sln-set in H. For each $v \in V(G)$, let $T_v \subseteq V(H^v)$ with $\langle T_v \rangle \cong \langle T \rangle$. Then $S = \bigcup_{v \in V(G)} T_v$ is a strictly locating set in $G \circ H$ by Theorem 2.14. Therefore, $|V(G)|\, ln(H) \leq sln(G \circ H) \leq |S| = |V(G)|\, sln(H)$. □

Corollary 2.16. *Let G and H be non-trivial connected graphs with $ln(H) = sln(H)$. Then $sln(G \circ H) = |V(G)|\, sln(H)$.*

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